Math 3210 **Answers** **Assignment #8**

1. Outline a proof of Gauss' Mean Value Theorem.

   This question was not marked. The proof is in the book and lecture notes. Please, come to discuss if you have questions.

2. Explain how Maximum Modulus Principle follows from Gauss' Mean Value Theorem.

   This question was not marked. The explanation is in the book and lecture notes. Please, come to discuss if you have questions.

3. Evaluate the contour integral.

   Note that the sign of the result will depend on the orientation of the contour. Answers given below are for the positive (counterclockwise) orientation of the contour. Otherise they would have an opposite sign.

   (a) $\oint_{|z|=3} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\, dz = 4\pi i.$
   Hint: Use partial fractions and then Cauchy's Integral Formula.

   (b) $\oint_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{8\pi i}{3e^2}$, where C is a rectangle with vertices at $\pm 2 \pm i$.

   (c) $\oint_{|z|=1} \frac{e^{2z}}{(z+1+i)^4}\, dz = 0.$

   (d) $\oint_{|z|=2} \frac{z^5}{(z+i)^3}\, dz = -20\pi$

   (e) $\oint_{|z|=3} \frac{z^3}{(z+2i)^5}\, dz = 0$

4. Let m, n be natural numbers, R be a positive real number and w be a complex number such that $|w| \neq R$. Evaluate

$$\oint_{|z|=R} \frac{z^n}{(z+w)^m}\, dz$$

   Answer: $\frac{2\pi i n(n-1)...(n-m+2)(-w)^{n-m+1}}{(m-1)!}$ for $|w| < R$ and $m \leq n+1$, otherwise the integral is 0.

5. Let $F(z)$ be entire function and $|F(z)| \leq a|z|$ for some real positive number a and for all $z \in C$. Show that $F(z) = bz$, where b is a complex number.

   Hint: show that $F''(z) = 0$.

   Solution. Use Lemma p.159 for function analytic inside the contour: $F''(z) = \frac{1}{\pi i} \int_{|z-z_0|=R} \frac{F(s)\, ds}{(s-z)^3}$.
   Then the estimate gives $|F''(z)| \leq \frac{(2\pi R)a(|z_0|+R)}{\pi R^3} \to 0$ as $R \to \infty$. So, $F''(z) = 0$ and thus $F(z) = A + Bz$. Since $|F(z)| \leq a|z|$, $A = 0$.

6. Let $G(z)$ be entire function and Re$G(z) < a$. Show that $G(z)$ =const.

   Hint: Show that $\exp(G(z))$ is const.

   Solution. Note that $|\exp(G(z))| = |\exp(\mathrm{Re}G(z))| < \exp(a)$. An analytic bounded function is a constant so $\exp(G(z))$ is contant, and thus $G(z)$ is constant.

7. Prove the *Minimum Modulus Principle*: Let $f(z)$ be analytic continuous function in a closed bounded region D. Assume $f(z) \neq 0$ in D. Then $|f(z)|$ reaches its minimum value on the boundary of D, but not in the interior of D.

   Solution. Consider $g(z) = \dfrac{1}{f(z)}$. Since $f(z) \neq 0$, $g(z)$ is analitic in D, and reaches its maximum at the boundary. Thus f reaches its minimum at the same point, which is on the boundary.

8. Evaluate by Cauchy's integral formula

$$\oint_{|z|=2004} \frac{e^z}{z}\, dz,$$

   and use this result to evaluate

$$\int_0^\pi e^{2004\cos t} \cos(2004 \sin t)\, dt.$$

   Solution.

$$\oint_{|z|=2004} \frac{e^z}{z}\, dz = 2\pi i$$

   Now take $z(t) = 2004e^{it}$, $-\pi \leq t \leq \pi$ and evaluate

$$2\pi i = \oint_{|z|=2004} \frac{e^z}{z}\, dz = \int_{-\pi}^{\pi} e^{2004(\cos t + i \sin t)} i\, dt = \int_{-\pi}^{\pi} e^{2004\cos t}(\cos(2004\sin t) + i\sin(2004\sin t))\, i\, dt.$$

   This gives

$$\int_0^\pi e^{2004\cos t} \cos(2004 \sin t)\, dt = \pi.$$