Assignment 5:

Problem 1: (Section 2.5 Exercise 1) Determine whether or not each of the following relations is a partial order and state whether or not each partial order is a total order.

(d.)$(N \times N, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$.

(e.) $(N \times N, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \geq d$.

Solution:

(d.) This is not a partial order because the relation is not antisymmetric; for example, $(1, 4) \preceq (1, 8)$ because $1 \leq 1$ and similarly, $(1, 8) \preceq (1, 4)$, but $(1, 4) \neq (1, 8)$.

(e.) This is a partial order.

Reflexive: For any $(a, b) \in N \times N$, $(a, b) \preceq (a, b)$ because $a \leq a$ and $b \geq b$.

Antisymmetric: If (a, b), $(c, d) \in N \times N$, $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$, then $a \leq c$, $b \geq d$, $c \leq a$ and $d \geq b$. So $a = c$, $b = d$ and hence, $(a, b) = (c, d)$.

Transitive: If $(a, b), (c, d), (e, f) \in N \times N$, $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$, then $a \leq c$, $b \geq d$, $c \leq e$ and $d \geq f$. So $a \leq e$ (because $a \leq c \leq e$) and $b \geq f$ (because $b \geq d \geq f$) and, therefore, $(a, b) \preceq (e, f)$.

This is not a total order; for example, $(1, 4)$ and $(2, 5)$ are incomparable.

Problem 2: (Section 2.5 Exercise 2) List the elements of the set

$$\{11, 1010, 100, 1, 101, 111, 110, 1001, 10, 1000\}$$

in lexicographic order, assuming $1 \preceq 0$.

Solution:

$$1, 11, 111, 110, 10, 101, 1010, 100, 1001, 1000$$

Problem 3: (Section 2.5 Exercise 5b) Draw the Hasse diagram for the following partial order.

(b.) $(\{\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, c\}, \{c, d\}, \}, \subseteq)$.

Solution:

The sets $\{a, b, c, d\}$, $\{a, b, c\}$, $\{a, b\}$, $\{a\}$ are all connected.

The sets $\{a, b, c, d\}$ and $\{c, d\}$ are connected.

The sets $\{a, b, c\}$, $\{a, c\}$ are connected and $\{a\}$, $\{a, c\}$ are connected.

Problem 4: (Section 2.5 Exercise 6) List all minimal, maximal, and maximum elements for each of the partial orders in Exercise 5.

Solution:

$\{a\}$ and $\{c, d\}$ are minimal; there is no minimum.

The set $\{a, b, c, d\}$ is maximal and maximum.

Problem 5: (Section 2.5 Exercise 12b) Prove that $A \vee B = A \cup B$.

Solution:

Assuming it exists, the least upper bound of A and B has two properties:

$$(1.) A \subseteq L, B \subseteq L;$$

$$(2.) if A \subseteq C \text{ and } B \subseteq C, \text{ then } L \subseteq C.$$

We must prove that $A \cup B$ has these properties. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $A \cup B$ satisfies (1.) Also, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, so $A \cup B$ satisfies (2) and $A \cup B = A \vee B$.

Problem 6: (Section 2.5 Exercise 16b) Give an example of a totally ordered set which has no maximum or minimum elements.

Solution:

$(Z, \leq)$ or $(R, \leq)$ are obvious examples.

Problem 7: (Section 3.1 Exercise 4) Give an example of a function $N \to N$ which is:

(b.) onto but not one-to-one;

(c.) neither one-to-one nor onto;

(d.) both one-to-one and onto.

Solution:

(b.) the function defined by $f(1) = 1$ and for $n > 1$, $f(n) = n - 1$, for example.

(c.) the constant function $f(n) = 107$ for all n, for example.

(d.) the identify function $F(n) = n$, for all n, for example.

Problem 8: (Section 3.1 Exercise 7) Let $S = \{1, 2, 3, 4\}$ and define $f : S \to Z$ by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \text{ is even} \\ 2x - 5 & \text{if } x \text{ is odd} \end{cases}$$

Express f as a subset of $S \times Z$. Is f one-to-one?

Solution:
We have $f(1) = 2(1) - 5 = -3$, $f(2) = 2^2 + 1 = 5$, $f(3) = 1$, $f(4) = 17$, $f(5) = 5$, and so, as a subset of $S \times Z$, $f = \{(1, -3), (2, 5), (3, 1), (4, 17), (5, 5)\}$.

No, f is not one-to-one because $f(2) = f(5)$ but $2 \neq 5$
(equivalently, $(2, 5)$ and $(5, 5)$ are both in f).

Problem 9: (Section 3.1 Exercise 13c) Define $f : A \to B$ by $f(x) = x^2 + 14x - 51$. Determine(with reasons) whether or not f is one-to-one and whether or not it is onto in each of the following cases.

(c.) $A = R$, $B = \{b \in R | b \geq -100\}$

Solution:
This function is not one-to-one; as in (b.), $f(0) = f(-14)$. But it is onto since for any $y \geq -100$, $x = \sqrt{100 + y} - 7$ is a solution to $y = f(x)$.

Problem 10: (Section 3.1 Exercise 18b) For each of the following, find the largest subset A of R such that the given formula for $f(x)$ defines a function f with domain A. Give the range of f in each case.

(b.) $f(x) = \frac{1}{\sqrt{1-x}}$

Solution:
We require $1 - x > 0$, so we take $A = \{x \in R | x < 1\}$.
Then the range is $f = \{y | y > 0\}$ because for any $y > 0$, $y = f(x)$ for $x = 1 - \frac{1}{y^2} \in A$.

Problem 11: (Section 3.1 Exercise 21) Let A be a set and let $f : A \to A$ be a function. For $x, y \in A$, define $x \sim y$ if $f(x) = f(y)$.

(a.) Prove that $\sim$ is an equivalence relation on A.

(b.) For $A = R$ and $f(x) = \lfloor x \rfloor$, find the equivalence classes of 0, $\frac{7}{5}$, and $\frac{-3}{4}$.

(c.) Suppose $A = \{1,2,3,4,5,6\}$ and $f = \{(1,2),(2,1),(3,1),(4,5),(5,6),(6,1)\}$. Find all equivalence classes.

Solution:
(a.) **Reflexive:** If $a \in A$, then $a \sim a$ because $f(a) = f(a)$.

Symmetric: If $a, b \in A$ and $a \sim b$, then $f(a) = f(b)$, so $f(b) = f(a)$; hence, $b \sim a$.

Transitive: If $a, b, c \in A$, $a \sim b$ and $b \sim c$, then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$, implying $a \sim c$.

(b.) $\overline{0} = \{x \in R | f(x) = f(0) = 0\} = [0,1)$; $\overline{\frac{7}{5}} = [1,2)$ since $\lfloor \frac{7}{5} \rfloor = 1$; $\overline{\frac{-3}{4}} = [-1,0)$ since $\lfloor \frac{-3}{4} \rfloor = -1$.

(c.) $\overline{1} = \{1\}$; $\overline{2} = \{2,3,6\}$; $\overline{4} = \{4\}$; $\overline{5} = \{5\}$.

Problem 12: (Section 3.2 Exercise 3) Show that each of the following functions $f : A \to R$ is one-to-one. Find the range of each function and a suitable inverse.

(b.) $A = \{x \in R | x \neq -1\}$, $f(x) = 5 - \frac{1}{1+x}$.

(d.) $A = \{x \in R | x \neq -3\}$, $f(x) = \frac{x-3}{x+3}$.

Solution:
(b.) Suppose $f(x_1) = f(x_2)$. Then

$$5 - \frac{1}{1+x_1} = 5 - \frac{1}{1+x_2},$$

so

$$\frac{1}{1+x_1} = \frac{1}{1+x_2},$$
$$1 + x_1 = 1 + x_2$$

and

$$x_1 = x_2.$$

Thus f is one-to-one. Next we find inverse. Start with

$$y = 5 - \frac{1}{1+x}$$

and solve for x

$$\frac{1}{1+x} = 5 - y$$

$$1 + x = \frac{1}{5-y}$$

$$x = \frac{1}{5-y} - 1.$$

Thus the inverse function is

$$f^{-1}(y) = \frac{1}{5-y} - 1.$$

Its domain, which is the same as the range of f is $y \neq 5$.

(d.)Suppose $f(x_1) = f(x_2)$. Then

$$\frac{x_1 - 3}{x_1 + 3} = \frac{x_2 - 3}{x_2 + 3},$$

so

$$x_1 x_2 + 3x_1 - 3x_2 - 9 = x_1 x_2 - 3x_1 + 3x_2 - 9$$

$6x_1 = 6x_2$ and $x_1 = x_2$. Thus f is one-to-one.

Next we find the inverse. Set

$$y = \frac{x-3}{x+3}$$

And solve for x to get after algebraic manipulations

$$x = f^{-1}(y) = \frac{3(1+y)}{1-y}.$$

Its domain, which is the same as the range of f is $y \neq 1$.

Problem 13: (Section 3.2 Exercise 5) Suppose A is the set of all married people, mother:$A \rightarrow A$ is the function which assigns to each married person his/her mother, and father and spouse have similar meanings. Give sensible intrepretations of each of the following:

(e.) spouse $\circ$ mother

(f.) father $\circ$ spouse

(h.) (spouse $\circ$ father) $\circ$ mother

(i.) spouse $\circ$ (father $\circ$ mother)

Solution:
(e.) father

(f.) father-in-law

(h.) maternal grandmother

(i.) maternal grandmother

Problem 14: (Section 3.2 Exercise 15) Let $S = \{1, 2, 3, 4, 5\}$ and let f, g, $h : S \to S$ be the functions defined by

$$f = \{(1, 2), (2, 1), (3, 4), (4, 5), (5, 3)\}$$

$$g = \{(1, 3), (2, 5), (3, 1), (4, 2), (5, 4)\}$$

$$h = \{(1, 2), (2, 2), (3, 4), (4, 3), (5, 1)\}$$

(b.) Explain why f and g have inverses but h does not. Find f^{-1} and g^{-1}.

(c.) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$.

Solution:
(b.) $f^{-1} = \{(1, 2), (2, 1), (3, 5), (4, 3), (5, 4)\}$; $g^{-1} = \{(1, 3), (2, 4), (3, 1), (4, 5), (5, 2)\}$

Functions f and g have inverses because they are one-to-one and onto while h does not have an inverse because it is not one-to-one(equally because it is not onto).

(c.)
$(f \circ g)^{-1} = \{(1, 4), (2, 3), (3, 2), (4, 1), (5, 5)\}$

$g^{-1} \circ f^{-1} = \{(1, 4), (2, 3), (3, 2), (4, 1), (5, 5)\} = (f \circ g)^{-1}$

$f^{-1} \circ g^{-1} = \{(1, 5), (2, 3), (3, 2), (4, 4), (5, 1)\} \neq (f \circ g)^{-1}$

Problem 15: (Section 3.2 Exercise 18) Suppose $f : A \to B$ and $g : B \to C$ are functions.

(b.) If $g \circ f$ is onto and g is one-to-one, show that f is onto.

Solution:
Given $b \in B$, we must find $a \in A$ such that $f(a) = b$. Consider $g(b) \in C$. Since $g \circ f : A \to C$ is onto, there is some $a \in A$ with $g \circ f(a) = g(b)$; that is, $g(f(a)) = g(b)$. But

g one-to-one implies $f(a) = b$, so we have the desired element a.

Problem 16: (Section 3.3 Exercise 7) Suppose S is a set and for A, $B \in \mathcal{P}(\mathcal{S})$, we define $A \preceq B$ to mean $|A| \leq |B|$. Is this relation a partial order on $\mathcal{P}(\mathcal{S})$? Explain.

Solution:

Case 1: $S = \emptyset$ and $\mathcal{P}(\mathcal{S})$ contains a single element, $\emptyset$. In this case $\preceq$ defines a partial order.

Reflexive: Certainly $A \preceq A$ for all $A \in \mathcal{P}(\mathcal{S})$ since $0 = |\emptyset| \leq |\emptyset|$.

Antisymmetric: If $A \preceq B$ and $B \preceq A$, then $A = B$ since there is only one set in $\mathcal{P}(\mathcal{S})$.

Transitive: If $A \preceq B$ and $B \preceq C$, then $A \preceq C$ since necessarily $A = B = C$ and for the single set A in $\mathcal{P}(\mathcal{S})$, $A \preceq A$.

Case 2: S contains just one element, so $\mathcal{P}(\mathcal{S}) = \{\emptyset, \mathcal{S}\}$ contains two elements. Again, $\preceq$ defines a partial order.

Reflexive: $A \preceq A$ for each $A \in \mathcal{P}(\mathcal{S})$ because $|A| \leq |A|$.

Antisymmetric: If A, $B \in \mathcal{P}(\mathcal{S})$, $A \preceq B$ and $B \preceq A$, then we have $|A| \leq |B|$ and $|B| \leq |A|$, so $|A| = |B|$. Since $\mathcal{P}(\mathcal{S})$ does not contain different sets of the same cardinality, it follows that $A = B$.

Transitive: Suppose $A \preceq B$ and $B \preceq C$. If $A = \emptyset$, then $|A| = 0 \leq |C|$ no matter what C is, so we'd have $A \preceq C$. If $A = S$, then $A \preceq B$ means $B = S$ and $B \preceq C$ means $C = S$, so $A = B = C = S$ and $A \preceq C$.

Case 3: S has more than 1 element. In this case $\preceq$ is not a partial order because it is not antisymmetric. For if $a, b \in S$ and $a \neq b$, then $\{a\} \preceq \{b\}$ because $|\{a\}| \leq |\{b\}|$ and for the same reason, $\{b\} \preceq \{a\}$; however $\{a\} \neq \{b\}$.

Problem 17: (Section 3.3 Exercise 8) Show that for any sets A and B, $|A \times B| = |B \times A|$.

Solution:

$f : A \times B \to B \times A$ defined by $f(a, b) = (b, a)$ is a one-to-one onto function.

Problem 18: (Section 3.3 Exercise 18) Determine, with justification, whether each of the following sets is finite, countably infinite, or uncountable.

(c.) $\{\frac{m}{n} | m, n \in N, m < 100, 5 < n < 105\}$

(d.) $\{\frac{m}{n} | m, n \in Z, m < 100, 5 < n < 105\}$

Solution:

(c.) This set is **finite**. In fact, it contains at most 99^2 elements since there are 99 possible numerators and, for each numerator, 99 possible denominators.

(d.) This set is **countably infinite**. List the elements as follows(deleting any repetitions such as $\frac{5}{100} = \frac{1}{20}$):

$$\frac{99}{6}, \frac{99}{7}, \cdots, \frac{99}{104}, \frac{98}{6}, \frac{98}{7}, \cdots, \frac{98}{104}, \cdots$$

Problem 19: (section 3.3 Exercise 23) Prove that the points of a plane and the points of a sphere are stes of the same cardinality.

Solution:

We employ a concept known as **stereographic projection**. Imagine the sphere sitting on the Cartesian plane with south pole at the origin. Any line from the north pole to the plane punctures the sphere at a unique point and the collection of such lines establishes a one-to-one correspondence between the points of the plane and the sphere except for the north pole. A small modification of this correspondence finishes the job. Suppose $p_0, p_1, p_2, \cdots$ are the points of the sphere which correspond to the points $(0,0), (1,0), (2,0), \cdots$ in the plane; thus, the line from the north pole to $(n, 0)$ punctures the sphere at p_n (in particular, $p_0 = (0,0)$), the origin to $(1,0)$, p_1 to $(2,0)$, and so forth and let all other points of the sphere go to the same points as before.