Undergraduate Mathematics Competition, Winter 2004
Solutions by Dr. Sergey Sadov

Problem 1
Two evenly matched teams are engaged in a best four-of-seven series of games with each other. Is it more likely for the series to end in six games than in seven games?

Solution 1. If the series isn't decided in 5 games, then the score before Game 6 is $3:2$. There are two possibilities:
1) The leading team wins, the series ends in 6 games.
2) The leading team loses, the series to be decided in Game 7.
The probability of each of the two events is $1/2$, so it is equally likely for the series to end in six and in seven games.

This proof employs conditional probabilities: we assume that something had happened, in this case — that 5 games didn't determine the winner — and find probabilities of subordinate events. A direct comparison of the probabilities to end the series in a certain number of games is more laborious (see below).

Solution 2. A complete series can be coded by a sequence of 0 and 1 of length 4,5,6, or 7, where zeros (ones) denote the games lost (won) by the ultimate winner. The last symbol in every admissible sequence is 1 and the total number of ones is exactly 4.

Therefore, the number of admissible sequences of length $n = 4 \ldots 7$ is $\binom{n-1}{3}$ (but not $\binom{n}{4}$; the last symbol is fixed!).

Thus, there exist $\binom{5}{3} = 10$ sequences of length 6 and $\binom{6}{3} = 20$ sequences of length 7.

It doesn't mean however that the length 7 is more probable than the length 6. One can think of a series of 6 games as of a series of 7 games, in which the result of Game 7 has no effect. Therefore, the probability of any admissible sequence of length 6 is two times the probability of any admissible sequence of length 7. (Formally: there are two ways to extend a sequences of length 6 by one element.)

The total number of such extensions of sequences of length 6 is $2 \times 10 = 20$, i.e. the same as the number of sequences of length 7.

Therefore, series of lengths 6 and 7 can occur with equal probability.

A generalization of this argument to a "n of $2n-1$" series amounts to the combinatorial identity

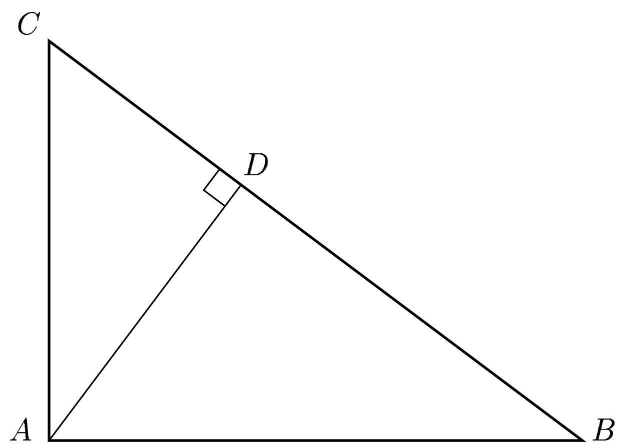
$$\binom{2n-2}{n-1} = 2\binom{2n-3}{n-1}.$$

Problem 2
Let ABC be a right triangle with right angle A. Let D be the foot of the perpendicular from A to the hypotenuse BC. Denote the inradius of ABC by r, inradius of ABD by r_B and inradius of ADC by r_C. Prove that $r^2 = r_B^2 + r_C^2$.

Solution. The triangles ABC, DBA, and DAC are similar (by two angles). Their linear dimensions are proportional in the ratios $BC : AB : AC$. In particular,

$$\left(\frac{r_B}{r}\right)^2+\left(\frac{r_C}{r}\right)^2=\left(\frac{AB}{BC}\right)^2+\left(\frac{AC}{BC}\right)^2=1,$$

the last "=" being Pithagorean Theorem.



Problem 3

Let k, l, m, n be positive integers such that $k+l+m \geq n$. Prove the following relation for binomial coefficients

$$\sum_{p+q+r=n}\binom{k}{p}\binom{l}{q}\binom{m}{r}=\binom{k+l+m}{n}$$

The summation in the left-hand side runs over all partitions of n into three non-negative integers p, q, r, such that $p \leq k,\ q \leq l,\ r \leq m$.

Solution. Consider a set of $h = k+l+m$ elements (call them, say, holes). Let there be k red holes, l white holes, and m black holes. We'll prove the formula by counting in two ways the number $N(n)$ of subsets that contain exactly n holes. The first, straightforward answer is $\binom{h}{n}$, which is the RHS of the stated formula.

On the other hand, let $N(p,q,r)$ be the number of subsets that contain exactly p red holes, exaclty q white holes, and exaclty r black holes. An obviuos necessary condition for $N(p,q,r)>0$ is

$$\{p \leq k,\ q \leq l,\ r \leq m\}. \qquad (*)$$

If the inequalities $(*)$ hold, then holes of each color can be chosen independently and we have

$$N(p,q,r)=\binom{k}{p}\binom{l}{q}\binom{m}{r}.$$

It is clear that

$$N(n)=\sum N(p,q,r)$$

where the summation runs over non-negative integers p, q, r satisfying $(*)$ and such that $p+q+r=n$. So $N(n)$ is equal the LHS of the stated formula.

The same counting idea can be expressed as an equation for generating functions

$$(1+x)^{k+l+m}=(1+x)^k\,(1+x)^l\,(1+x)^m.$$

It is a good exercise to work out details, if you don't have an experience with generating functions.

Problem 4
Fibbonacci numbers are defined by the recurrence $F_{n+1} = F_n + F_{n-1}$ with initial terms $F_1 = F_2 = 1$.

a) Show that every third Fibbonacci number is even;

b) Show that every fifth Fibbonacci number is divisible by 5;

Solution.

a) We have

$F_1 = 1$ (odd), $F_2 = 1$ (odd), $F_3 = 2$ (even), $F_4 = 3$ (odd), $F_5 = 5$ (odd), $F_6 = 8$ (even).

Let's prove by induction that the pattern "odd, odd, even" repeats periodically. It is true for the beginnig of the Fibonacci sequence. Now suppose that F_{3n-2} and F_{3n-1} are odd and F_{3n} even for some $n \geq 1$. Then

$$F_{3n+1} = F_{3n-1} + F_{3n} = \text{odd} + \text{even} = \text{odd},$$

$$F_{3n+2} = F_{3n} + F_{3n+1} = \text{even} + \text{odd} = \text{odd},$$

$$F_{3(n+1)} = F_{3n+1} + F_{3n+2} = \text{odd} + \text{odd} = \text{even}.$$

The induction step is thus proven.

b) The proof is concise if formulated in terms of arithmetic of conguences modulo 5. [1]

Consider remainders of Fibonacci numbers modulo 5. For the first ten members (which we divide in two groups of five, for convenience) they are

$$\{1, 1, 2, 3, 0\}, \{3, 3, 1, 4, 0\}$$

The two groups are different, so we can't repeat the argument of part (a) literally. However, we'll prove by induction that $F_{5n} \equiv 0 \pmod 5$ for $n \geq 1$. Suppose it is so for the given n. Denote $F_{5n-1} = a$. Step of induction:

$$F_{5n+1} = a + F_{5n} \equiv a + 0 \equiv a \pmod 5,$$

$$F_{5n+2} = F_{5n} + F_{5n+1} \equiv 0 + a \equiv a \pmod 5,$$

$$F_{5n+3} = F_{5n+1} + F_{5n+2} \equiv a + a \equiv 2a \pmod 5,$$

$$F_{5n+4} = F_{5n+2} + F_{5n+3} \equiv a + 2a \equiv 3a \pmod 5,$$

$$F_{5(n+1)} = F_{5n+3} + F_{5n+4} \equiv 2a + 3a \equiv 5a \equiv 0 \pmod 5,$$

Q.E.D.

Generalization. *F_{kn} is divisible by F_k for any integers $n, k \geq 0$.* ($k = 2$ in part (a), and $k = 5$ in part (b).)

[1] see e.g. PM 3370 Course Notes, 2002, by Don Rideout

Define $F_0 = 0$ and $F_{-1} = 1$ for convenience. The Fibonacci recurrence isn't damaged by this extension.

Lemma. If $F_p \equiv 0 \,(\mathrm{mod}\, F_k)$ for some p, then $F_{p+l} \equiv F_{p-1}F_l \,(\mathrm{mod}\, F_k)$ for all $l \geq 0$.

Proof: by induction. Case $l = 0$ is trivial. For $l = 1$ we have

$$F_{p+1} = F_p + F_{p-1} \equiv 0 + F_{p-1} \equiv F_{p-1}F_1 \,(\mathrm{mod}\, F_k).$$

If Lemma is true for $l - 1$ and l, then

$$F_{p+(l+1)} = F_{p+(l-1)} + F_{p+l} \equiv F_{p-1}F_{l-1} + F_{p-1}F_l \equiv F_{p-1}(F_{l-1} + F_l) \equiv F_{p-1}F_{l+1} \,(\mathrm{mod}\, F_k).$$

(We have used the distributive law for congruences. Recall that congruences modulo F_k form a commutative ring.) The step of induction is done, hence the Lemma holds.

Proof of Generalization. Let k be fixed. Prove Generalization by induction in n. Case $n = 0$ is trivial. Assuming Generalization is true for n, take $p = kn$ in Lemma and obtain

$$F_{k(n+1)} = F_{kn+k} \equiv F_{kn-1}F_k \equiv 0 \,(\mathrm{mod}\, F_k).$$

Therefore, $F_{k(n+1)}$ is divisible by F_k and the induction works.

Remarks. 1. The same general statement with the same proof is true for any sequence C_n satisfying a second-order recurrence $C_{n+1} = aC_{n-1} + bC_n$ with constant integer coefficients and initial values $C_0 = 0$, $C_1 = 1$.

2. The proof becomes perhaps more illuminating if, in addition to the language of congruences, one uses the language of linear algebra. The Fibonacci recurrence can be stated as the vector equation

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}.$$

Denote the 2×2 matrix in this equation Φ. Then

$$\begin{pmatrix} F_{k-1} \\ F_k \end{pmatrix} = \Phi^k \begin{pmatrix} F_{-1} \\ F_0 \end{pmatrix}.$$

Denote $\Phi^k = M$. Since $F_{-1} = 1$ and $F_0 = 0$, we obtain, comparing the second components of the vectors in both sides: $F_k = M_{12}$. Thus

$$M_{12} \equiv 0 \,(\mathrm{mod}\, F_k),$$

that is *matrix M is upper-triangular modulo F_k*. Upper-triangular matrices with elements from any given ring form a *semigroup*, that is the product of two upper-triangular matrices is also an upper-triangular matrix. The ring of coefficients in our case is the ring of congruences mod F_k. So for any integer $n \geq 1$

$$\Phi^{nk} = M^n \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mathrm{mod}\, F_k,$$

where $*$ denote some (undetermined) congruences mod F_k. Finally,

$$\begin{pmatrix} F_{nk-1} \\ F_{nk} \end{pmatrix} = \Phi^{nk} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} * \\ 0 \end{pmatrix} \mod F_k.$$

so $F_{nk} \equiv 0 \pmod{F_k}$.

3. The closed form of the Fibonacci numbers given by the Binet-Cauchy formula

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}}, \qquad \lambda_1 = \frac{1+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}, \text{ [2]}$$

can also be used to prove the general statement. We need to prove that F_k divides F_{nk}. Write

$$\frac{F_{kn}}{F_k} = \frac{\lambda_1^{kn} - \lambda_2^{kn}}{\lambda_1^k - \lambda_2^k} = \lambda_1^{k(n-1)} + \lambda_1^{k(n-2)}\lambda_2^k + \ldots + \lambda_2^{k(n-1)}.$$

The expression in the right side is a *symmetric function* of λ_1 and λ_2, who, in their turn, are the roots of a monic quadratic equation with integer coefficients (explicitly: $\lambda^2 - \lambda - 1 = 0$). It follows from the Main Theorem for symmetric functions [3] that any symmetric function of roots of such an equation is an integer.

Problem 5

Find

$$L = \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{dx}{1+x^{2n}}.$$

Solution.

Denote $f_n(x) = (1+x^{2n})^{-1}$. The limit L exists, because the sequences $\{\int_{|x|<1} f_n\}$ and $\{\int_{|x|>1} f_n\}$ are monotone and bounded. As $n \to \infty$,

$$f_n(x) \to g(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 1/2, & \text{if } |x| = 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Interchanging formally the integration and passing to the limit, we have

$$L \overset{\#}{=} \int_{-\infty}^{\infty} \left(\lim_{n\to\infty} \frac{1}{1+x^{2n}} \right) dx = \int_{-1}^{1} 1\, dx = 2. \qquad (*)$$

Step (#) requires justification. There are several ways, of which we present three.

(A) A common technique. Let $0 < \varepsilon < 1$ and $0 < \delta < 1$ be two arbitrary parameters. Later we'll let $\varepsilon \to 0$, $\delta \to 0$. Divide the given integral into three parts

$$I_n = \int_{-1+\delta}^{1-\delta} f(x)\, dx, \qquad J_n = \int_{-1-\delta}^{-1+\delta} + \int_{1-\delta}^{1+\delta}, \qquad K_n = \int_{-\infty}^{-1-\delta} + \int_{1+\delta}^{\infty}.$$

[2] $\lambda_{1,2}$ are eigenvalues of the matrix Φ.

[3] See e.g. J.W. Archbold, Algebra, § 11.2

Find n_0 such that

$$(1-\delta)^{2n_0} < \varepsilon \qquad \text{and} \qquad (1+\delta)^{2n_0-2} > \varepsilon^{-1}. \tag{A.1}$$

It is enough to take $n_0 = 1 + \lceil \frac{1}{2} |\log \varepsilon| / |\log(1-\delta)| \rceil$. [4] Explicit expression for n_0 is not too important, we only need the existence of such n_0.

Let $n \geq n_0$. Then

1) for any $x \in [-1+\delta,\ 1-\delta]$ we have $0 < x^{2n} \leq \varepsilon$, so $(1+\varepsilon)^{-1} \leq f_n(x) < 1$ and

$$(2-2\delta)(1+\varepsilon)^{-1} \leq I_n < 2-2\delta; \tag{A.2}$$

2) for any x s.t. $|x| > 1+\delta$, we have $x^{2n} \geq \varepsilon^{-1}x^2$, so $0 < f_n(x) \leq (1+\varepsilon^{-1}x^2)^{-1} < \varepsilon x^{-2}$ and

$$0 < K_n \leq 2\varepsilon \int_1^\infty x^{-2}\,dx = 2\varepsilon. \tag{A.3}$$

Also there is a trivial inequality

$$0 < J_n < 4\delta, \tag{A.4}$$

Adding up (A.2)–(A.4), we see: if $n \geq n_0$, then

$$\frac{2-2\delta}{1+\varepsilon} \;<\; I_n + J_n + K_n \;<\; 2 - 2\delta + 2\varepsilon + 4\delta.$$

Therefore,

$$\frac{2-2\delta}{1+\varepsilon} \;\leq\; L \;\leq\; 2 + 2\varepsilon + 2\delta.$$

Since δ and ε can be arbitrarily small, $L = 2$.

(B) Using specific of the integrand. Using the inequalities

$$1 - x^{2n} < (1+x^{2n})^{-1} < 1 \tag{B.1}$$

when $0 < x < 1$, and the inequality

$$(1+x^{2n})^{-1} < x^{-2n} \tag{B.2}$$

when $x \geq 1$, we have

$$1 - \frac{1}{2n+1} \;<\; \int_0^1 f_n(x)\,dx \;<\; 1 \tag{B.3}$$

and

$$0 \;<\; \int_1^\infty f_n(x)\,dx \;<\; \frac{1}{2n-1} \tag{B.4}$$

By the Squeeze Theorem, the integral in (B.3) tends to 1, the integral in (B.4) tends to 0, and the result $(*)$ follows.

(C) A journal-style proof. Define $h(x) = 1$ if $|x| \leq 1$ and $h(x) = x^{-2}$ if $|x| > 1$. Then $|f_n(x)| < h(x)$ for any $n \geq 1$ and any x. Since $\int_{-\infty}^{\infty} h(x)\,dx < \infty$, Eq. (#) holds by Lebesgue's Dominated Convergence Theorem [of Lebesgue's integration theory]. [5].

[4] The symbol $\lceil a \rceil$ denotes the smallest integer $\geq x$ and is called the *ceiling* of a.

[5] See e.g. W. Rudin, Principles of Mathematical Analysis, § 10.32

Remarks. 1. As a matter of fact, the integrals $L_n = \int_0^\infty (1+x^{2n})^{-1}\,dx$ can be evaluated in a closed form for every n. Trying to find the limit through such an evaluation is not a good way to approach the proposed problem, but the result is interesting by itself:

$$L_n = \frac{\pi/(2n)}{\sin(\pi/2n)}.$$

This result can be readily obtained through the theory of Euler's Integrals — Beta and Gamma functions. [6]

Problem 6

A function $f(x)$ is called *logarithmically convex* on the interval I if $f > 0$ and the function $F(x) = ln f(x)$ is concave upward on I. For example, $f(x) = \frac{1}{x}$ is logathmically convex on $(0, +\infty)$, but $f(x) = x^2$ is not.

Prove that if $f(x)$ is logarithmically convex on I, then for any real a the function $g(x) = f(x) + e^{ax}$ is also logarithmically convex on I.

Solution. We'll assume that $f(x)$ is twice differentiable on I. Then the property of f being logarithmically convex is equivalent to

$$(\log f)'' > 0 \quad\Leftrightarrow\quad ff'' - f'^2 > 0 \quad \text{on } I. \tag{1}$$

It is sufficient to prove the same for g, i.e. $gg'' - g'^2 > 0$ on I. Denote for brevity $e^{ax} = t$. We have

$$gg'' - g'^2 = (f'' + a^2 t)(f + t) - (f' + at)^2 = a^2\, ft - \; 2a\, f't \;+ (ff'' - f'^2 + f''t).$$

The RHS is a quadratic polynomial in a with positive coefficient at a^2 and the quarter-discriminant

$$(f't)^2 - (ft)(ff'' - f'^2 + f''t) = -(ff'' - f'^2)(ft + t^2) < 0.$$

Therefore the RHS is positive for any real a, as desired.

Remark. The existence of f'', while a reasonable assumption in a time-costrained competition situation, is not a part of the original question. We demonstrate how the general case can be derived from the considered "smooth" case.

> The definition of concavity needs some comments. In general Calculus texts (Larson et al., Stewart), concavity is only defined for differentiable functions: $f(x)$ is concave upward $\Leftrightarrow$ $f'(x)$ is increasing. Existence of f'' is not assumed. A more general definition of concavity doesn't assume even continuity. It says: $f(x)$ is (strictly) concave upward in I, if
>
> $$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y) \tag{$*$}$$
>
> whenever $x, y \in I$, $x \neq y$, $0 < \lambda < 1$. For continuous functions, it is enough to require $(*)$ with $\lambda = 1/2$. Conversely, if $(*)$ holds with any $\lambda \in (0,1)$, then $f(x)$ is continuous. (Cf. [Rudin] cited in footnote 5, Ch. IV, Ex. 18.)

[6] See e.g. M. Abramowitz, I. Stegun, Handbook of Mathematical Functions

Let $f(x)$ be logarithmically convex, continuous, but not necessarily differentiable. Suppose that for some a the function $g(x)$ defined in the Problem is not logarithmically convex. Then there exist distinct $x_1, x_2 \in I$ such that

$$\ln g(x_3) > \frac{1}{2}(\ln g(x_1) + \ln g(x_2)), \qquad x_3 = \frac{x_1 + x_2}{2}. \tag{2}$$

This inequality refers only to the values of $g(x)$ at three points, therefore it remains valid if one replaces $f(x)$ by any other function $\tilde{f}$ such that

$$f(x_j) = \tilde{f}(x_j), \qquad j = 1, 2, 3. \tag{3}$$

According to the first part of the solution, $\tilde{f}''$ can not exist everywhere in I.

We'll obtain a contradiction by constructing a smooth log-convex function $\tilde{f}$ that satisfies (2). Denote $y_j = \ln f(x_j)$, $j = 1, 2, 3$. Consider *Lagrange's interpolation polynomial*

$$p(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} y_3$$

It is easy to verify that $p(x_j) = y_j$ for each j. The coefficient at x^2 in $p(x)$ is

$$\frac{y_1}{\frac{1}{2}(x_1 - x_2)^2} + \frac{y_2}{\frac{1}{2}(x_1 - x_2)^2} + \frac{y_3}{\frac{-1}{4}(x_1 - x_2)^2} = \frac{2}{(x_1 - x_2)^2}\,(y_1 + y_2 - 2y_3) > 0,$$

because f is log-convex. Therefore, $p(x)$ is concave up everywhere. The function $\tilde{f}(x) = e^{p(x)}$ is smooth, log-convex, and coincides with f at $x_{1,2,3}$.

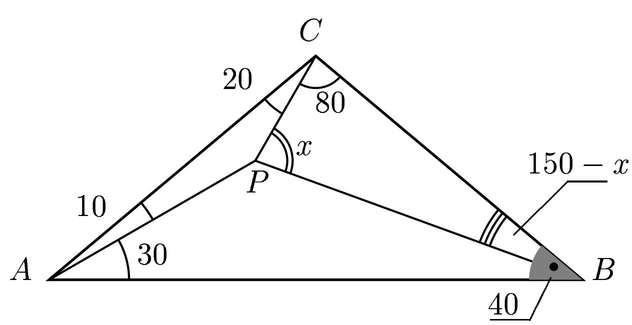
Problem 7

Let P be a point inside the triange ABC such that $\angle PAC = 10^o$, $\angle PCA = 20^o$, $\angle PAB = 30^o$, and $\angle ABC = 40^o$. Determine $\angle BPC$.

Solution 1. 1) $AC = BC$, because $\angle CAB = \angle CBA = 40°$. Note also that $\angle ACB = 100°$ and $\angle PCB = 80°$.

2) $\angle APC = 180° - 10° - 20° = 150°$. By Law of Sines for $\triangle ACP$,

$$\frac{CP}{AC} = \frac{\sin 10°}{\sin 150°}.$$



3) Denote $\angle BPC = x$. Then $\angle CBP = 100° - x$. By Law of Sines for $\triangle BCP$,

$$\frac{CP}{BC} = \frac{\sin(100° - x)}{\sin x}.$$

4) From 1)–3) it follows the equation

$$\frac{\sin x}{\sin(100° - x)} = \frac{\sin 150°}{\sin 10°}.$$

Making trigonometric transformations

$$\frac{\sin(100^\circ - x)}{\sin x} = \frac{\sin 80^\circ \ \cos x + \cos 80^\circ \ \sin x}{\sin x} = \sin 80^\circ \ \cot x + \cos 80^\circ$$

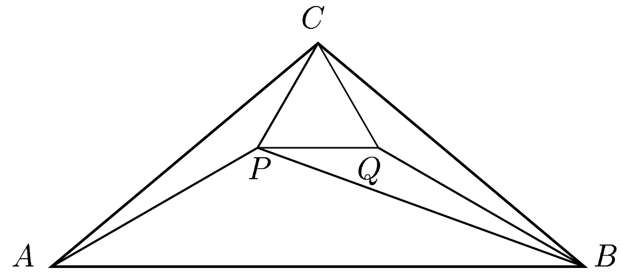
and

$$\frac{\sin 10^\circ}{\sin 150^\circ} \ = \ 2\sin 10^\circ \ = \ 2\cos 80^\circ,$$

we obtain

$$\cot x = \cot 80^\circ \quad \Rightarrow \quad \boxed{x = 80^\circ\,.}$$

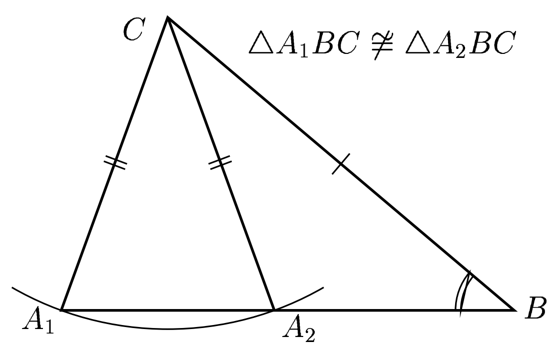
Solution 2. Let Q be the point symmetric to P with respect to the axis of the isosceles $\triangle ABC$. Then $\angle BCQ = \angle ACP = 20^\circ$ and $\angle PCQ = (100 - 40)^\circ = 60^\circ$. Therefore, the isosceles $\triangle PCQ$ is equilateral.



Since $\angle QBC = \angle PAC = 10^\circ$, the angles QBC and QBP are equal and it follows that $QPB \cong QCB$ by two sides and an angle. (See Remark after the proof).

So $\angle QPB = \angle QCB = 20^\circ$ and $\angle CPB = 60^\circ + 20^\circ = 80^\circ$.

Remark. If two triangles have two pairs of equal sides and equal angles opposite to the sides of one of the pairs, then such triangles are not necessarily congruent, as the figure shows. However, if in both triangles the angles opposite to the other equal sides are either both acute or both obtuse, then such triangles *are* congruent.



In our case, $\angle QCB = 20^\circ$, and it suffices to prove that $\angle QPB$ is acute. But $\angle APB > 180 - 30 - 40 = 110^\circ$, $\angle CPB < 360 - 150 - 110 = 100^\circ$, so $\angle QPB < 100 - 60 = 40^\circ$.