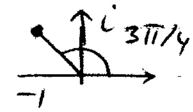
Assignment #5



1 (a) $(i-1)^{1+2i} = \exp\left((1+2i)\log(i-1)\right) =$

$= \exp\left((1+2i)\left(\ln\sqrt{2} + i\left(\frac{3\pi}{4} + 2\pi k\right)\right)\right)$

$= \exp\left(\ln\sqrt{2} - \left(\frac{3\pi}{2} + 4\pi k\right) + i\left(2\ln\sqrt{2} + \frac{3\pi}{4} + 2\pi k\right)\right)$

$= \sqrt{2}\, e^{-\left(\frac{3\pi}{2} + 4\pi k\right)} e^{i\left(\ln 2 + \frac{3\pi}{4} + 2\pi k\right)}$

$= \sqrt{2}\, e^{\frac{\pi}{2}} \cdot e^{+4\pi k} e^{i\left(\ln 2 + \frac{3\pi}{4}\right)}, \quad k \in \mathbb{Z}$

$= \sqrt{2}\, e^{\pi/2} \cdot e^{4\pi k}\left(\cos\left(\ln 2 + \frac{3\pi}{4}\right) + i\sin\left(\ln 2 + \frac{3\pi}{4}\right)\right).$

(b) $\frac{d}{dz}\left((-5)^z\right) = \frac{d}{dz}\left[e^{z\log(-5)}\right] = \log(-5)\cdot e^{z\log(-5)} =$

$= \log(-5)\cdot(-5)^z \;; \quad \log(-5) = \ln 5 + i(\pi + 2\pi k).$

$(-5)^{i\pi} = e^{i\pi\log(-5)} = e^{i\pi(\ln 5 + i(\pi + 2\pi k))} = e^{-\pi^2(1+2k)} e^{i\pi\ln 5}$

$\frac{d}{dz}(-5)^z\Big|_{z=i\pi} = \log(-5)\cdot(-5)^{i\pi} = \left(\ln 5 + i(\pi + 2\pi k)\right) e^{-\pi^2(1+2k)} e^{i\pi\ln 5}$

$= e^{-\pi^2(1+2k)}\left(\ln 5 + i\pi(1+2k)\right)\left(\cos(\pi\ln 5) + i\sin(\pi\ln 5)\right)$

$= e^{-\pi^2(1+2k)}\Big[\ln 5\cdot\cos(\pi\ln 5) - \pi(1+2k)\sin(\pi\ln 5) +$

$+ i\left\{\ln 5\cdot\sin(\pi\ln 5) + \pi(1+2k)\cos(\pi\ln 5)\right\}\Big].$

(c) $\left(-\sqrt{3} - 3i\right)^3 = (-1)\cdot(\sqrt{3})^3\left(1 + \sqrt{3}i\right)^3 = -3\sqrt{3}\cdot 8\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3$

$= -3\sqrt{3}\cdot 8\left(e^{i\pi/3}\right)^3 = -3\sqrt{3}\cdot 8\cdot e^{i\pi} = 24\sqrt{3}.$

2 (a) $\cos^2 z - \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz} - e^{iz}}{2i}\right)^2 =$

$\frac{1}{4}\left(\left(e^{2iz} + e^{-2iz} + 2\right) + \left(e^{2iz} - 2 + e^{-2iz}\right)\right) = \frac{e^{2iz} + e^{-2iz}}{2} = \cos(2z)$

2 (b) $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right)$

$= \frac{1}{2}\left(e^{-y} + e^{y}\right)\sin x - \frac{i}{2}\left(\cos x(e^{-y} - e^{y}\right)$

$= \cosh y\cdot\sin x + i\sinh y\cdot\cos x$

2(b) cont'd

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \left(+ \sinh^2 y \sin^2 x - \sinh^2 y \sin^2 x \right)$$
$$= \sin^2 x + \sinh^2 y.$$

(c) $\sin(iz) = \dfrac{e^{-z} - e^{z}}{2i} = +i\, \dfrac{e^{z} - e^{-z}}{2} = i \sinh z$

(d) $(\tanh z)' = \left[(e^{z} - e^{-z})(e^{z} + e^{-z})^{-1} \right]' =$

$$= (e^{z} + e^{-z})(e^{z} + e^{-z})^{-1} - (e^{z} - e^{-z})^2 (e^{z} + e^{-z})^{-2}$$
$$= 1 - \left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} \right)^2 = \frac{(e^{z} + e^{-z})^2 - (e^{z} - e^{-z})^2}{(e^{z} + e^{-z})^2} = \frac{4}{(e^{z} + e^{-z})^2} = \operatorname{sech}^2 z$$

(e) $(\sec z)' = \left(\dfrac{2}{e^{iz} + e^{-iz}} \right)' = -2i (e^{iz} + e^{-iz})^{-2} (e^{iz} - e^{-iz})$

$$= \frac{4}{(e^{iz} + e^{-iz})^2} \cdot \frac{e^{iz} - e^{-iz}}{2i} = \frac{\sin z}{\cos^2 z} = \tan z \sec z$$

3. $\tan w = z$

(a) $e^{iw} - e^{-iw} = iz\,(e^{iw} + e^{-iw})$

$$e^{2iw} - 1 = iz\,(e^{2iw} + 1) \quad ; \quad e^{2iw}(1 - iz) = 1 + iz$$
$$e^{2iw} = \frac{1+iz}{1-iz} \quad ; \quad w = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz} \right) = -\frac{i}{2} \log\left(\frac{1+iz}{1-iz} \right).$$

$$\boxed{w = \arctan(z) \equiv \tan^{-1}(z) = -\frac{i}{2} \log\left(\frac{1+iz}{1-iz} \right)}$$

(b) $\dfrac{d}{dz} w \equiv \dfrac{d}{dz}\left(\tan^{-1} z \right) = -\dfrac{i}{2} \dfrac{d}{dz} \log\left(\dfrac{1+iz}{1-iz} \right) =$

$$= -\frac{i}{2} \, \frac{\cancel{1-iz}}{1+iz} \cdot \frac{i(1-iz) + i(1+iz)}{(1-iz)^{\cancel{2}}} = \frac{1}{(1+iz)(1-iz)} = \frac{1}{1+z^2}$$

$$\boxed{\frac{d}{dz}\left(\tan^{-1} z \right) = \frac{1}{1+z^2}}$$

4 (a) $\frac{d}{dz}\left(\sinh^2(\tan(3t)-it^3)\right) =$

$2\sinh(\tan(3t)-it^3)\cdot\cosh(\tan(3t)-it^3)\cdot(3\sec^2(3t)-3it^2)$

$= \sinh\left[2(\tan(3t)-it^3)\right]\cdot(3\sec^2(3t)-3it^2)$

(b) $\frac{d}{dz}(\sec t + i\,\mathrm{sech}\,t)^{n^2} = n^2(\sec t+i\,\mathrm{sech}\,t)^{n^2-1}\cdot(\sec t\tan t -$

$- i(\mathrm{sech}\,t\cdot\tanh t)$

5. (a) $\int_0^\pi e^{(a+ib)t}dt = \frac{1}{a+ib}e^{t(a+ib)}\Big|_0^\pi = \frac{a-ib}{a^2+b^2}\left(e^{\pi(a+ib)}-1\right)$

$= \frac{a-ib}{a^2+b^2}\left(e^{\pi a}\cos\pi b - 1 + ie^{\pi a}\sin\pi b\right) \equiv I(a,b)$

(b) $\int_0^\pi e^{at}\cos nt\,dt = \operatorname{Re} I(a,b)\Big|_{b=n\in\mathbb{N}} =$

$= \frac{a(e^{\pi a}\cos\pi n - 1) + ne^{\pi a}\sin n b}{a^2+n^2} = \boxed{\frac{a((-1)^n e^{\pi a}-1)}{a^2+n^2}}$

(c) $J \equiv \int_0^\pi e^{at}\cos nt\,dt = \left|\begin{array}{ll} u=e^{at} & du = ae^{at}dt \\ dv = \cos nt\,dt & v = \frac{1}{n}\sin(nt)\end{array}\right.$

$= \frac{1}{n}\cancel{e^{at}\sin(nt)\Big|_0^\pi}^{\,0} - \frac{a}{n}\int e^{at}\sin(nt)\,dt = \left|\begin{array}{l} u=e^{at} \\ dv=\sin(nt) \\ v = -\frac{1}{n}\cos nt\end{array}\right.$

$= -\frac{a}{n}\left[-\frac{1}{n}e^{at}\cos nt\Big|_0^\pi + \frac{a}{n}\int e^{at}\cos nt\,dt\right]$

$= +\frac{a}{n^2}\left(e^{a\pi}(-1)^n - 1\right) - \frac{a^2}{n^2}\cdot J \Rightarrow \frac{n^2+a^2}{n^2}J = \frac{a}{n^2}\left(e^{a\pi}(-1)^n-1\right)$

$\Rightarrow J = \frac{a}{a^2+n^2}\left((-1)^n e^{\pi a}-1\right)$ - same as in (b).

6 (a) $\frac{(t+i)^3}{t^{9/2}} = \frac{t^3+3it^2-3t-i}{t^4\cdot t^{1/2}} = \frac{1}{t^{3/2}} + \frac{3i}{t^{5/2}} - \frac{3}{t^{7/2}} - \frac{i}{t^{9/2}}$

$\int_1^\infty \frac{(t+i)^3}{t^{9/2}} = 2 - 3\cdot\frac{2}{5} + i\left(3\cdot\frac{2}{3} - \frac{2}{7}\right) = \boxed{\frac{4}{5} + i\,\frac{12}{7}}$

6(b) $$\int_0^\infty \frac{(1+i)^2}{(1+t)^2}\,dt = (1+i)^2\int_0^\infty \frac{dt}{(1+t)^2} =$$

$$= (1+2i-1)\int_1^\infty \frac{d\xi}{\xi^2} = 2i\cdot(0-(-1)) = \boxed{2i}$$

6(c) $$\int_{-\infty}^\infty \frac{2}{1+t^2} + \frac{2it}{1+t^6}\,dt = 2\int_{-\infty}^\infty \frac{dt}{1+t^2} + 2i\int\frac{t\,dt}{1+t^6}$$

$$\int_{-\infty}^\infty \frac{dt}{1+t^2} = \int_{-\infty}^0 \frac{dt}{1+t^2} + \int_0^\infty \frac{dt}{1+t^2} = -\lim_{u\to-\infty}\arctan u +$$

$$+\lim_{u\to\infty}\arctan u = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

$$\int_{-\infty}^\infty \frac{t\,dt}{1+t^6} = 0$$ b/c Integrant is an odd function and both integrals $\int_{-\infty}^0$ and $\int_0^\infty$ converge.

$\therefore$ Answer: $\boxed{2\pi}$