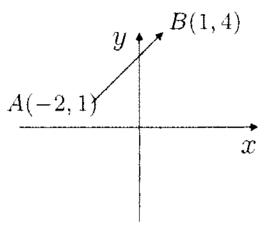
Extra Problems for Linear Algebra I

SOLUTIONS

1. $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$



2. Let $B = (x, y)$. Then $\overrightarrow{AB} = \begin{bmatrix} x-1 \\ y-4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, so $x - 1 = -1$, $y - 4 = 2$, and $B = (0, 6)$.

3. $\mathsf{x} = -\frac{1}{4}\mathsf{u} = -7\mathsf{w}$; x is not a scalar multiple of v.

4. The zero vector, u_4 is parallel to each of the others. The remaining pairs of parallel vectors are $\mathsf{u}_2, \mathsf{u}_5$ and $\mathsf{u}_1, \mathsf{u}_3$.

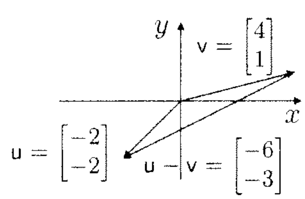
5. (a) $4\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \end{bmatrix}$

   (b) $3\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 11 \end{bmatrix}$

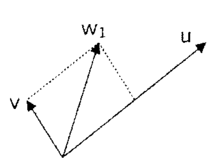
6. Since $2\mathsf{u} - \mathsf{v} = \begin{bmatrix} 2a - 1 \\ -10 - 6 + b \end{bmatrix} = \begin{bmatrix} 2a-1 \\ -16+b \end{bmatrix}$, we must have $2a - 1 = 3$ and $-16 + b = 1$, so $a = 2$ and $b = 17$.

7. $\mathsf{x} = \mathsf{y} + \mathsf{u}$, so $2(\mathsf{y} + \mathsf{u}) + 3\mathsf{y} = \mathsf{v}$. This is $5\mathsf{y} + 2\mathsf{u} = \mathsf{v}$, so $5\mathsf{y} = \mathsf{v} - 2\mathsf{u}$ and $\mathsf{y} = \frac{1}{5}(\mathsf{v} - 2\mathsf{u}) = -\frac{2}{5}\mathsf{u} + \frac{1}{5}\mathsf{v}$. So $\mathsf{x} = \mathsf{y} + \mathsf{u} = \frac{3}{5}\mathsf{u} + \frac{1}{5}\mathsf{v}$.

8. $\mathsf{u} - \mathsf{v} = \begin{bmatrix} -6 \\ -3 \end{bmatrix}$



9.



It appears that $\mathsf{w}_1 = \frac{1}{2}\mathsf{u} + \mathsf{v}$.

10. We wish to find a and b such that $\begin{bmatrix}7\\7\end{bmatrix} = a\begin{bmatrix}2\\3\end{bmatrix} + b\begin{bmatrix}-1\\2\end{bmatrix}$.

    Thus we should have $\begin{aligned} 2a - b &= 7 \\ 3a + 2b &= 7 \end{aligned}$.

    The first equation gives $b = 2a - 7$, so substituting in the second, $3a + 2(2a - 7) = 7$, $7a - 14 = 7$, $a = 3$, $b = -1$.

11. The typical linear combination of $\begin{bmatrix}-2\\-2\end{bmatrix}$ and $\begin{bmatrix}3\\3\end{bmatrix}$ is a vector of the form $a\begin{bmatrix}-2\\-2\end{bmatrix} + b\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}-2a+3b\\-2a+3b\end{bmatrix}$. This is a vector both of whose components are equal, so it can never be $\begin{bmatrix}1\\2\end{bmatrix}$. The answer is "no".

12. Setting $\begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}4\\5\\6\end{bmatrix} + c\begin{bmatrix}0\\1\\0\end{bmatrix}$ and equating components gives

$$\begin{aligned} a + 4b &= 1 \\ 2a + 5b + c &= 0 \\ 3a + 6b &= 0. \end{aligned}$$

    Substituting $a = -2b$ in the first equation gives $b = \frac{1}{2}$. The second equation then gives $c = -2a - 5b = -\frac{1}{2}$. Indeed, $\begin{bmatrix}1\\0\\0\end{bmatrix} = -\mathsf{v}_1 + \frac{1}{2}\mathsf{v}_2 - \frac{1}{2}\mathsf{v}_3$ is a linear combination of $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$.

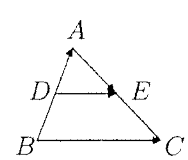
13. There are many ways to express $\begin{bmatrix}0\\0\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\4\end{bmatrix}$ and $\begin{bmatrix}-2\\-8\end{bmatrix}$. Here are some of these.

$$\begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}1\\4\end{bmatrix} + 0\begin{bmatrix}-2\\-8\end{bmatrix} = 2\begin{bmatrix}1\\4\end{bmatrix} + 1\begin{bmatrix}-2\\-8\end{bmatrix} = 4\begin{bmatrix}1\\4\end{bmatrix} + 2\begin{bmatrix}-2\\-8\end{bmatrix} = 20\begin{bmatrix}1\\4\end{bmatrix} + 10\begin{bmatrix}-2\\-8\end{bmatrix}.$$

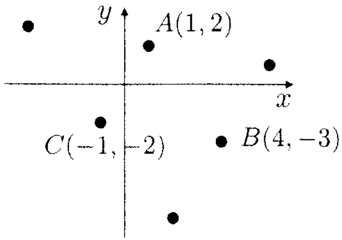
14. A linear combination of $2\mathsf{u}$ and $-3\mathsf{v}$ is a vector of the form $a(2\mathsf{u}) + b(-3\mathsf{v})$. This is the same as $(2a)\mathsf{u} + (-3b)\mathsf{v}$, and hence also a linear combination of u and v.

15. $\mathsf{u} = \begin{bmatrix}2\\2\end{bmatrix}$; $\mathsf{v} = \begin{bmatrix}2\\-1\end{bmatrix}$; $\overrightarrow{OD} = \begin{bmatrix}4\\1\end{bmatrix} = \mathsf{u} + \mathsf{v}$.

16. We wish to show that $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$. Now $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB}$ and $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$, so $\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = -\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB}) = \frac{1}{2}\overrightarrow{BC}$.



17. The picture at the right shows the position of the points. The answer is not unique. There are three possibilities for $D(x, y)$.



If $\overrightarrow{CA} = \overrightarrow{DB}$, then $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 - x \\ -3 - y \end{bmatrix}$ and D is $(2, -7)$.
If $\overrightarrow{DA} = \overrightarrow{CB}$, then $\begin{bmatrix} 1 - x \\ 2 - y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ and D is $(-4, 3)$.
If $\overrightarrow{AD} = \overrightarrow{CB}$, then $\begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ and D is $(6, 1)$.