MUN Undergraduate Mathematics Competition. September 24, 2009.

This competition consists of 7 problems and is open to all undergraduate students. Authors of the best papers will be considered for participation in the Fall 2009 APICS mathematics competition and conference.

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1. Show that

$$(\sqrt{5}+2)^{1/3}-(\sqrt{5}-2)^{1/3}$$

is a rational number and find its value.

2. Prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{99}{100}<\frac{1}{\sqrt{101}}$$

3. Find the greatest common divisor of (14!)+1 and (16!)+10. Justify your answer.

4. Let $ABCD$ be an isosceles trapezoid with sides $|AB|=|CD|=F_n$, $|BC|=F_{n-1}$, $|AD|=F_{n+1}$, where F_k are Fibonacci numbers: $F_1=F_2=1, F_{k+1}=F_k+F_{k-1}$.

Find the area of the trapezoid and express it in terms of a single Fibonacci number.

5. Consider two parallel planes Π_1 and Π_2. Let D be a disk in Π_1 of radius r and center O. Let I be a straight segment in Π_2 of length l and mid-point M. Assume that the segment OM is perpendicular to the planes and has length d. Consider the solid S formed by all segments XY, where X belongs to D and Y belongs to I:

(a) Describe the cross-section of S by the plane parallel to Π_1 and Π_2 half-way between them;

(b) Find the volume of S in terms of r, l, d.

6. Let A_n be the $n\times n$ matrix whose diagonal entries are 0 and all remaining entries are 1. Find $\det A_n$ as a function of n.

7. Suppose $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence such that $a_n\geq 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{n\to\infty} na_n=0.$$