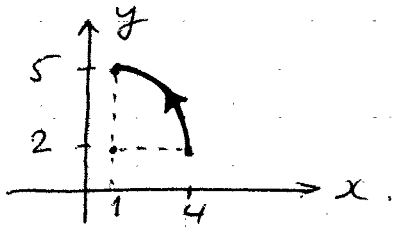
1 (a) $z(t) = 1+3\cos t + i\,(2+3\sin t) \quad t \in [0, \pi/2]$

$= (1+2i) + 3e^{it}$

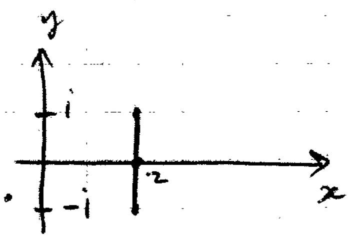
so the curve is an arc of a circle of radious 3 and center at $1+2i$ in the complex plane. The arc goes from point $4+2i$ to $1+5i$



This curve is simple and smooth.
$z'(t) = -3\sin t + 3i\cos t \neq 0.$

(b) $z(t) = 2+i\cos(2t), \quad t \in [0, \pi]$

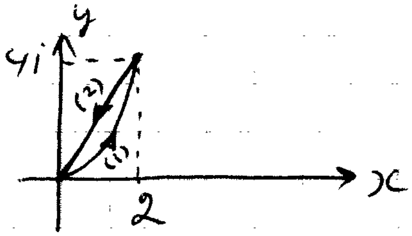
the curve is a vertical segment from $2+i$ to $2-i$ and back to $2+i$.



This curve is differentiable : $z' = -2i\sin(2t)$
but <u>not</u> smooth or simple. It is a closed contour.

(c) $$z = \begin{cases} t+it^2, & t \in [0,2] \quad (1) \\ (2-s)+i(4-2s), & s \in [0,2] \quad (2) \end{cases}$$

this curve consists of a fragment of parabole and a straight line from $2+4i$ to the origin.



It is a closed contour.
(simple)

2 (a) If $a \le t \le b$ then $-a \geq -t \geq -b$; call $\boxed{\tau = -t}$

Curve $\{ z(-\tau),\ -b \le \tau \le -a \}$ starts at $z(-(-b)) = z(b)$ and ends at $z(-(-a)) = z(a)$. Thus it is opposite to $\{z(t),\ a \le t \le b\}$

2(b) $\mathcal{C} = \{ z(t),\ a \le t \le b \}$

$-\mathcal{C} = \{ z(-\tau),\ -b \le \tau \le -a \}$

$$\int_{-\mathcal{C}} f(z)\,dz = \int_{-b}^{-a} f(z(-\tau))\,\frac{d}{d\tau}\big(z(-\tau)\big)\,d\tau =$$

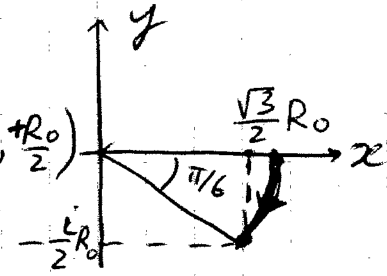
$$= -\int_{-b}^{-a} f(z(-\tau))\left(\frac{dz}{dt}\right)\Big|_{t=-\tau} d\tau = \left(\begin{matrix}\theta = -\tau \\ d\theta = -d\tau\end{matrix}\right) =$$

$$= -\int_a^b f(z(\theta))\left(\frac{dz}{dt}\right)\Big|_{t=\theta} d\theta = -\int_{\mathcal{C}} f(z)\,dz.$$

3(a) $z_1(t) = R_0 e^{-it} \qquad 0 < t < \pi/6$

$z(s) = x(s) + i y(s)\ ,\ \ y(s) = s \in (0, +\frac{R_0}{2})$

$x(s) = +\sqrt{R_0^2 - s^2}$



Thus $z_1(t) \cong \bar{z}_2(s)\ ,\ \ s \in [0, \frac{R_0}{2}]$

Also $z_1(t) \cong z_2(-s)\ ,\ \ s \in [0, \frac{R_0}{2}]$

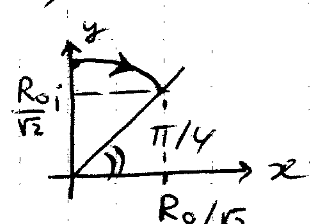
Similarly, $z_1(t) \cong \bar{z}_3(-q)\ ,\ \ q \in [-R_0, -\frac{\sqrt{3}}{2}R_0]$

3(b) $z_1(t) = R_0 e^{-it} \qquad -\frac{\pi}{2} < t < -\frac{\pi}{4}$

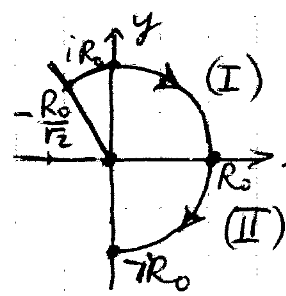
$z_1(t) \cong z_3(q)\ ,\ q \in [0, \frac{R_0}{\sqrt{2}}]$

or

$z_1(t) \cong z_2(-s)\ ,\ \ s \in [-R_0, -\frac{R_0}{\sqrt{2}}]$



3(c) $z_1(t) = R_0 e^{-it}\ ,\ \ -\frac{3\pi}{4} < t < \frac{\pi}{2}$



<u>union</u> of $z_3(q)\ ,\ q \in [-\frac{R_0}{\sqrt{2}}, R_0]$ (I)

and $\bar{z}_2(s)\ ,\ s \in [0, R_0]$ (II)

(or $z_2(-s)\ ,\ s \in [0, R_0]$)

4(a) $z(t) = 3e^{2it}, \quad 0 \le t \le \pi \quad ; \quad z'(t) = i6e^{2it}$

$$\int_C (z^3+3z-2)\,z^{-2}\,dz = \int_C z + \frac{3}{z} - \frac{2}{z^2}\,dz =$$

$$= \int_0^{\pi} \left[(3e^{2it}) + \frac{1}{e^{2it}} - \frac{2}{9e^{4it}}\right] \cdot i6e^{2it}\,dt$$

$$= i\int_0^{\pi} 18e^{4it}\,dt + i\int_0^{\pi} 6\,dt - i\int_0^{\pi} \frac{4}{3e^{2it}}\,dt$$

$$= 0 + 6i\pi + 0 = \boxed{6\pi i}$$

4(b) $z(t) = \sqrt{3}\,e^{-it}, \quad 0 \le t \le 4\pi \qquad dz = -i\sqrt{3}\,e^{-it}\,dt$

$z(t) = \sqrt{3}\cos t - \sqrt{3}\,i\sin t = x+iy$

$$\int_C x\,|z|^{-2}\,dz = \int_0^{4\pi} \frac{\sqrt{3}\cos t}{3} \cdot (-i\sqrt{3}\,e^{-it})\,dt$$

$$= -i\int_0^{4\pi} \cos t\,(\cos t - i\sin t)\,dt = -i\int_0^{4\pi}\left(\cos^2 t - \frac{i}{2}\sin(2t)\right)dt$$

$$= -i\,\frac{1}{2}\int_0^{4\pi} dt = \boxed{-2\pi i}$$

4(c) $$\int_C \left(\frac{1}{z} + \frac{1}{\bar z}\right) dz = \int_C \frac{\bar z + z}{|z|^2}\,dz = 2\int_C \frac{x}{|z|^2}\,dz = -4\pi i$$

(since we have the same contour as in 4(b))

4(d) $$xy + i\left(\frac{y^2-x^2}{2}\right) = -\frac{i}{2}\left(+i2xy + (x^2-y^2)\right) = -\frac{i}{2}z^2$$

$$\int_C xy + i\left(\frac{y^2-x^2}{2}\right) dz = -\frac{i}{2}\int_C z^2\,dz$$

C_1: segment from $2i$ to 1 : $x=t,\ y=2-2t,\ t\in[0,1]$

$z = t + 2i(1-t) \qquad z' = 1-2i$

$$\int_{C_1}\left[2t(1-t) + \frac{i}{2}\left(4(1-t)^2 - t^2\right)\right](1-2i)\,dt = \frac{8-i}{6}$$

C_2: segment from 1 to -1 : $x=-t,\ t\in[-1,1],\ y=0\ ;\ z'=-1$

4(d) cont-d.

$$-\frac{i}{2}\int_{C_2} z^2\,dz = \int_{-1}^{1} \frac{i}{2} t^2\,dt = i\,\frac{t^3}{6}\Big|_{-1}^{1} = \frac{i}{3}$$

Thus $-\frac{i}{2}\int_{C_1+C_2} z^2\,dz = \frac{8-i}{6} + \frac{i}{3} = \frac{4}{3} + \frac{i}{6}$

5(a) use $\big|\,|z_1| - |z_2|\,\big| \le |z_1 \pm z_2| \le |z_1| + |z_2|$

$$|z^2+z+2| \le |z^2| + |z| + 2 = 4+2+2 = 8$$

Since $z = 2e^{2it}$, $t\in[0,\pi/4]$ $\Rightarrow$ $|z| = 2$.

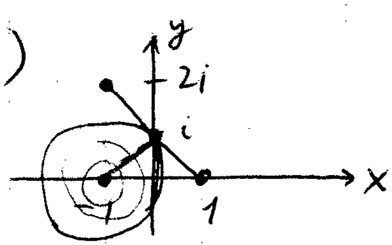
$$|z^5+z+1| \geqslant \big|\,|z^5| - |z+1|\,\big| \geqslant \big|\,|z^5| - (|z|+1)\big|$$

$$= 32-(2+1) = 29$$

$$\left|\frac{z^2+2+z}{z^5+z+1}\right| \le \frac{8}{29} \Rightarrow \left|\int_C \frac{z^2+2+z}{z^5+z+1}dz\right| \le \frac{2\cdot 2\pi}{4}\cdot\frac{8}{29} = \frac{8\pi}{29}$$

Since $\frac{8\pi}{29} > \frac{4\pi}{15}$, the estimate is not confirmed.

5(b)



$f(z) = \left|\frac{1}{(z+1)^4}\right|$ is constant on each circle $|z+1| = r$

$z = i$ is the nearest point on the segment $[-1+2i, 1]$ to $z=-1$.

Thus $\max_{[-1+2i,1]} |f(z)| = |f(i)| = \left|\frac{1}{(1+i)^4}\right| = \frac{1}{4}$

The segment has length $2\sqrt{2}$.

$$\left|\int_C \frac{dz}{(z+1)^4}\right| \le 2\sqrt{2}\cdot\frac{1}{|(1+i)|^4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}.$$

Thus the estimate is confirmed.