Assignment 2 - Answers

Problems 21 - 37

21. We are given that u and v are parallel. Thus $\mathsf{v} = c\mathsf{u}$ or $\mathsf{u} = c\mathsf{v}$ for some scalar c. Suppose $\mathsf{v} = c\mathsf{u}$. This can be rewritten $1\mathsf{v} - c\mathsf{u} = \mathsf{0}$. This expresses $\mathsf{0}$ as a linear combination of u and v in a nontrivial way since at least one coefficient, the coefficient of v, is not zero. The case $\mathsf{u} = c\mathsf{v}$ is similar.

22. $\|\mathsf{u}\| = \sqrt{0^2 + 3^2 + 4^2} = 5$, so $\frac{1}{5}\mathsf{u} = \frac{1}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ is a unit vector in the direction of u. Since $\|\mathsf{v}\| = \sqrt{1^2 + 2^2 + 2^2} = 3$, $\frac{1}{3}\mathsf{v}$ is a unit vector in the direction of v and

$-\frac{2}{3}\mathsf{v} = -\frac{2}{3}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}\\-\frac{4}{3}\\-\frac{4}{3}\end{bmatrix}$ is a vector of length 2 in the direction opposite to v.

23. A better approach would be to say that $\mathsf{u} = \frac{4}{7}\mathsf{v}$, with $\mathsf{v} = \begin{bmatrix}-2\\1\\5\end{bmatrix}$, so $\|\mathsf{u}\| = \frac{4}{7}\|\mathsf{v}\| = \frac{4}{7}\sqrt{30}$.

24. (a) Since $\mathsf{u}\cdot\mathsf{v} = 0$, the vectors are orthogonal: $\theta = \frac{\pi}{2}$.

    (b) $\mathsf{u}\cdot\mathsf{v} = -3+4 = 1$, $\|\mathsf{u}\| = \sqrt{3^2+4^2} = 5$, $\|\mathsf{v}\| = \sqrt{1^2+1^2} = \sqrt{2}$, so $\cos\theta = \dfrac{\mathsf{u}\cdot\mathsf{v}}{\|\mathsf{u}\|\,\|\mathsf{v}\|} = \dfrac{1}{5\sqrt{2}} \approx .1414$. $\theta \approx \arccos(.1414) \approx 1.43$ rads $\approx 82°$.

25. $\mathsf{u}\cdot\mathsf{v} = \|\mathsf{u}\|\,\|\mathsf{v}\|\cos\theta = 3\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$.

26. Let θ be the angle between u and v. If $\mathsf{u}\cdot\mathsf{v} = -7$, $\|\mathsf{u}\| = 3$, and $\|\mathsf{v}\| = 2$, then $\cos\theta = \frac{\mathsf{u}\cdot\mathsf{v}}{\|\mathsf{u}\|\|\mathsf{v}\|} = \frac{-7}{6}$. This is impossible since $-1 \le \cos\theta \le +1$ for any θ.

27. (a) $\|\mathsf{u}\| = \sqrt{3^2+4^2+0^2} = 5$; $\mathsf{u}+\mathsf{v} = \begin{bmatrix}5\\5\\2\end{bmatrix}$, so $\|\mathsf{u}+\mathsf{v}\| = \sqrt{5^2+5^2+2^2} = \sqrt{54}$; $\left\|\frac{\mathsf{w}}{\|\mathsf{w}\|}\right\| = 1$ (this is true for any $\mathsf{w} \neq \mathsf{0}$).

    (b) The answer is $\frac{1}{\|\mathsf{u}\|}\mathsf{u} = \frac{1}{5}\mathsf{u} = \begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\\0\end{bmatrix}$.

    (c) First we find a vector of norm 1 in the direction of v; this vector is $\frac{1}{\|\mathsf{v}\|}\mathsf{v} = \frac{1}{3}\mathsf{v}$. A vector of norm 4 in this direction is $\frac{4}{3}\mathsf{v}$, so a vector of norm 4 in the opposite direction is $-\frac{4}{3}\mathsf{v} = \begin{bmatrix}-\frac{8}{3}\\-\frac{4}{3}\\-\frac{8}{3}\end{bmatrix}$.

28. We are given $\mathsf{u}\cdot\mathsf{u} = 3^2 = 9$ and $\mathsf{v}\cdot\mathsf{v} = 5^2 = 25$. So $(\mathsf{u}-\mathsf{v})\cdot(2\mathsf{u}-3\mathsf{v}) = 2\mathsf{u}\cdot\mathsf{u} - 5\mathsf{u}\cdot\mathsf{v} + 3\mathsf{v}\cdot\mathsf{v} = 2(9) - 5(8) + 3(25) = 53$.

    $\|\mathsf{u}+\mathsf{v}\|^2 = (\mathsf{u}+\mathsf{v})\cdot(\mathsf{u}+\mathsf{v}) = \mathsf{u}\cdot\mathsf{u} + 2\mathsf{u}\cdot\mathsf{v} + \mathsf{v}\cdot\mathsf{v} = 9 + 2(8) + 25 = 50$.

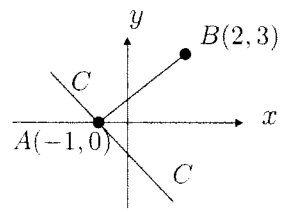
29. $\mathsf{u} + k\mathsf{v} = \begin{bmatrix}1+k\\2-k\end{bmatrix}$, so

$$\|\mathsf{u}+k\mathsf{v}\| = \sqrt{(1+k)^2+(2-k)^2} = \sqrt{k^2+2k+1+4-4k+k^2} = \sqrt{2k^2-2k+5}.$$

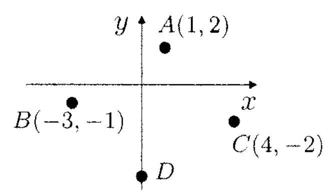
    We want $2k^2-2k+5 = 3^2$, so $2k^2-2k-4 = 0$, $k^2-k-2 = 0$, $(k+1)(k-2) = 0$. Thus $k = -1, 2$.

30. $(\mathsf{u} + k\mathsf{v}) \cdot \mathsf{u} = \mathsf{u} \cdot \mathsf{u} + k\mathsf{v} \cdot \mathsf{u} = \mathsf{u} \cdot \mathsf{u}$ since $\mathsf{v} \cdot \mathsf{u} = 0$. Since $\mathsf{u} \neq 0$, we know that $\mathsf{u} \cdot \mathsf{u} \neq 0$, so $(\mathsf{u} + k\mathsf{v}) \cdot \mathsf{u} \neq 0$.

31. There are two possible answers, as the figure to the right shows. Since $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and we wish $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$, we must have $\overrightarrow{AC} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for some t. Since $\left\|\overrightarrow{AC}\right\| = |t|\sqrt{2}$, we want $|t|\sqrt{2} = 2$, so $|t| = \sqrt{2}$. If $t = \sqrt{2}$ and $C = (x, y)$, then $\overrightarrow{AC} = \begin{bmatrix} x+1 \\ y \end{bmatrix} = \sqrt{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, giving $x + 1 = -\sqrt{2}$, $y = \sqrt{2}$, and $C = (-1 - \sqrt{2}, \sqrt{2})$. If $t = -\sqrt{2}$ and $C = (x, y)$, then $\overrightarrow{AC} = \begin{bmatrix} x+1 \\ y \end{bmatrix} = -\sqrt{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, giving $x + 1 = \sqrt{2}$, $y = -\sqrt{2}$, and $C = (-1 + \sqrt{2}, -\sqrt{2})$.



32. The picture at the right shows the approximate position of the points. There is only one possible location for D. Let D have coordinates (x, y). Since $\overrightarrow{CA} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \overrightarrow{DB}$, we need $\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 - x \\ -1 - y \end{bmatrix}$, $-3 - x = -3$ and $4 = -1 - y$. Thus $x = 0$, $y = -5$, and D is $(0, -5)$.



33. u is orthogonal to v since $\mathsf{u} \cdot \mathsf{v} = 12 - 8 - 4 = 0$.
u is orthogonal to w since $\mathsf{u} \cdot \mathsf{w} = -6 + 20 - 14 = 0$.

34. Commutativity: Let $\mathsf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} z \\ w \end{bmatrix}$ be vectors. Then $\mathsf{u} \cdot \mathsf{v} = xz + yw$ and $\mathsf{v} \cdot \mathsf{u} = zx + wy = xz + yw$, so $\mathsf{u} \cdot \mathsf{v} = \mathsf{v} \cdot \mathsf{u}$.
For the second part, for any vector $\mathsf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, we have $\mathsf{v} \cdot \mathsf{v} = x^2 + y^2$, the sum of squares of real numbers. Since $a^2 \geq 0$ for any real number a, $x^2 + y^2 \geq 0$. Moreover, $x^2 + y^2$ is positive if either $x \neq 0$ or $y \neq 0$; that is, $x^2 + y^2 = 0$ if and only if $x = y = 0$. This says precisely that $\|\mathsf{v}\| = 0$ if and only if $\mathsf{v} = 0$.

35. We have $\mathsf{u} \cdot \mathsf{v} = -6 + 5 = -1$, $\|\mathsf{u}\| = \sqrt{5}$ and $\|\mathsf{v}\| = \sqrt{34}$. The Cauchy–Schwarz inequality says that $1 \leq \sqrt{170}$.

36. Let $\mathsf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Then $\mathsf{u} \cdot \mathsf{v} = a\cos\theta + b\sin\theta$, $\|\mathsf{u}\|^2 = a^2 + b^2$ and $\|\mathsf{v}\|^2 = \cos^2\theta + \sin^2\theta = 1$. Squaring the Cauchy–Schwarz inequality gives $(\mathsf{u} \cdot \mathsf{v})^2 \leq \|\mathsf{u}\|^2 \|\mathsf{v}\|^2$, which is the given inequality.

37. Let $\mathsf{u} = \begin{bmatrix} \sqrt{3}a \\ b \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} \sqrt{3}c \\ d \end{bmatrix}$. We have $\mathsf{u} \cdot \mathsf{v} = 3ac + bd$, $\|\mathsf{u}\| = \sqrt{3a^2 + b^2}$, and $\|\mathsf{v}\| = \sqrt{3c^2 + d^2}$, so the result follows upon squaring the Cauchy–Schwarz inequality: $(\mathsf{u} \cdot \mathsf{v})^2 \leq \|\mathsf{u}\|^2 \|\mathsf{v}\|^2$.