Math 3210 - Assignment #8

① Liouville's Theorem: If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

The Fundamental Theorem of Algebra: Any polynomial $P(z) = a_0 + a_1 z + \dots + a_n z^n$ $(a_n \neq 0)$ of degree n $(n \geq 1)$ has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

Outline of Proof: Suppose that $P(z)$ is not zero for any value of z. Then the reciprocal $f(z) = \frac{1}{P(z)}$ is clearly entire, and it is also bounded in the complex plane.

By Liouville's Theorem, $f(z)$ is constant, so $P(z)$ is constant. But, $P(z)$ is not constant, so we have a contradiction. ■

It follows from The fundamental theorem of alg. that any polynomial $P(z)$ of degree n $(n \geq 1)$ can be expressed as a product of linear factors: $P(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n)$, where a_n and z_k $(k=1,2,\dots,n)$ are complex constants. (z_k, $k=1,2,\dots,n$ are the roots, a_n is the leading coefficient)

② Gauss's Mean Value Theorem: When a function $f(z)$ is analytic within and on the circle given by $z_0 + \rho e^{i\theta}$, $\theta \in [0,2\pi]$ its value at the centre is the arithmetic mean of its values on the circle. That is,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta.$$

Maximum Modulus Principle: If a function f is analytic and not constant in a given domain D then $|f(z)|$ has no maximum value in D.

That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.

Outline of proof: Suppose that $|f(z)|$ reaches its maximum at z_0 in the domain D, that is, $|f(z_0)| \geq |f(z)|$ for all $z \in D$. But since $f(z)$ is analytic, by Gauss's th., $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$ thus, $|f(z_0)| \leq \max_{z \in C_\rho(z_0)} |f(z)|$. Here $C_\rho(z_0)$ is a circle of radius ρ centered at z_0. That would imply that $|f(z_0)| = |f(z)|$ for $z \in C_\rho(z_0)$. Since that must be true for any circle of radius $\rho > 0$, we find that $|f(z)| = \text{const} = f(z_0)$, so z_0 is not a maximum point for $|f(z)|$

③ If $f(z)$ is analytic within and on a closed contour C (that contains z_0),

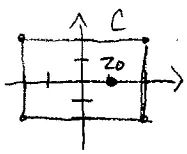
$$2\pi i\, f^{(n)}(z_0) = n! \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad n = 0, 1, 2, \ldots$$

a) $$\oint_{|z|=3} \frac{\cosh(z)}{(z-i)^4}\,dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3}(\cosh z)\Big|_{z=i} = \frac{\pi i}{3}\sinh(i) = \frac{\pi i}{3}\left(\frac{e^{i} - e^{-i}}{2}\right)$$

$$= \frac{\pi i}{6}\Big[(\cos(1) + i\sin(1)) - (\cos(1) - i\sin(1))\Big]$$

$$= \boxed{\frac{-\pi \sin(1)}{3}}$$

b) $$\oint_C \frac{e^{-2z}}{(z-1)^3}\,dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left(e^{-2z}\right)\Big|_{z=1} = \pi i\,(-2)^2\, e^{-2(1)} = \boxed{4\pi i\, e^{-2}}$$



c) $\dfrac{\tan(z/2)}{(z+\pi/2)^2}$ is analytic within and on C given by $|z|=1$, so

$$\oint_C \frac{\tan(z/2)}{(z+\pi/2)^2}\,dz = \boxed{0} \text{ by Cauchy-Goursat Theorem}$$

d) $$\oint_{|z|=2} \frac{\tan(z/2)}{(z+\pi/2)^2}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left(\tan(z/2)\right)\Big|_{z=-\frac{\pi}{2}} = \frac{2\pi i}{2}\left[\sec\left(\tfrac{-\pi}{4}\right)\right]^2 = \boxed{2\pi i}$$

④ $$\oint_{|z|=R} \frac{1}{(z-a)^n(z-b)}\,dz = \oint_{|z|=R} \frac{1/(z-b)}{(z-a)^n}\,dz$$

Since $b > R$, $f(z) = \frac{1}{z-b}$ is analytic in and on C, so:

$$= \frac{2\pi i}{(n-1)!} f^{(n-1)}(a) = \frac{2\pi i}{(n-1)!}\frac{(-1)^{n-1}(n-1)!}{(a-b)^n} = \boxed{\frac{-2\pi i}{(b-a)^n}}$$

5. a) $F_n(z) = \frac{1}{n!\,2^n}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{n!\,2^n}\frac{d^n}{dz^n}\left(z^{2n}+\cdots+(-1)^n\right)$

$= \frac{1}{n!\,2^n}\left[\frac{(2n)!}{n!}z^n+\cdots\right]$ which is a polynomial of degree n.

b) Recall: $2\pi i f^{(n)}(z_0) = n!\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ if $f(z)$ is analytic in & on C & z_0 is inside C.

Let $f(z) = (z^2-1)^n$, then

$$\frac{1}{2^{n+1}\pi i}\oint_{|z-z_0|=r}\frac{(z^2-1)^n}{(z-z_0)^{n+1}} = \frac{1}{2^{n+1}\pi i(n!)}\cdot 2\pi i\,\frac{d^n}{dz^n}(z^2-1)^n$$

$$= \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n = F_n(z)$$

c) $F_n(+1) = \frac{1}{2^{n+1}\pi i}\oint_{|z-1|=r}\frac{(z^2-1)^n}{(z-1)^{n+1}}dz = \frac{1}{2^{n+1}\pi i}\oint_{|z-1|=r}\frac{(z-1)^n(z+1)^n}{(z-1)^{n+1}}dz$

$= \frac{1}{2^{n+1}\pi i}\oint_{|z-1|=r}\frac{(z+1)^n}{z-1}dz = \frac{2\pi i}{2^{n+1}\pi i}(1+1)^n = \boxed{1}$

Similarly, $F_n(-1) = \frac{1}{2^{n+1}\pi i}\oint_{|z+1|=r}\frac{(z-1)^n}{z+1}dz = \frac{2\pi i}{2^{n+1}\pi i}(-1-1)^n = \boxed{(-1)^n}$

6. $f(z) = e^z$ and $R: 0\le x\le 1,\ 0\le y\le \pi$.

$u(x,y) = e^x\cos y$, $v(x,y) = e^x\sin y$

a) $\frac{du}{dx} = e^x\cos y = 0 \Rightarrow y = \pi/2$ $\qquad u(x,\pi/2) = 0$

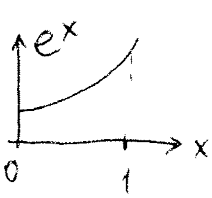
$\frac{du}{dy} = -e^x\sin y = 0 \Rightarrow y = 0,\pi$ $\qquad u(x,0) = e^x$, $\quad u(x,\pi) = -e^x$

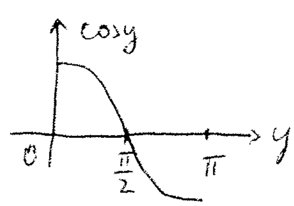
$u(x,y)$ takes its maximum e at $x=1, y=0$ and its minimum $-e$ at $x=1, y=\pi$.

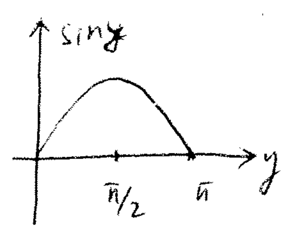
b) $\frac{dv}{dx} = e^x\sin y = 0 \Rightarrow y = 0,\pi$ $\qquad v(x,0) = 0$, $\quad v(x,\pi) = 0$

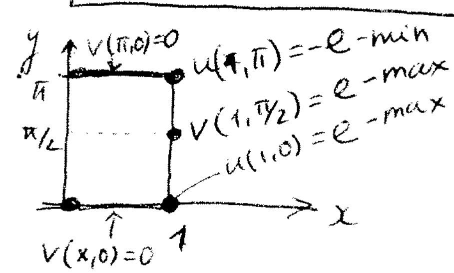
$\frac{dv}{dy} = e^x\cos y = 0 \Rightarrow y = \pi/2$ $\qquad v(x,\pi/2) = e^x$

$v(x,y)$ takes its maximum e at $x=1, y=\pi/2$ and its minimum 0 on the lines $y=0$ & $y=\pi$.









#7. Let $f(z) = u(x,y) + iv(x,y)$ be analytic in R

Then $g(z) = \exp(f(z))$ is also analytic in region R.

Thus, by Maximum Modulus Principle,

$|g(z)|$ reaches its maximum on the boundary of R but not inside. Considering that

$$|g(z)| = |e^{u(x,y)}| \cdot \underbrace{|e^{iv(x,y)}|}_{=1} = |e^{u(x,y)}| = e^{u(x,y)},$$

we have that function $e^{u(x,y)}$ has its maximum at the boundary of R but not inside. Due to the property of exponent: $a < b \iff e^a < e^b$ for any real #s a, b, if $e^{u(x,y)}$ takes maximum at a point then $u(x,y)$ also has maximum at this point and thus $u(x,y)$ reaches its maximum at the boundary but not in the interior of R.

Similar considerations involving function $h(z) = \exp(if(z))$, with $|h(z)| = e^{v(x,y)}$, lead to the fact that $v(x,y)$ reaches its maximum at the boundary of R.

Consideration of functions $\frac{1}{\exp(f)}$ and $\frac{1}{\exp(if)}$ leads to the statements conserning minimum values of $u(x,y)$ and (x,y) occuring at a point of the boundary.