Undergraduate Mathematics Competition, Winter 2004

Problem 1
Two evenly matched teams are engaged in a best four-of-seven series of games with each other. Is it more likely for the series to end in six games than in seven games?

Problem 2
Let ABC be a right triangle with right angle A. Let D be the foot of the perpendicular from A to the hypotenuse BC. Denote the inradius of ABC by r, inradius of ABD by r_B and inradius of ADC by r_C. Prove that $r^2 = r_B^2 + r_C^2$.

Problem 3
Let k, l, m, n be positive integers such that $k + l + m \geq n$. Prove the following relation for binomial coefficients

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}$$

The summation in the left-hand side runs over all partitions of n into three integers p, q, r, such that $0 \leq p \leq k$, $0 \leq q \leq l$, $0 \leq r \leq m$.

Problem 4
Fibbonacci numbers are defined by the recurrence $F_{n+1} = F_n + F_{n-1}$ with initial terms $F_1 = F_2 = 1$.

a) Show that every third Fibbonacci number is even;
b) Show that every fifth Fibbonacci number is divisible by 5;

Problem 5
Find

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{dx}{1+x^{2n}}.$$

Problem 6
A function $f(x)$ is called *logarithmically convex* on the interval I if $f > 0$ and the function $F(x) = \ln f(x)$ is concave upward on I. For example, $f(x) = \frac{1}{x}$ is logarithmically convex on $(0, +\infty)$, but $f(x) = x^2$ is not.

Prove that if $f(x)$ is logarithmically convex on I, then for any real a the function $g(x) = f(x) + e^{ax}$ is also logarithmically convex on I.

Problem 7
Let P be a point inside the triange ABC such that $\angle PAC = 10^o$, $\angle PCA = 20^o$, $\angle PAB = 30^o$, and $\angle ABC = 40^o$. Determine $\angle BPC$.