195. $\begin{vmatrix} a & -g & 2d \\ b & -h & 2e \\ c & -i & 2f \end{vmatrix} = -2 \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$ factoring -1 from the second column and 2 from the third column

$= -2 \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$ since the determinant of a matrix is the determinant of its transpose

$= +2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(-3) = -6$ interchanging rows two and three.

196. Suppose $A = LDU$. Each of L and U is triangular with 1s on the diagonal, so $\det L = \det U = 1$, the product of the diagonal entries in each case. Thus $\det A = \det L \det D \det U = \det D$. Using just the third elementary row operation to reduce $A = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 1 & -10 & -2 \\ 4 & 0 & -7 & 6 \\ 6 & -3 & 0 & 1 \end{bmatrix}$ to an upper triangular matrix, we have

$$\begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 1 & -10 & -2 \\ 4 & 0 & -7 & 6 \\ 6 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & \frac{3}{2} & -12 & -\frac{5}{2} \\ 0 & 2 & -15 & 4 \\ 0 & 0 & -12 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & \frac{3}{2} & -12 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{22}{3} \\ 0 & 0 & -12 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & \frac{3}{2} & -12 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{22}{3} \\ 0 & 0 & 0 & 86 \end{bmatrix} = U' = DU$$

with $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 86 \end{bmatrix}$. Thus $\det A = \det D = 2(\frac{3}{2})(1)(86) = 258$.

197. We have $\frac{1}{2}(I - A)A = I$, so $(I - A)A = 2I$. Taking the determinant of each side gives $\det(I - A) \det A = 2^n$. If the product of two numbers is a power of 2, each number itself is a power of 2, so the result follows.

198. $A\mathsf{v} = \begin{bmatrix} 8 \\ 16 \\ 0 \end{bmatrix} = 4\mathsf{v}$. Thus v is an eigenvector of A corresponding to the eigenvalue 4.

199. $A\mathsf{v}_1 = 5\mathsf{v}_1$, so v_1 is an eigenvector corresponding to $\lambda = 5$.

$A\mathsf{v}_2 = 0\mathsf{v}_2$, so v_2 is an eigenvector corresponding to $\lambda = 0$.

$A\mathsf{v}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is not $\lambda\mathsf{v}_3$ for any λ, so v_3 is not an eigenvector.

$A\mathsf{v}_4 = \begin{bmatrix}-3\\-6\end{bmatrix}$ is not $\lambda\mathsf{v}_4$ for any λ, so v_4 is not an eigenvector.

v_5 is not an eigenvector since an eigenvector is a **nonzero** vector.

200. (a) It is easy to compute and factor $\det(A - \lambda I)$ when A is 2×2, so this is the most straightforward way the question for this particular A. The matrix $A - \lambda I = \begin{bmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{bmatrix}$ has determinant $(1-\lambda)(4-\lambda)-4 = \lambda^2 - 5\lambda = \lambda(\lambda-5)$. Thus $\lambda = 0$ and $\lambda = 5$ are eigenvalues, while -1, 1, and 3 are not.

(b) When A is larger than 2×2, it is not so easy to find $\det(A - \lambda I)$ and its roots, so we answer the question in this instance by computing $\det(A - \lambda I)$ for each given value of λ and determining whether or not the matrix has 0 determinant. The matrix $A - \lambda I = \begin{bmatrix}5-\lambda & -7 & 7\\ 4 & -3-\lambda & 4\\ 4 & -1 & 2-\lambda\end{bmatrix}$. When $\lambda = 1$, $A - \lambda I = \begin{bmatrix}4 & -7 & 7\\ 4 & -4 & 4\\ 4 & -1 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -1 & 1\\ 4 & -7 & 7\\ 4 & -1 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -1 & 1\\ 0 & -3 & 3\\ 0 & 3 & -3\end{bmatrix}$ has determinant 0, so $\lambda = 1$ is an eigenvalue.

When $\lambda = 2$, $A - \lambda I = \begin{bmatrix}3 & -7 & 7\\ 4 & -5 & 4\\ 4 & -1 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}3 & -7 & 7\\ 4 & -5 & 4\\ 0 & 4 & -4\end{bmatrix} \longrightarrow \begin{bmatrix}12 & -28 & 28\\ 12 & -15 & 12\\ 0 & 1 & -1\end{bmatrix} \longrightarrow \begin{bmatrix}12 & -28 & 28\\ 0 & 13 & -16\\ 0 & 1 & -1\end{bmatrix}$ has nonzero determinant, so $\lambda = 2$ is not an eigenvalue.

When $\lambda = 4$, $A - \lambda I = \begin{bmatrix}1 & -7 & 7\\ 4 & -7 & 4\\ 4 & -1 & -2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -7 & 7\\ 0 & 21 & -24\\ 0 & 27 & -30\end{bmatrix}$ has nonzero determinant, so $\lambda = 4$ is not an eigenvalue.

When $\lambda = 5$, $A - \lambda I = \begin{bmatrix}0 & -7 & 7\\ 4 & -8 & 4\\ 4 & -1 & -3\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -2 & 1\\ 0 & -1 & 1\\ 0 & 7 & -7\end{bmatrix}$ has determinant 0, so $\lambda = 5$ is an eigenvalue.

When $\lambda = 6$, $A - \lambda I = \begin{bmatrix}-1 & -7 & 7\\ 4 & -9 & 4\\ 4 & -1 & -4\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & -7 & 7\\ 0 & -37 & 31\\ 0 & -29 & 24\end{bmatrix}$ has nonzero determinant, so $\lambda = 6$ is not an eigenvalue.

201. (a) $A\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}8\\12\end{bmatrix} = 4\begin{bmatrix}2\\3\end{bmatrix}$, so $\begin{bmatrix}2\\3\end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$.

(b) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix}1-\lambda & 2\\ 3 & 2-\lambda\end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4.$$

(c) Since $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$ for $\lambda = 4$ and $\lambda = -1$, the eigenvalues are 4 and -1.

202. (a) The characteristic polynomial of A is $\begin{vmatrix}5-\lambda & 8\\ 4 & 1-\lambda\end{vmatrix} = (5-\lambda)(1-\lambda) - 32 = \lambda^2 - 6\lambda - 27 = (\lambda - 9)(\lambda + 3)$, so $\lambda = 9$ and $\lambda = -3$ are the eigenvalues of A.

To find the eigenspace corresponding to $\lambda = 9$, we must solve the homogeneous system $(A - \lambda I)\mathsf{x} = 0$ with $\lambda = 9$. We have

$$A - \lambda I = \begin{bmatrix} -4 & 8 \\ 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

The variable $x_2 = t$ is free and $x_1 = 2x_2 = 2t$. The eigenspace corresponding to $\lambda = 9$ is the set of vectors of the form of $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda = -3$, we must solve the homogeneous system $(A - \lambda I)\mathsf{x} = 0$ with $\lambda = -3$. We have

$$A - \lambda I = \begin{bmatrix} 8 & 8 \\ 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Again, $x_2 = t$ is free, but $x_1 = -x_2 = -t$. The eigenspace corresponding to $\lambda = -3$ is the set of vectors of the form of $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(b) The characteristic polynomial of $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$ is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 & 3 \\ 2 & 6-\lambda & -6 \\ 1 & 2 & -1-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3.$$

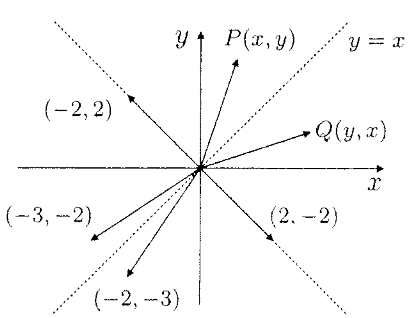
The only eigenvalue of A is $\lambda = 2$. To find the corresponding eigenspace, we solve the homogeneous system $(A - \lambda I)\mathsf{x} = 0$ for $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $\lambda = 2$. We have

$$A - \lambda I = \begin{bmatrix} -1 & -2 & 3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solutions are $x_3 = t$, $x_2 = s$, $x_1 = -2s + 3t$. The eigenspace consists of vectors of the form $\begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

203. (a) Since $A\mathsf{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$, Q is (y, x).

(b) The vector $\overrightarrow{PQ} = \begin{bmatrix} y - x \\ x - y \end{bmatrix}$. The dot product of $\overrightarrow{PQ}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (which is the direction of ℓ) is $y - x + x - y = 0$, so PQ and ℓ are perpendicular. Since the midpoint of PQ, which is $(\frac{x+y}{2}, \frac{x+y}{2})$, is on ℓ, ℓ is the right bisector of PQ.



(c) Multiplication by A is reflection in the line with equation $y = x$.

(d) Reflection in a line fixes the line; in fact, it fixes every vector on the line:

$$A\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}.$$

Thus every vector on this line is an eigenvector corresponding to $\lambda = 1$. This reflection also fixes the line with equation $y = -x$ because every vector on this line is moved to its negative (which is still on the line):

$$A\begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} -x \\ x \end{bmatrix} = -\begin{bmatrix} x \\ -x \end{bmatrix}.$$

Every vector on the line with equation $y = -x$ is an eigenvector corresponding to $\lambda = -1$. The matrix A has two eigenspaces, the lines with equations $y = x$ and $y = -x$.

204. Since multiplication by A is reflection in a line, any vector on the line will be fixed by A, so we seek solutions to $A\mathsf{x} = \mathsf{x}$:

$$A\mathsf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ gives } x = 0,$$

so our matrix reflects vectors in the y-axis. Any nonzero vector on the y-axis is an eigenvector corresponding to $\lambda = 1$ and any nonzero vector on the x-axis is an eigenvector corresponding to $\lambda = -1$ since these vectors are mapped to their negatives.

205. We have $A(a\mathsf{u} + b\mathsf{v}) = aA\mathsf{u} + bA\mathsf{v} = a\lambda\mathsf{u} + b\lambda\mathsf{v} = \lambda(a\mathsf{u} + b\mathsf{v})$. This shows that $a\mathsf{u} + b\mathsf{v}$ is an eigenvector corresponding to λ (provided it is not $\mathsf{0}$).

206. Let $\mathsf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and observe that $A\mathsf{x} = 17\mathsf{x}$.

207. Let λ be an eigenvalue of A and let $\mathsf{x} \neq \mathsf{0}$ be a corresponding eigenvector. Then $A\mathsf{x} = \lambda\mathsf{x}$. Multiplying on the left by P gives $PA\mathsf{x} = \lambda P\mathsf{x}$. Since $PA = BP$, we get $BP\mathsf{x} = \lambda P\mathsf{x}$, that is, $B(P\mathsf{x}) = \lambda(P\mathsf{x})$. Since P is invertible, $P\mathsf{x} \neq \mathsf{0}$. So λ is an eigenvalue of B with corresponding eigenvector $P\mathsf{x}$.

208. The only eigenvalue is $\lambda = 2$. The corresponding eigenspace is R^3.

209. The answer is yes. Since $A\mathsf{v} = \lambda\mathsf{v}$, $(5A)\mathsf{v} = 5\lambda\mathsf{v} = (5\lambda)\mathsf{v}$, so v is an eigenvector of $5A$ with eigenvalue 5λ.

210. (a) A similar to I means $A = P^{-1}IP$ for some invertible matrix P. But $P^{-1}IP = P^{-1}P = I$, so $A = I$.

(b) No, by 210(a). The given matrix is not I.

(c) The characteristic polynomial of A is $(\lambda - 1)^2$, so $\lambda = 1$ is the only eigenvalue of A. If A were diagonalizable, it would be similar to I, hence equal to I by 210(a). Since this is not the case, A is not diagonalizable.

211. We have equations of the type $B = P^{-1}AP$ and $C = Q^{-1}BQ$ for invertible matrices P and Q. Thus $C = Q^{-1}P^{-1}APQ = R^{-1}AR$, with R the invertible matrix PQ. Thus A is similar to C.

212. Since B is similar to A, the determinant and characteristic polynomial of B are, respectively, the determinant and characteristic polynomial of A. Thus $\det B = -2$ and the characteristic polynomial of B is $\lambda^2 - 5\lambda - 2$.

213. (a) $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2$ so $\lambda = 0$ is the only eigenvalue. Corresponding eigenvectors are obtained by solving $(A - \lambda I)\mathsf{x} = 0$ for $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $\lambda = 0$. Since A is already in row echelon form, we have $x_1 = t$ is free and $x_2 = 0$, so $\mathsf{x} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) If $P^{-1}AP = D$, the columns of P are eigenvectors. The only possibility for P here is a matrix of the form $\begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}$ and such a matrix is not invertible.

214. (a) Since $A - 2I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$ is not invertible, we can find nonzero vectors x with $(A - 2I)\mathsf{x} = 0$ using Gaussian elimination: $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$, so if $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_2 = t$ is free and $x_1 = 2x_2 = 2t$. Thus $\mathsf{x} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The eigenspace corresponding to $\lambda = 2$ is the set of scalar multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) The characteristic polynomial of A is $(\lambda - 2)(\lambda + 3) = \lambda^2 + \lambda - 6$.

(c) A is diagonalizable because A is 2×2 and A has two different eigenvalues.

(d) The columns of P should be eigenvectors corresponding to -3 and 2, respectively. So $P = \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$.

(e) The columns of Q are eigenvectors of A corresponding to eigenvalues -3 and 2, respectively. So $Q^{-1}AQ$ is the diagonal matrix with these numbers as the diagonal entries, in the given order: $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

215. (a) The characteristic polynomial of A is $\begin{vmatrix} 4-\lambda & 2 & 2 \\ -5 & -3-\lambda & -2 \\ 5 & 5 & 4-\lambda \end{vmatrix}$

$$= (4-\lambda)[(-3-\lambda)(4-\lambda) + 10] + 5[2(4-\lambda) - 10] + 5[-4 + 2(3+\lambda)]$$
$$= -\lambda^3 + 5\lambda^2 - 2\lambda - 8 = -(\lambda - 4)(\lambda + 1)(\lambda - 2).$$

(b) Since the 3×3 matrix A has three different eigenvalues, 4, -1, and 2, A is diagonalizable by Theorem 12.12.

(c) The desired matrix P is a matrix whose columns are eigenvectors corresponding to 4, -1 and 2, **in that order**. To find the eigenspace for $\lambda = 4$, we solve $(A - \lambda I)\mathsf{x} = \mathsf{0}$ with $\lambda = 4$. Gaussian elimination proceeds

$$\begin{bmatrix} 0 & 2 & 2 \\ -5 & -7 & -2 \\ 5 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

With $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have $x_3 = t$ free, $x_2 = -x_3 = -t$, and $x_1 = -x_2 = t$, so $\mathsf{x} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. To find the eigenspace for $\lambda = -1$, we solve $(A - \lambda I)\mathsf{x} = \mathsf{0}$ with $\lambda = -1$. Gaussian elimination proceeds $\begin{bmatrix} 5 & 2 & 2 \\ -5 & -2 & -2 \\ 5 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

With $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have $x_3 = t$ free, $x_2 = -x_3 = -t$, and $x_1 = -x_2 - x_3 = 0$, so $\mathsf{x} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. To find the eigenspace for $\lambda = 2$, we solve $(A - \lambda I)\mathsf{x} = \mathsf{0}$ with $\lambda = 2$. Gaussian elimination proceeds $\begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -2 \\ 5 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ -5 & -5 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

With $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have $x_2 = t$ free, $x_3 = 0$, and $x_1 = -x_2 - x_3 = -t$, so $\mathsf{x} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. We obtain $P = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

216. (a) The characteristic polynomial of A is $\begin{vmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)$, so A has two distinct eigenvalues and hence is diagonalizable. For $\lambda = 1$, we find that $\mathsf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector and, for $\lambda = 3$, $\mathsf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For $P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ we have $P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) The characteristic polynomial of A is

$$\begin{vmatrix} -1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 2 & 1 & -1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = (2-\lambda)(\lambda+1)^2.$$

There are two eigenvalues, $\lambda = -1$ and $\lambda = 2$. For $\lambda = 2$, the eigenspace is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = -1$, the eigenspace is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. There are just two linearly independent eigenvectors. The matrix is not diagonalizable.

(c) The characteristic polynomial of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$
$$= (1-\lambda)(\lambda^2 - 4\lambda + 3) = (1-\lambda)(\lambda-1)(\lambda-3),$$

so A has eigenvalues $\lambda = 1$ and $\lambda = 3$. The eigenvectors for $\lambda = 1$ are found by solving $(A - \lambda I)\mathsf{x} = 0$ with $\lambda = 1$. Gaussian elimination proceeds

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $x_3 = t$ is free, $x_2 = 0$, and $x_1 = -x_2 - x_3 = -t$. Eigenvectors are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. To find the eigenvectors for $\lambda = 3$, we solve $(A - \lambda I)\mathsf{x} = 0$ with $\lambda = 3$. This time, Gaussian elimination proceeds $\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_3 = t$ is free, $x_2 = 0$, and $x_1 = x_2 + x_3 = t$. Eigenvectors are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Since there are only two linearly independent eigenvectors, the matrix is not diagonalizable.

217. (a) The eigenvalues of A are 1 and 2 with corresponding eigenvectors, respectively, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Let $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Then $P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) Following the hint, we note that $D = D_1^2$ where $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. Now $P^{-1}AP = D_1^2$, so $A = PD_1^2P^{-1} = (PD_1P^{-1})(PD_1P^{-1}) = B^2$, with $B = PD_1P^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2\sqrt{2} & 2-2\sqrt{2} \\ -1+\sqrt{2} & 2-\sqrt{2} \end{bmatrix}$.

218. The characteristic polynomial of A is

$$\begin{vmatrix} 103-\lambda & -96 \\ -96 & 47-\lambda \end{vmatrix} = \lambda^2 - 150\lambda - 4375 = (\lambda - 175)(\lambda + 25).$$

So A has two eigenvalues, $\lambda = 175$ and $\lambda = -25$. The eigenspace for $\lambda = 175$ is the set of solutions to the homogeneous system $(A - \lambda I)\mathsf{x} = \mathsf{0}$ with $\lambda = 175$. Gaussian elimination proceeds

$$\begin{bmatrix} -72 & -96 \\ -96 & -128 \end{bmatrix} \to \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}.$$

With $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_2 = t$ is free and $x_1 = -\frac{4}{3}x_2 = -\frac{4}{3}t$, so the eigenspace consists of all vectors of the form $t\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$, that is, multiples of $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

The eigenspace for $\lambda = -25$ is the set of solutions to the homogeneous system $(A - \lambda I)\mathsf{x} = \mathsf{0}$ with $\lambda = -25$. Gaussian elimination proceeds

$$\begin{bmatrix} 128 & -96 \\ -96 & 72 \end{bmatrix} \to \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}.$$

With $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_2 = t$ is free and $x_1 = \frac{3}{4}x_2 = \frac{3}{4}t$, so the eigenspace consists of all vectors of the form $t\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$, that is, multiples of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The matrix $P = \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}$ has orthogonal columns and $P^{-1}AP = \begin{bmatrix} 175 & 0 \\ 0 & -25 \end{bmatrix}$.

219. The characteristic polynomial of A is $\begin{vmatrix} 3-\lambda & -5 \\ -5 & 3-\lambda \end{vmatrix} = 9 - 6\lambda + \lambda^2 - 25 = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$. The eigenvalues are $\lambda = -2$ and $\lambda = 8$. When $\lambda = -2$, $A - \lambda I = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$, the eigenspace is the set of vectors of the form $\begin{bmatrix} t \\ t \end{bmatrix}$, which is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. When $\lambda = 8$, $A - \lambda I = \begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix}$, the eigenspace is the set of vectors of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$, which is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are orthogonal. We can take $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, in which case $P^{-1}AP = D$ with $D = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}$.

220. First multiply (†) by A and use the fact that $A\mathsf{x}_1 = \lambda_1\mathsf{x}_1$, $A\mathsf{x}_2 = \lambda_2\mathsf{x}_2$, and so on. We obtain

$$c_1\lambda_1\mathsf{x}_1 + c_2\lambda_2\mathsf{x}_2 + \cdots + c_\ell\lambda_\ell\mathsf{x}_\ell = \mathsf{0}. \qquad (*)$$

Now we multiply (†) by λ_1 and obtain

$$c_1\lambda_1\mathsf{x}_1 + c_2\lambda_1\mathsf{x}_2 + \cdots + c_\ell\lambda_1\mathsf{x}_\ell = \mathsf{0}. \qquad (**)$$

The first terms of equations (*) and (**) are the same, so subtracting (**) from (*), we obtain

$$c_2(\lambda_2 - \lambda_1)\mathrm{x}_2 + c_3(\lambda_3 - \lambda_1)\mathrm{x}_3 + \cdots + c_\ell(\lambda_\ell - \lambda_1)\mathrm{x}_\ell = 0.$$

This is an equation just like (†) but with one less term. Also, each coefficient is nonzero since $c_i \neq 0$ and $\lambda_i \neq \lambda_1$ (the λs were all different). We have contradicted the fact that (†) was the shortest dependence relation with all coefficients nonzero and the result follows.

221. Similar matrices have the same characteristic polynomial, so the characteristic polynomial of A is the same as that of the given matrix. The given matrix is triangular; its characteristic polynomial is $(-1-\lambda)(1-\lambda)(-3-\lambda)^2$.

222. For some invertible matrix P, we have $P^{-1}AP = D$ where D is diagonal with diagonal entries the eigenvalues of A. Let D_1 be the diagonal matrix whose diagonal entries are the square roots of those of D. Thus $D_1^2 = D$. Let $B = PD_1P^{-1}$. Then $B^2 = (PD_1P^{-1})(PD_1P^{-1}) = PD_1P^{-1}PD_1P^{-1} = PD_1^2P^{-1} = PDP^{-1} = A$.