Assignment 3 Solutions

Problem1: (Section 1.2 Exercise 8) Consider the following assertions.

A:"There exists a real number y such that $y > x$ for every real number x"

B:"For every real number x, there exists a real number y such that $y > x$"

Solution:

A: **False.** Since there is no such a number y which is larger than all of real numbers.

B: **True.** Since for each individual number x we can always find $y > x$.

Problem 2: (Section 1.2 Exercise 10) Answer true or false and supply a direct proof or a counterexample to each of the following assertions.

(a.)There exists a positive integer n such that nq is an integer for every rational number q.

(b.) For every rational number q, there exists an integer n such that nq is an integer.

Solution:

(a.) **False.** The original statement $P : \exists n, \forall q\ nq$ is an integer. Then its negation is: $\neg P : \forall n\, \exists q$ such that nq is not an integer.

The negation $\neg P$ is true. Indeed, for any integer n take $q = \frac{a}{b}$ such that n is not an integer multiple of b. For example, take $q = \frac{1}{n+1}$. Then nq in not an integer.

Since $\neg P$ can be proved to be **True**, that means P is **False**.

(b.) **True.** q is a rational number. Let $q = \frac{a}{b}$, where a, b are integers and $b \neq 0$. So $nq = n\frac{a}{b}$

Choose $n = bk$ where k is some integer, so n is an intger. Then $nq = n\frac{a}{b} = bk\frac{a}{b} = (b\frac{a}{b})k = ak$. The product of two integers a, k is also an integer.

Therefore, for every rational number q, there exists an integer n such that nq is an integer.

Problem 3: (Section 1.2 Exercise 12) Provide a direct proof that $n^2 - n + 5$ is odd, for all integers n.

Solution:

Proof: $n^2 - n + 5 = n(n-1) + 5$ Since $(n-1)$ and n are two consecutive integers, therefore, one of them must be even, and the other must be odd. So the product $n(n-1)$ must contain a factor 2.

Let $n(n-1) = 2k$, where k is an integer. Then

$$n^2 - n + 5 = n(n-1) + 5 = 2k + 5 = 2k + 4 + 1 = 2(k+2) + 1 = 2m + 1,$$

where $m = k + 2$. m is an integer. Therefore, $n^2 - n + 5 = 2m + 1$ is odd.

Problem 4: (Section 1.2 Exercise 14) Let a and b be integers. By examining the four cases:
(i.) a, b both even
(ii.) a, b both odd
(iii.) a even, b odd
(iv.) a odd, b even

Find a necessary and sufficient condition for $a^2 - b^2$ to be odd.

Solution:
The necessary and sufficient condition for $a^2 - b^2$ to be odd is: one of a or b is odd and another is even. This conclution follows from consideration of cases:
(i.) Let $a = 2k$, $b = 2m$, where k and m are integers.

$$a^2 - b^2 = (2k)^2 - (2m)^2 = 4k^2 - 4m^2 = 4(k^2 - m^2).$$

Let $k^2 - m^2 = n$, so n is an integer. Then

$$a^2 - b^2 = 4n = 2(2n)$$

So $a^2 - b^2$ is even. Therefore, case(i.) is not what we need.

(ii.)Let $a = 2k + 1$, $b = 2m + 1$, where k and m are integers.

$$a^2-b^2 = (2k+1)^2-(2m+1)^2 = 4k^2+4k+1-(4m^2+4m+1) = 4k^2+4k-4m^2-4m = 2(2k^2+2k-2m^2-2m)$$

Since k and m are integers, k^2, m^2 are also integers. Let $2k^2 + 2k - 2m^2 - 2m = P$, then P is an integer and $a^2 - b^2 = 2P$ is even. Therefore, case(ii.) is not what we need, either.

(iii.)Let $a = 2k$, $b = 2m + 1$, where k and m are integers.

$$a^2-b^2 = (2k)^2-(2m+1)^2 = 4k^2-4m^2-4m-1 = 4k^2-4m^2-4m-2+1 = 2(2k^2-2m^2-2m-1)+1.$$

Let $P = 2k^2 - 2m^2 - 2m - 1$, so P is an integer, then $a^2 - b^2 = 2P + 1$ is odd. Therefore, in case(iii.) $a^2 - b^2$ is odd.

(iv.) Let $a = 2k + 1$, $b = 2m$, where k, m are integers.

$$a^2-b^2 = (2k+1)^2-(2m)^2 = 4k^2+4k+1-4m^2 = 4k^2+4k-4m^2+1 = 2(2k^2+2k-2m^2)+1.$$

Let $2k^2 + 2k - 2m^2 = P$, then $a^2 - b^2 = 2P + 1$, which is odd. Therefore, in case (iv.) $a^2 - b^2$

is odd.

Problem 5: (Section 1.2 Exercise 16) Let x be a real number. Find a necessary and sufficient condition for $x + \frac{1}{x} \geq 2$. Prove your answer.

Solution:

First, we notice that $x \neq 0$ otherwise the function $\frac{1}{x}$ is undefined. Trying several values of x we can make a conjecture that the condition is $x > 0$. To prove the statement : $x > 0$ *is necesary and sufficient condition for* $x + \frac{1}{x} \geq 2$ we need two parts.

Part i: ($x > 0$ is sufficient.)

Assume that $x > 0$ and show that then $x + \frac{1}{x} \geq 2$.

proof: since $x > 0$, multiply both sides of the inequality we wish to prove by x and simplify. We get

$$x + \frac{1}{x} \geq 2$$

$$x(x + \frac{1}{x}) \geq 2x$$

$$x^2 + 1 \geq 2x$$

$$x^2 - 2x + 1 \geq 0$$

$$(x - 1)^2 \geq 0$$

The last inequality is true for any x, and since for $x > 0$ the last one is equivalent to the first one $x + \frac{1}{x} \geq 2$ then the first one is also true for $x > 0$.

Part ii. ($x > 0$ is necessary.)

Assuume that $x + \frac{1}{x} \geq 2$ and show that $x > 0$. We would rather proof the contrapositive: $x < 0$ implies $x + \frac{1}{x} < 2$. But this one it true because for $x < 0$ $x + \frac{1}{x} < 0$ and $0 < 2$.

Problem 6: (Section 1.2 Exercise 21) Let $n = ab$ be the product of positive integers a and b. Prove that either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution:

Proof: Suppose $a \leq b$, a and b are positive integers. Then

$$aa \leq ab = n$$

$$a^2 \leq n$$

$$a \leq \sqrt{n}$$

Note that since $a > 0$ then $n > 0$. Since we arbitrarily assigned $a \leq b$, $b \leq a$ is also possible. If $b \leq a$, the proof is exactly that same as the aboveexcept that we need to switch the notation a and b. The conclusion will become $b \leq \sqrt{n}$. Therefore, either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Problem 7: (Section 1.2 Exercise 25) Find a proof or exhibit a counterexample to each of the following statements.

(b.) a an even integer $\rightarrow \frac{1}{2}a$ an even integer.
(d.) If a and b are real numbers with $a + b$ rational, then a and b are rational.

Solution:

(b.) **Counterexample:** Let $a = 6$, then $\frac{1}{2}(a) = \frac{1}{2}(6) = 3$, which is an integer but it is odd.

(d.) **Counterexample:** Let $a = \sqrt{2}+1$, $b = -\sqrt{2}+1$, then $a+b = (\sqrt{2}+1)+(-\sqrt{2}+1) = 2$, which is rational. But obviously neither a nor b is rational.

Problem 8: (Section 1.2 Exercise 26) Suppose ABC and $A'B'C'$ are triangles with pairwise equal angles; that is $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$. Then it is a well-known result in Euclidean geometry that the triangles have pairwise proportional sides (the triangles are similar). Does the same property hold for polygons with more than three sides? Give a proof or provide a counterexample.

Solution:

Counterexample: Square and rectangle have same angles all equal to $\pi/2$ but sides are not proportional.

Problem 9: (Section 5.1 Exercise 3) Prove that it is possible to fill an order for $n \geq 32$ pounds of fish given bottomless wheelbarrows full of 5-pound and 9-pound fish.

Solution:

Proof: P(n):"$n = 5m + 9l$, m and l are some non-negative integers, $\forall n \geq 32$"

$$P(32) : 32 = 5(1) + 9(3) \quad m = 1, l = 3$$

$$P(33) : 33 = 5(3) + 9(2) \quad m = 3, l = 2$$
$$P(34) : 34 = 5(5) + 9(1) \quad m = 5, l = 1$$
$$P(35) : 35 = 5(7) + 9(0) \quad m = 7, l = 0$$
$$P(36) : 36 = 5(0) + 9(4) \quad m = 0, l = 4$$

We've shown that $P(32) \wedge P(33) \wedge P(34) \wedge P(35) \wedge P(36)$ is **True.**

We assume that $P(k) \wedge P(k+1) \wedge P(k+2) \wedge P(k+3) \wedge P(k+4)$ is **True**, which means:

$$P(k) : k = 5m + 9l,$$
$$P(k+1) : k+1 = 5m + 9l,$$
$$P(k+2) : k+2 = 5m + 9l,$$
$$P(k+3) : k+3 = 5m + 9l,$$
$$P(k+4) : k+4 = 5m + 9l,$$

where m and l are some non-negative integers. Then

$$P(k+5) : k+5 = 5m + 9l + 5 = 5(m+1) + 9l = 5m' + 9l',$$

where $m' = m+1$, $l' = l$

$$P(k+6) : k+6 = (k+1) + 5 = 5m + 9l + 5 = 5m' + 9l'$$

where $m' = m+1$, $l' = l$

$$P(k+7) : k+7 = (k+2) + 5 = 5m + 9l + 5 = 5m' + 9l'$$

where $m' = m+1$, $l' = l$

$$P(k+8) : k+8 = (k+3) + 5 = 5m + 9l + 5 = 5m' + 9l'$$

where $m' = m+1$, $l' = l$

$$P(k+9) : k+9 = (k+4) + 5 = 5m + 9l + 5 = 5m' + 9l'$$

where $m' = m+1$, $l' = l$

That is, $P(k+5) \wedge P(k+6) \wedge P(k+7) \wedge P(k+8) \wedge P(k+9)$ is also **True.** Therefore, $P(n)$ is **True** for $\forall n \geq 32$

Problem 10: (Section 5.1 Exercise 4) Use mathematical induction to prove the truth of each of the following assertions for all $n \geq 1$.

(b.) $n^3 + 2n$ is divisible by 3.
(d.) $5^n - 1$ is divisible by 4.
(e.) $8^n - 3^n$ is divisible by 5.

Solution:

(b.)
Proof:

Step 1:
$P(n): n^3 + 2n$ is divisible by 3.

$$1^3 + 2(1) = 3$$
$$\frac{3}{3} = 1$$

So $P(1)$ is **True**
Step 2:
Assume $P(k)$ is True. That is $k^3 + 2k = 3q, q$ is an integer. Then

$$(k+1)^3 + 2(k+1)$$
$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= (k^3 + 2k) + (3k^2 + 3k + 3)$$
$$= 3q + 3(k^2 + k + 1)$$
$$= 3(q + k^2 + k + 1)$$

$q + k^2 + k + 1$ is also an integer, which shows that $(k+1)^3 + 2(k+1)$ is also divisible by 3.
Step3:

Since $P(1)$ is True and all implications $P(k) \to P(k+1)$ are True, then $P(n)$ is True for all $n \geq 1$.
(d.)

Proof:

Step 1:
$P(n): 5^n - 1$ is divisible by 4.

$$5^1 - 1 = 4$$
$$\frac{4}{4} = 1$$

So $P(1)$ is **True**
Step 2:
Assume $P(k)$ is True. That is $5^k - 1 = 4P$, P is an integer. Then

$$5^{k+1} - 1 = (5)5^k - 1$$
$$= 5(4P + 1) - 1$$

$$= 20P + 4$$
$$= 4(5P + 1)$$

Obviously, $5P + 1$ is an integer. Therefore, $5^{k+1} - 1$ is also divisible by 4.

Step 3:

Since $P(1)$ is True, and all implications $P(k) \to P(k+1)$ are True, then $P(n)$ is generally True for all $n \geq 1$.

(e.)

Proof:

Step 1:

$P(n) : 8^n - 3^n$ is divisible by 5.

$$8^1 - 3^1 = 8 - 3$$
$$= 5$$
$$\frac{5}{5} = 1$$

So $P(1)$ is True.

Step 2:

Assume $P(k)$ is true. Therefore, $8^k - 3^k = 5m$, m is an integer. Then

$$8^{k+1} - 3^{k+1} = (8)8^k - (3)3^k$$
$$= (3 + 5)8^k - (3)3^k$$
$$= (3)8^k - (3)3^k + (5)8^k$$
$$= 3(8^k - 3^k) + (5)8^k$$
$$= 3(5m) + (5)8^k$$
$$= 5(3m + 8^k)$$

$3m + 8^k$ is an integer, provided that m and k are integers. Thus, $8^{k+1} - 3^{k+1}$ is divisible by 5.

Step 3:

Since $P(1)$ is true and all of the implications $P(k) \to P(k + 1)$ are true, thus $P(n)$ is True for all $n \geq 1$.

Problem 11: (Section 5.1 Exercise 5)

(b.) Prove by mathematical induction that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}$$

for any natural number n.

(c.) Use the results of (a.) and (b.) to establish that

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+\cdots+n^3$$

for all $n \geq 1$.

Solution:

(b.)

Proof:

Step 1:
$P(n): 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

$$\frac{1^2(1+1)^2}{4} = \frac{1^2 2^2}{4}$$

$$= 1$$

$$= 1^3$$

So $P(1)$ is True.

Step 2:
Assume that $P(k)$ is True, that is

$$1^3+2^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4}.$$

Then

$$1^3+2^3+\cdots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$= (k+1)^2(\frac{k^2}{4}+k+1)$$

$$(k+1)^2(\frac{1}{4}(k^2+4k+4)$$

$$\frac{(k+1)^2(k+2)^2}{4}$$

$$\frac{(k+1)^2[(k+1)+1]^2}{4}$$

We've shown if $P(k)$ holds then $P(k+1)$ holds.

Step 3:
Because $P(1)$ is True and all the implications $P(k) \to P(k+1)$ are True. $P(n)$ is True for all natural number n.

(c.)

From (a.) we know that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Therefore,

$$(1 + 2 + 3 + \cdots + n)^2 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$$

for $n \geq 1$

From (b.), we know that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for $n \geq 1$

Thus,

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for $n \geq 1$.