Math 3210 **Due Nov 24** **Assignment #9**

1. Write down Laurent or Taylor series expansion for a function $f(z)$. What statements do you know about their convergence? What are the properties of the function to which they converge?

2. Let $\sum_{n=0}^{\infty} b_n z_0^n = S$ and $\sum_{n=0}^{\infty} c_n z_0^n = T$. Find the sum $\sum_{n=0}^{\infty}(\sum_{k=0}^{n}(\bar{b}_k \bar{c}_{n-k})\bar{z}_0^n$.

3. True of false? Explain.

   Function defined as $f(z) = \dfrac{sinz}{z^2 - \pi^2}$ for $z \neq \pm\pi$ and $f(z) = -(2\pi)^{-1}$ for $z = \pm\pi$ is entire.

4. Find series expansion in terms of integer powers of $z - a$ for given function $f(z)$ and complex number a. Find domain of convergence. Which of them are power series?

   (a) $f(z) = \dfrac{z^2 + 1}{z^6 + 2}$, $a = 0$.

   (b) $f(z) = z^5 \cosh(z^{-4})$, $a = 0$.

   (c) $f(z) = \sin(z)$, $a = \pi/2$.

   (d) $f(z) = \sinh z$, $a = -i\pi/2$.

   (e) $f(z) = (5z + z^2)^{-1}$, $a = 0$.

5. Find Laurent series representation centered at zero in each domain of analyticity for the function
$$f(z) = \frac{z^2 + 8z + 2}{z^2 + 7z + 6}.$$

6. Let $f(z) = (z - 1)^{-1} Log(z)$ for $z \neq 1$ and $f(1) = 1$. Explain why this function is analytic for $-\pi < Arg(z) < \pi$ and $0 < |z| < \infty$.

7. Consider Bessel functions of the first kind $J_n(a)$ defined as
$$J_n(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int + ia\sin t}\, dt, \quad n = 0, \pm 1, \pm 2, ...$$
where a is a complex number. Show that the following series expansion is valid
$$\exp\left(b\left(\frac{z^2 - 1}{z}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(2b) z^n, \quad 0 < |z| < \infty$$
Hint: Use integral formula for coefficients in Laurent series and unit contour of integration $z = e^{it}$.

Extra point problem. Show that Bessel function $J_n(z)$ solves the Bessel equation

$$z^2y'' + zy' + (z^2 - n^2)y = 0.$$

Show that Bessel equation appears when you solve the following Helmholz equation

$$U_{xx} + U_{yy} = -k^2U, \quad k = \text{const}$$

by the separation of variables method in polar coordinates.