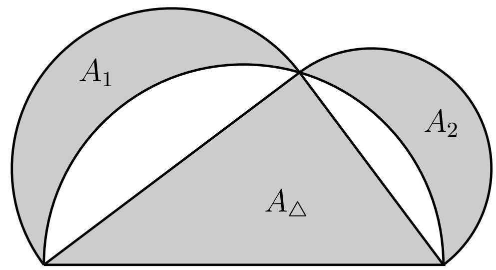
MUN Undergraduate Mathematics Competition. September 23, 2010.

1. How many arrangements of the word **graceful** are there with the letter **g** preceding the letter **f**? (An arrangement doesn't have to be a meaningful word.)

2. Three semicircles have the sides of a right triangle as their diameters as shown. Prove the following relation between the areas of the painted regions:

$$A_1 + A_2 = A_\triangle.$$



3. Determine all integer solutions (x, y) of the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{6}.$$

4. In a group of nine students nobody is taking more than 3 courses. Among any three students, two have one course in common. Show that there is a course taken by at least three students.

5. Given that a series with positive terms $\sum_{n=1}^{\infty} a_n$ converges, prove that the series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges as well.

6. Let $f_1, f_2, \ldots, f_n$ be differentiable linearly independent functions on the real line. Prove that it is possible to choose $n-1$ linearly independent functions among f_1', $f_2', \ldots, f_n'$. (Functions $f_1, \ldots, f_n$ are called *linearly independent* if the equality $\sum_{k=1}^{n} c_k f_k(x) = 0$ for all x with some constants $c_1, c_2, \ldots, c_n$ is only possible when $c_1 = c_2 = \ldots c_n = 0$.)

7. Let A be an $n \times n$ matrix with diagonal entries $A_{ii} = 1$. Suppose A has n real nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n \geq 0$. Prove that

$$0 \leq \det A \leq 1.$$