Math 3210 **Answers** **Assignment #5**

1. Find *all values* and *the principal value* of the complex expression

   (a) $(-2)^{2/\pi} = e^{2/\pi(\log(-2))} = e^{2/\pi(\ln 2 + i\pi + 2\pi ni)}$, $P.V. = 4^{1/\pi}e^{2i}$

   (b) $(1+i)^{i-1} = e^{(i-1)(\ln\sqrt{2} + i\pi/4 + 2\pi ni)}$

   (c) $(1-i\sqrt{3})^{5/2} = e^{5/2(\ln 2 - \pi i/3 + 2\pi ni)}$, $P.V. = 2^{5/2}e^{-5\pi i/6}$

   (d) This little poem is dedicated to the days when all formulae were written in words:
   i to the i times i to the i
   times e to the π – please simplify.
   $i^i i^i e^\pi = e^{2\pi m}$. $P.V. = 1$

2. Given $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$, $\cosh z = \dfrac{e^z + e^{-z}}{2}$ $\tanh z = \dfrac{\sinh z}{\cosh z}$, verify that

   (a) $\sin(z+w) = \sin z \cos w + \cos z \sin w$
   $\sin z \cos w + \cos z \sin w = \frac{(e^{iz}-e^{-iz})}{2i}\frac{(e^{iw}+e^{-iw})}{2} + \frac{(e^{iz}+e^{-iz})}{2}\frac{(e^{iw}-e^{-iw})}{2i} = \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} = \sin(z+w).$

   (b) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

   (c) $\sin(iz) = i\sinh z$

   (d) $(\tanh z)' = \text{sech}^2 z$

   (e) $(\tanh z)' = -\text{sech} z \tanh z$

3. Show that $2\tanh^{-1}(e^{i\theta}) = \log(i\cot(\theta/2))$.

$$\tanh^{-1}(e^{i\theta}) = \frac{1}{2}\log\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) = \frac{1}{2}\log\left(\frac{e^{-i\theta/2}+e^{i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}}\right) = \frac{1}{2}\log(i\cot\theta/2).$$

4. Find derivative $w'(t)$ of the complex valued function w w.r.t. real variable t.

   (a) $w(t) = \cosh(\sin(3t) + it^2)$
   $w'(t) = (3\cot(3t) + i2t)\sinh(\sin(3t) + it^2)$

   (b) $w(t) = (u(t) + iv(t))^n$, $n = 1, 2, 3...$
   $w'(t) = n(u(t) + iv(t))^{n-1}(u'(t) + iv'(t))$

5. (a) Evaluate integral $\int_0^\pi e^{(a+ib)t}dt$, where a, b are real numbers and t is real variable.

$$\int_0^\pi e^{(a+ib)t}dt = \frac{e^{(a+ib)\pi} - 1}{a+ib}$$

   (b) Use result from (a) to find $\int_0^\pi e^{at}\sin nt$, where a is real and n is a natural number.

$$\int_0^\pi e^{at}\sin(nt)dt = \text{Im}\frac{e^{(a+in)\pi} - 1}{a+in} = \frac{((-1)^n e^{a\pi} - 1)n}{a^2+n^2}$$

   (c) Confirm your result in (b) using integration by parts.

6. Envaluate improper integral

(a) $\displaystyle\int_1^\infty \frac{(t+2i)^2}{t^4}\,dt = -\frac{1}{3} + 2i$

(b) $\displaystyle\int_0^\infty \frac{200+300i}{\sqrt{t}(1+t)}\,dt = 200\pi + i300\pi$

(c) $\displaystyle\int_{-\infty}^\infty \frac{t}{1+t^4} + \frac{i}{1+t^2}\,dt = i\pi$

7. **Extra Points Problem** Find simple algebraic condition for real numbers k such that the principal value of the expression $(ik)^{ik}$ is real? Give few (approximate) numerical values of such numbers k.