Math 3210 - Assignment #7

(1) <u>Cauchy-Goursat Theorem</u>: If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

<u>Outline of Proof</u>: We assume also that f' is continuous in R. This was Cauchy's original formulation of the theorem, but Goursat proved that this assumption was unnecessary.

Write $f(z) = u(x,y) + iv(x,y)$ and $z(t) = x(t) + iy(t)$, $a \le t \le b$.

Now, $\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt = \int_a^b [u(x(t),y(t)) + iv(x(t),y(t))][x'(t) + iy'(t)]\,dt$

$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt = \int_a^b (u\,dx - v\,dy) + i\int_a^b (v\,dx + u\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Using Green's Theorem $\left(\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA \right)$,

we get $\int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA$.

But, if f is analytic and f' is continuous in R, v_x, v_y, u_x, u_y exist and are continuous, and we have the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$, making each double integral $= 0$, and thus

$$\int_C f(z)\,dz = 0.$$

(2) <u>Cauchy Integral Formula</u>: Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense.

If z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz$. (1)

Equivalently, $2\pi i f(z_0) = \int_C \frac{f(z)}{z-z_0}\,dz$ (2)

<u>Outline of Proof (for (2)):</u>

Let C_ρ denote a positively oriented circle $|z-z_0| = \rho$, where ρ is small enough that C_ρ is interior to C. Since $f(z)/(z-z_0)$ is analytic between and on the contours C_ρ and C, we have that

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz.$$

Thus, $\int_C \frac{f(z)}{z-z_0} - f(z_0) \int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz$.

But $\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i$, so we have $\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz$.

Since f is analytic, it is continuous: $\forall \varepsilon > 0$, $\exists \delta$ such that $|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$.

Now, let $\rho < \delta$ so that $|z-z_0| = \rho < \delta$, then $\left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| < 2\pi\varepsilon$

$\Rightarrow \left| \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right| < 2\pi\varepsilon$. $\forall \varepsilon > 0$, so $\int_C \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0)$.

③ Morera's Theorem: Let f be continuous on a domain D. If $\int_C f(z)\,dz = 0$ for every closed contour C in D, then f is analytic throughout D.

Outline of Proof:

Recall two results:

Theorem 1: Suppose that a function $f(z)$ is continuous on a domain D. The following are equivalent:

(a) $f(z)$ has an antiderivative $F(z)$ throughout D

(b) the integrals of $f(z)$ along contours lying entirely in D and extending from z_1 to z_2 all have the same value, namely,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1) \quad (\text{where } F'(z) = f(z) \text{ in } D)$$

(c) $\int_C f(z)\,dz = 0$ where C is any closed contour lying entirely in D.

Theorem 2: If a function is analytic at a given point, then its derivatives of all orders are analytic there too.

Now, given the hypotheses of Morera's Theorem, Theorem 1 tells us that $f(z)$ has an antiderivative $F(z)$ throughout D. That is, there is an analytic function $F(z)$ such that $F'(z) = f(z)$ at each point in D. Theorem 2 now tells us that $f(z)$ is analytic throughout D.

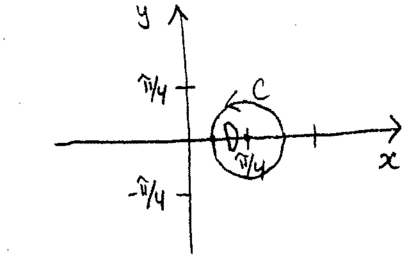
④ a) $\displaystyle\oint_{|z|=5} \frac{z^3}{z+4i}\,dz = \oint_{|z|=5} \frac{z^3}{z-(-4i)}\,dz$ Using Cauchy-Integral Formula (CIF) with $f(z) = z^3$ & $z_0 = -4i$,

this integral $= 2\pi i(-4i)^3 = \boxed{-128\pi}$

b) Again, using CIF with $f(z) = z^2$ & $z_0 = -i-1$,

$$\oint_{|z|=\sqrt{3}} \frac{z^2}{z+i+1}\,dz = \oint_{|z|=\sqrt{3}} \frac{z^2}{z-(-i-1)}\,dz = 2\pi i(-i-1)^2 = \boxed{-4\pi}$$

c) $D: |z-\pi/4| = \pi/8$



$\cot z = \dfrac{\cos z}{\sin z}$ is analytic at all points interior to and on the simple closed contour C, so

$$\oint_{|z-\pi/4|=\pi/8} \cot z \, dz = 0,$$ by the Cauchy-Goursat Theorem (CGT).

d) Let $g(z) = \dfrac{e^z}{z-3i}$. It is analytic every where except at $z = 3i$.

Since C is given by $|z| = e < |3i| = 3$, $g(z)$ is analytic at all points interior to and on C, so

$$\oint_C g(z)\,dz = \oint_{|z|=e} \frac{e^z}{z-3i}\,dz = 0,$$ by the CGT.

e) $\dfrac{1}{(z+2i)(z+5)} = \dfrac{A}{z+2i} + \dfrac{B}{z+5} \Rightarrow 1 = A(z+5) + B(z+2i)$

$\Rightarrow \left.\begin{array}{l} A+B=0 \\ 5A+2iB=1 \end{array}\right\} \begin{array}{l} B=-A \\ A(5-2i)=1 \end{array}$

$A = \dfrac{1}{5-2i}$, $B = \dfrac{-1}{5-2i}$

So, $$\oint_{|z|=4} \frac{z^2}{(z+2i)(z+5)}\,dz = \underbrace{\frac{1}{5-2i}\oint_{|z|=4}\frac{z^2}{z+2i}}_{I_1} - \underbrace{\frac{1}{5-2i}\oint_{|z|=4}\frac{z^2}{z+5}}_{I_2}$$

Since $\dfrac{z^2}{z+5}$ is analytic everywhere except $z=-5$, and $z=-5$ is out side the contour $|z|=4$, $I_2 = 0$ by CGT.

With $f(z) = z^2$ and $z_0 = -2i$, $I_1 = \dfrac{2\pi i(-2i)^2}{5-2i} = \dfrac{-8\pi i}{5-2i}$ by CIF

So, $$\oint_{|z|=4} \frac{z^2}{(z+2i)(z+5)} = \frac{-8\pi i}{5-2i} = \frac{-8\pi i(5+2i)}{5^2+2^2} = \boxed{\frac{8\pi}{29}(2-5i)}$$

f) $\cosh^2 z + \sinh^2 z = \frac{1}{4}(e^{z}+e^{-z})^2 + \frac{1}{4}(e^{z}-e^{-z})^2$

$= \frac{1}{4}\left[(e^{2z}+2+e^{-2z}) + (e^{2z}-2+e^{-2z})\right]$

$= \frac{1}{4}(e^{2z}+e^{-2z}) = \frac{1}{2}\cosh 2z$, which is analytic everywhere

Thus, by CGT, $\oint_{|z|=2} (\cosh^2 z + \sinh^2 z)\,dz = 0$

g) $\oint_{|z|=2} \frac{\cosh^2 z + \sinh^2 z}{z-\ln 2}dz = \frac{1}{4}\oint_{|z|=2} \frac{2e^{2z}+2e^{-2z}}{z-\ln 2}dz = \frac{2\cdot 1}{4}(2\pi i)\left[e^{2\ln 2}+e^{-2\ln 2}\right]$ (by CIF)

$= 2\cdot\frac{\pi i}{2}\left(4+\frac{1}{4}\right) = \boxed{\frac{17\pi i}{\cancel{8}4}} = \frac{17\pi i}{4}$

h) Log$(z-2)\,dz$ is analytic everywhere except $z=2$. Since $z=2$ is not inside or on the contour $|z-5|=2$,

$\oint_{|z-5|=2} \text{Log}(z-2)\,dz = 0$, by CGT.

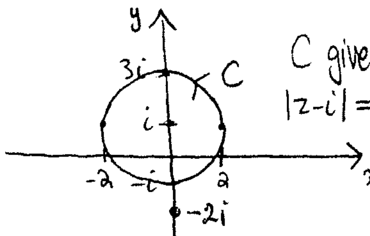
i) $\frac{2z+3}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \Rightarrow 2z+3 = A(z+2)+B(z+1)$

$\Rightarrow \left.\begin{array}{l} A+B=2 \\ 2A+B=3 \end{array}\right\} \begin{array}{l} A=2-B \\ 2(2-B)+B=3 \end{array}\Bigg\} \begin{array}{l} 4-2B+B=3 \\ \Rightarrow B=1 \\ \Rightarrow A=1 \end{array}$

$\Rightarrow \oint_{|z|=3}\frac{2z+3}{(z+1)(z+2)}dz = \oint_{|z|=3}\frac{1}{z+1}dz + \oint_{|z|=3}\frac{1}{z+2}dz = 2\pi i(1) + 2\pi i(1)$ (By CIF w/ $f(z)=1$)

$= \boxed{4\pi i}$

j)



C given by $|z-i|=2$

$\frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)} = \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right)$

$\Rightarrow \oint_{|z-i|=2}\frac{\sin z}{z^2+4}dz = \frac{1}{4i}\oint_{|z-i|=2}\frac{\sin z}{z-2i}dz - \frac{1}{4i}\underbrace{\oint_{|z-i|=2}\frac{\sin z}{z+2i}dz}_{=0}$

$= \frac{1}{4i}(2\pi i)\sin(2i) - \boxed{\frac{1}{4i}(2\pi i)\sin(-2i)} = \pi\sin(2i) = \boxed{\frac{\pi}{2i}(e^{-2}-e^{2})}$

$= \frac{\pi}{2}\sin(2i)$

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$z=-2i$ outside of $|z-i|\leq 2$.

(5.) Since f(z) is analytic with and on C, f'(z) is analytic within and on C, so we can use CIF on the LHS:

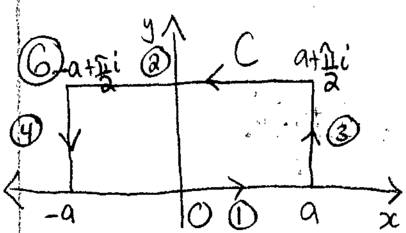
$$\int_C \frac{f'(z)}{z-z_0}\,dz = 2\pi i f'(z_0)$$

Since we have that $n!\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = 2\pi i f^{(n)}(z_0)$, $n=0,1,\ldots$

we know that the RHS is: $\int_C \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0)$

Thus, LHS $= 2\pi i f'(z_0) =$ RHS.

(6)



By Cauchy-Goursat, $\oint_C e^{-z^2}dz = 0$.

$$f(z) = e^{-z^2} = e^{-(x^2-y^2)}\cdot e^{-2ixy}$$

Side 1: $y=0 \Rightarrow e^{-x^2} = f(z)$

Side 2: $y=\frac{\pi}{2} \Rightarrow e^{\frac{\pi^2}{4}-x^2}\cdot e^{-\pi i x} = f(z)$

Side 3: $x=a \Rightarrow f(z) = e^{-(a^2-y^2)}\cdot e^{-2aiy}$

Side 4: $x=-a \Rightarrow f(z) = e^{-(a^2-y^2)}\cdot e^{2aiy}$

So, $0 = \oint_C e^{-z^2}dz = \int_{-a}^{a} e^{-x^2}dx - \int_{-a}^{a} e^{-x^2}\cdot e^{\pi^2/4}(\cos(\pi x) - \cancelto{0}{i\sin(\pi x)})dx$

$$+ \int_0^{\pi} e^{-a^2}e^{y^2}(e^{-2aiy} - e^{2aiy})\,dy$$

Let $a \to \infty$

$$e^{\pi^2/4}\int_{-\infty}^{\infty} e^{-x^2}\cos\pi x\,dx = \int_{-\infty}^{\infty} e^{-x^2}dx - \cancelto{0}{e^{-\infty}}\int_0^1 e^{y^2}\,2i\sin(2ay)\,dy$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\cos\pi x\,dx = \frac{\sqrt{\pi}}{e^{\pi^2/4}}}$$