Assignment #4

1. (a) $v = 3xy + 5x$

$v_{xx} + v_{yy} = 0 + 0 = 0 \Rightarrow v(x,y)$ is harmonic

$v_x = +3y + 5 \equiv -u_y \Rightarrow u = -3\frac{y^2}{2} - 5y + C(x)$

$v_y = 3x \equiv u_x = C'(x) \Rightarrow C(x) = 3\frac{x^2}{2} + \text{const}$

$\Rightarrow \boxed{u = \frac{3}{2}(x^2 - y^2) - 5y + \text{const.}}$ ← harmonic conjugate to $v(x,y)$

(b) $v = y^4 - 6x^2y^2 + x^4 - y$

$v_x = -12xy^2 + 4x^3$; $v_y = 4y^3 - 12x^2y - 1$

$v_{xx} + v_{yy} = (-12y^2 + 12x^2) + (12y^2 - 12x^2) = 0$

$\Rightarrow v$ is harmonic

$v_x = -u_y \Rightarrow u = \int 12xy^2 - 4x^3\, dy = 4xy^3 - 4x^3y + C(x)$

$v_y = u_x \Rightarrow 4y^3 - 12x^2y + C'(x) = 4y^2 - 12x^2y - 1$

$\Rightarrow C(x) = -x \Rightarrow \boxed{u(x,y) = 4xy^3 - 4x^3y - x + \text{const}}$

(c) $v = e^{-2x}\cos(2y)$

$v_x = -2e^{-2x}\cos(2y) \qquad v_y = -2e^{-2x}\sin(2y)$

$v_{xx} + v_{yy} = 4e^{-2x}\cos 2y - 4e^{-2x}\cos(2y) = 0$

$\Rightarrow v$ is harmonic

$v_x = -u_y \Rightarrow u = \int 2e^{-2x}\cos(2y)\,dy = \frac{2e^{-2x}\sin(2y)}{2} + C(x)$

$= e^{-2x}\sin(2y) + C(x)$

$v_y = u_x \Rightarrow -2e^{-2x}\sin(2y) + C'(x) = -2e^{-2x}\sin(2y)$

$\Rightarrow C'(x) = 0 \Rightarrow C = \text{const} \Rightarrow \boxed{u = e^{-2x}\sin(2y) + \text{const}}$

2. $F(z)$ is analytic in D $\Rightarrow$ $F(z) = u(x,y) + iv(x,y)$ is differentiable in D $\Rightarrow$ (C.R.) $u_x = v_y$, $u_y = -v_x$

If $F(z)$ is real valued in D then $v(x,y) = 0$ in D

Thus $v_x = v_y = 0$ in D. Thus, by (C.R.) $\underline{\underline{u_x = u_y = 0}}$ in D. Therefore $f(z) = u(x,y) = \text{const}$. QED.

3. $f(z) = \dfrac{z-1}{z+1}$, $z \neq -1$.

$$f(z) = \frac{(z-1)(\bar{z}+1)}{|z+1|^2} = \frac{z\bar{z} + z - \bar{z} - 1}{|x+1+iy|^2} = \frac{x^2+y^2+i2y-1}{(x+1)^2+y^2}$$

$$u = \frac{x^2+y^2-1}{(x+1)^2+y^2} \qquad v = \frac{2y}{(x+1)^2+y^2}$$

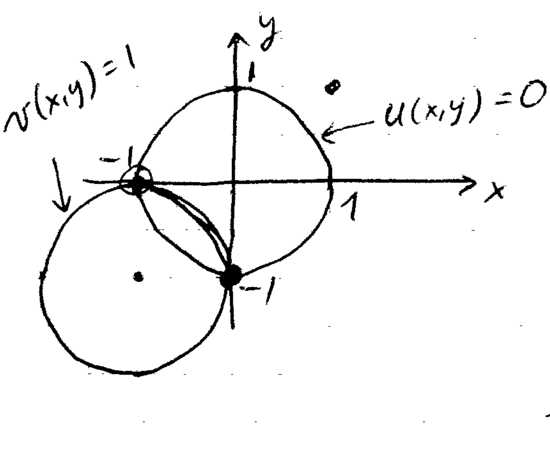
a) $u(x,y) = C_1 \iff x^2+y^2-1 = C_1(x^2+2x+1+y^2)$

$(1-C_1)(x^2+y^2) - 2C_1 x = 1+C_1$ (take $C_1 = +0$)

$x^2+y^2 = 1$ circle for $C_1 = 0$. (P.S. it is a circle for $C_1 \neq 1$.)

b) $v(x,y) = C_2 \iff x^2+2x+1+y^2 + \frac{2}{C_2} y = 0$.

take $C_2 = 1 \Rightarrow (x+1)^2 + (y+1)^2 = 1$ - circle.



Two level curves intersect at points $(-1,0)$ and $(0,-1)$ at right angle.

Note that point $(-1,0)$ is not in the domain of $f(z)$ thus we ignore this point. For point $(0,-1)$

$f'(-i) = u'(0,-1) + iv'(0,-1) = -i$

(see next page for calculations)

3 (cont-d)

$$u = \frac{x^2+y^2-1}{x^2+2x+y^2+1} \Rightarrow u_x = \frac{2x\cdot(x^2+2x+y^2+1)-(2x+2)(x^2+y^2-1)}{(x^2+2x+y^2+1)^2} =$$

$$= \frac{2x^2+4x\cancel{-2x^2}-2y^2+2}{(x^2+2x+y^2+1)^2} = \frac{2(x^2-y^2+2x+1)}{(x^2+2x+y^2+1)^2} = v_y$$

$$u_y = \frac{2y(x^2+2x+y^2+1)-(x^2+y^2-1)\cdot 2y}{(x^2+2x+y^2+1)^2} = \frac{4xy+4y}{(x^2+2x+y^2+1)^2} = -v_x$$

$$f'(-i)=f'(0,-1) = 0+i\frac{(-4)}{4} = -i \neq 0.$$

This illustrates the Lemma at point $(0,-1)$ in $\mathbb{C}$. and level curves $u(x,y)=0$ $v(x,y)=1$.

4. Th $f(z)$ analytic in $\mathbb{C}$. Then $\overline{f(z)} = -f(\bar z) \Leftrightarrow f(x) \in i\mathbb{R}$.

<u>Proof</u>: $(\Rightarrow)$ $\overline{f(z)} = -f(\bar z)$

$u(x,y) - iv(x,y) = -u(x,-y) - iv(x,-y)$.

take $y=0$ $\quad u(x,0) - iv(x,0) = -u(x,0) - iv(x,0)$

Thus, $u(x,0) = -u(x,0) \Rightarrow u(x,0) = 0$.

therefore, $f(x) = iv(x,0) \in i\mathbb{R}$.

$(\Leftarrow)$ We know that $f(x) = iv(x,0)$ and $u(x,0)=0$

Introduce $F(z) = -\overline{f(\bar z)} = -u(x,-y) + iv(x,-y)$

$\equiv U(x,y) + iV(x,y)$

Let $t=-y$; $u_x(x,t) = v_t(x,t)$ and $u_t(x,t) = -v_x(x,t)$

$U(x,y) = -u(x,t)$; $U_x = -u_x = -v_t = V_y(x,y)$

$V(x,y) = v(x,t)$; $V_x = v_x = -u_t = -U_y$

Thus $F(z)$ is analytic in $\mathbb{C}$.

$F(x,0) = U(x,0) + iV(x,0) = iv(x,0) = f(x,0)$

Thus these two function coincide on line $y=0$.

Because they both are analytic ~~they~~ $F(z) = f(z)$ in $\mathbb{C}$

Recall, $F(z) = -\overline{f(\bar z)}$. Thus $\overline{f(z)} = -f(\bar z)$ in $\mathbb{C}$.

5 (a) $f(z) = z^8$

(1) $f(\bar{z}) = (\bar{z})^8 = \overline{z^8} = \overline{f(z)}$ | use $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(2) $f(x) = (x+i\,0)^8 = x^8 \in \mathbb{R}$
function is analytic (polynomial) and $f(x)$ is real
Thus $f(\bar{z}) = \overline{f(z)}$. by Reflexion principle

(b) $f(z) = iz^7$

(1) $f(\bar{z}) = i\bar{z}^7 = i\overline{z^7} = -\left(\overline{iz^7}\right) = -\overline{f(z)}$

(2) $f(x) = ix^7 \in i\mathbb{R}$
function is analytic and $f(x)$ is pure imaginary. Thus $f(\bar{z}) = -\overline{f(z)}$ by Anty-Reflexion principle

(c) $f(z) = \cos(z)$

(1) $f(\bar{z}) = \cos(\bar{z}) = \dfrac{e^{i\bar{z}} + e^{-i\bar{z}}}{2} =$

$= \frac{1}{2}\left(e^{y}(\cos x + i\sin x) + e^{-y}(\cos x - i\sin x)\right)$

$f(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right)$

$\overline{f(z)} = \frac{1}{2}\left(e^{-y}(\cos x - i\sin x) + e^{y}(\cos x + i\sin x)\right) = f(\bar{z})$.

(2) $f(0) = \cos x \in \mathbb{R}$; $f(z)$ is analytic in $\mathbb{C}$.
thus by reflection principle $f(\bar{z}) = \overline{f(z)}$.

(d) $f(z) = \cos iz = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{z} + e^{-z}}{2} = \cosh z$.

(1) $f(\bar{z}) = \dfrac{e^{\bar{z}} + e^{-\bar{z}}}{2} = \dfrac{\overline{e^{z}} + \overline{e^{-z}}}{2} = \overline{f(z)}$.

(2) $f(x) = \cosh(x) \in \mathbb{R}$, and analytic.
$\Rightarrow f(\bar{z}) = \overline{f(z)}$. (linear combin. of exponents).

6. a) $\exp\left(\frac{3+i\pi}{6}\right) = \exp\left(\frac{1}{2}\right)\cdot\exp\left(\frac{i\pi}{6}\right) = e^{1/2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$= \frac{e^{1/2}}{2}\left(\sqrt{3}+i\right).$

b) $\frac{d}{dz}\exp(z^5)\Big|_{z=2i} = 5z^4\exp(z^5)\Big|_{z=2i} = 80(\cos 32 + i\sin 32)$

c) $\log(i^2) = \log(-1) = \ln(1) + i(\pi+2\pi k) = i(\pi+2\pi k)$
$k\in\mathbb{Z}.$

d) $\log(i^{1/2}) = \log\left(e^{\frac{i\pi/2+2\pi k}{2}}\right) = i\left(\frac{\pi}{4}+\pi k\right),\quad k\in\mathbb{Z}$

e) $\log(e) = \ln e + i\cdot 2\pi k = 1 + i\cdot 2\pi k,\ k\in\mathbb{Z}.$

f) $\log(1+\sqrt{3}i) = \log\left(2\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right) = \ln 2 + \log e^{i\left(\frac{\pi}{3}+2\pi k\right)}$

$= \ln 2 + i\left(\frac{\pi}{3}+2\pi k\right),\quad k\in\mathbb{Z}.$

7. a) $e^z = e^x(\cos y + i\sin y)$

e^z is pure imaginary $\iff \cos y = 0 \iff \boxed{y = \frac{\pi}{2}+\pi k.}$

b) $e^{i\bar{z}} = \overline{e^{iz}} \iff e^{+y}(\cos x + i\sin x) = e^{-y}(\cos x - i\sin x)$

$\iff e^{y} = e^{-y}$ and $\sin x = -\sin x \iff \boxed{y=0,\ x=\pi k}$

8 (a) $\log(2z) = i\pi/2$

$2z = e^{i\pi/2} = i \implies \boxed{z = i/2}$

(b) $\tan(z/2) = 2i$; Denote $a = z/2.$

$\tan(a) = 2i \implies \sin a = (2i)\cos a \implies$

$\frac{e^{ia}-e^{-ia}}{2i} = 2i\left(\frac{e^{ia}+e^{-ia}}{2}\right)$; $e^{ia}(1+2) = e^{-ia}(2-1)$

8(b) (cont'd) Denote $w = e^{ia}$

$$3w = \frac{1}{w} \Rightarrow w = \sqrt{-\tfrac{1}{3}} = \frac{\pm i}{\sqrt{3}} \Rightarrow ia = \log\sqrt{-\tfrac{1}{3}}$$

$$a = -i\left(\ln\sqrt{\tfrac{1}{3}} + i\left(\cancel{2}\pi k + \tfrac{\pi}{2}\right)\right) = \frac{i}{2}\ln 3 + \cancel{2}\pi k + \frac{\pi}{2} = \frac{z}{2}$$

$$\Rightarrow z = i\ln 3 + 2\pi k + \pi$$

$$\boxed{z = i\ln 3 + (2k+1)\pi}, \quad k \in \mathbb{Z}.$$

(c) $\sinh z = 2i$

$$e^{z} - e^{-z} = 4i \; ; \; (e^{z})^2 - 4ie^{z} - 1 = 0$$

Denote $w = e^{z}$; $w^2 - 4iw - 1 = 0$

$$w = 2i \pm \sqrt{-3} = i(2 \pm \sqrt{3})$$

$$z = \log(i(2\pm\sqrt{3})) = \ln(2\pm\sqrt{3}) + i\left(\tfrac{\pi}{2} + 2\pi k\right)$$

$$\boxed{z = \ln(2\pm\sqrt{3}) + i\left(\tfrac{\pi}{2} + 2\pi k\right)}, \quad k \in \mathbb{Z}$$

(d) $\sinh(\cos z) = 0$

$$e^{\cos z} - e^{-\cos z} = 0 \; ; \; (e^{\cos z})^2 = 1$$

$$2\cos z = 0 \; ; \; \boxed{z = \tfrac{\pi}{2} + \pi k} \quad k \in \mathbb{Z}$$

$$\Updownarrow$$

$$e^{iz} + e^{-iz} = 0$$

$$e^{iz\cdot 2} = -1$$

$$2iz = i(\pi + 2\pi k)$$