Math 3210 **Answers** **Assignment #9**

1. Write down the most important statements you know about series representation for analitic functions. Type your favorite formula from the course in the Discussion (Message) Board at my web page.

   http://www.math.mun.ca/ mkondra/okno/index.php Please, select Math 3210

2. Let $\sum_{n=0}^{\infty} z_n = S$ and $\sum_{n=0}^{\infty} w_n = T$. Find $\sum_{n=0}^{\infty} \bar{z}_n$, $\sum_{n=0}^{\infty} aw_n$, $\sum_{n=0}^{\infty}(iz_n + 3\bar{w}_n)$.

   Answer: $\sum_{n=0}^{\infty} \bar{z}_n = \bar{S}$, $\sum_{n=0}^{\infty} aw_n = aT$, $\sum_{n=0}^{\infty}(iz_n + 3\bar{w}_n) = iS + 3\bar{T}$.

3. True of false?

$$\sum_{n=1}^{\infty} r^n \sin(nt) = \frac{r \sin t}{1 - 2r \cos t + r^2}.$$

   Answer: The series $\sum_{n=1}^{\infty} r^n \sin(nt)$ converges to $\operatorname{Im} \dfrac{1}{1 - re^{it}} = \dfrac{r \sin t}{1 - 2r \cos t + r^2}$ only for $|r| < 1$.

   So the statement as it is, is FALSE, but with the restriction $|r| < 1$ it is true.

4. Find series expantion in terms of integer powers of $z-a$ for given function $f(z)$ and complex number a. Find domain of convergence. Which of them are power series?

   (a) $f(z) = e^z$, $a = \pi i$.

   Answer: $f(z) = e^z = e^{\pi i} e^{z - \pi i} = -\sum_{n=0}^{\infty} \dfrac{(z - \pi i)^n}{n!}$

   Converges in the whole complex plane. It is a power series.

   (b) $f(z) = z^6 \sinh(z^{-3})$, $a = 0$.

   Answer: $z^6 \sinh(z^{-3}) = \sum_{n=0}^{\infty} \dfrac{z^{3-6n}}{(2n+1)!}$

   Does not exist at $z = 0$. Converges at any other point. It is not a power series.

   (c) $f(z) = \sin(z - \pi/2)$, $a = 0$.

   Answer: $\sin(z - \pi/2) = -\cos z = -\sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n}}{(2n)!}$

   Converges in the whole complex plane. It is a power series.

   (d) $f(z) = \cosh z$, $a = i\pi/2$.

   Answer:

$$\cosh z = \frac{e^z + e^{-z}}{2} = \frac{ie^{z - i\pi/2} - ie^{-(z - i\pi/2)}}{2} = \frac{i}{2} \sum_{n=0}^{\infty} (z - i\pi/2)^n \frac{1 - (-1)^n}{n!} = i \sum_{n=0}^{\infty} \frac{(z - i\pi/2)^{2n+1}}{(2n+1)!}$$

   Converges in the whole complex plane. It is a power series.

   (e) $f(z) = (5z - z^2)^{-1}$, $a = 0$.

   Answer: $(5z - z^2)^{-1} = \dfrac{1}{5z} \dfrac{1}{(1 - z/5)} = \sum_{n=-1}^{\infty} \dfrac{z^n}{5^{n+2}}$ for $0 < |z| < 5$;

   $(5z - z^2)^{-1} = -\dfrac{1}{z^2} \dfrac{1}{(1 - 5/z)} = -\sum_{n=0}^{\infty} \dfrac{5^n}{z^{n-2}}$ for $|z| > 5$;

   In both cases it is not a power series.

5. Find Laurent series representation centered at zero in each domain of analitycity for the function

$$f(z) = \frac{2z-3}{(z-5)(z+2)}.$$

Answer: $f(z) = \dfrac{2z-3}{(z-5)(z+2)} = \dfrac{1}{z-5} + \dfrac{1}{z+2}$. Thus

$f(z) = -\dfrac{1}{5}\sum_{n=0}^{\infty} \dfrac{z^n}{5^n} + \dfrac{1}{2}\sum_{n=0}^{\infty} \dfrac{(-1)^n z^n}{2^n}$ for $|z| < 2$;

$f(z) = -\dfrac{1}{5}\sum_{n=0}^{\infty} \dfrac{z^n}{5^n} + \dfrac{1}{z}\sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n}{z^n}$ for $2 < |z| < 5$;

$f(z) = \dfrac{1}{z}\sum_{n=0}^{\infty} \dfrac{5^n}{z^n} + \dfrac{1}{z}\sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n}{z^n}$ for $|z| > 5$;

6. Show that function $f(z) = (1+z^2)^{-1}$, $z \neq \pm i$ is an analytic continuation of $g(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ from the disk $|z| < 1$ to the complex plane without points $\pm i$.

   Answer: Note that $f(z) = (1+z^2)^{-1}$ is an analytic function in the complex plane without points $\pm i$ since it is differentiable at each point. Also $g(z)$ is analytic in the disk $|z| < 1$ since it is given by a convergent power series there. The domains of analyticity of two functions do intersect and $f = g$ on the intersection, thus f is an analytic continuation of g.

7. Consider function $f(z) = \dfrac{\cos z}{z^2 - (\pi/2)^2}$. This function has removable discontinuity at points $z = \pm\pi/2$. What values should be assigned at the poins to make the function continuous? Does the continuous function become entire? Explain.

   Answer: $\lim_{z\to\pm\pi/2} f(z) = -1/\pi$. Introduce function $g(z)$ which is equal to $-1/\pi$ at $z = \pm\pi/2$, and is equal to $f(z)$ otherwise. This function is continuous and is given by a convergent power series at each point in the complex plane. In particulat, at $z = \pm\pi/2$. Thus it is entire function.

8. Let z be a complex number. Show that

$$\exp\left(\frac{z}{2}\left(w - \frac{1}{w}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(z) w^n, \quad 0 < |w| < \infty$$

   where $J_n(z)$ are Bessel functions of the first kind, namely

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(nt - z\sin t)}\, dt, \quad n = 0, \pm 1, \pm 2, ...$$

   Hint: Use integral formula for coefficients in Laurent series and unit contour of integration $w = e^{it}$.

   Answer: Denote $f(w) = \exp\left(\dfrac{z}{2}\left(w - \dfrac{1}{w}\right)\right)$. By formula on page 191, $f(w) = \sum_{n=-\infty}^{\infty} c_n w^n$, where $c_n = \dfrac{1}{2\pi i}\oint \dfrac{f(w)dw}{w^{n+1}}$. Take $w = e^{it}$, $-\pi \le t \le \pi$. Then $dw = iw$ and $w + 1/w = 2i\sin t$. Substitute all this to get $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(nt - z\sin t)}\, dt = J_n$.

Extra point problem. Show that Bessel function $J_n(z)$ solves the Bessel equation

$$z^2y'' + zy' + (z^2 - n^2)y = 0.$$

Show that Bessel equation appears when you solve the following Helmholz equation

$$U_{xx} + U_{yy} = -k^2U, \quad k = \text{const}$$

by the separation of variables method in polar coordinates.