Math 3210 - Assignment #9

1. **Taylor Series**: If $f(z)$ is analytic throughout a disk $|z-z_0|<R_0$, then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad (|z-z_0|<R_0)$$

Laurent Series: If $f(z)$ is analytic throughout an annular domain $R_1<|z-z_0|<R_2$, and if C is a positively oriented simple closed contour around z_0 and lying in that domain, then at each point in the domain, $f(z)$ has the series representation $f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$ where $C_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$, $n\in\mathbb{Z}$.

- If a Taylor series at z_0 converges when $z=z_1$ $(z_1\neq z_0)$, then it is absolutely convergent at each point z in the open disk $|z-z_0|<R_1$ where $R_1=|z_1-z_0|$.
- If z_1 is a point inside the circle of convergence $|z-z_0|=R$ of a power series at z_0, then that series must be uniformly convergent in the closed disk $|z-z_0|\leq R_1$, where $R_1=|z_1-z_0|$.

- A power series represents a continuous function at each point inside its circle of convergence.
- In fact, if $S(z)$ is the sum of a power (Taylor) series, it is analytic at each point z interior to the circle of convergence of the series it represents.
- $S(z)$ can be differentiated term by term.

(2.) $\overline{S}\cdot\overline{T} = \left(\sum_{n=0}^{\infty}\overline{b_n}\,\overline{z_0}^{\,n}\right)\left(\sum_{n=0}^{\infty}\overline{c_n}\,\overline{z_0}^{\,n}\right) = \left(\overline{b_0}+\overline{b_1}\,\overline{z_0}+\overline{b_2}\,\overline{z_0}^{\,2}+\cdots\right)\left(\overline{c_0}+\overline{c_1}\,\overline{z_0}+\overline{c_2}\,\overline{z_0}^{\,2}+\cdots\right)$

$$= \overline{b_0}\,\overline{c_0} + (\overline{b_0}\,\overline{c_1}+\overline{b_1}\,\overline{c_0})\overline{z_0} + (\overline{b_0}\,\overline{c_2}+\overline{b_1}\,\overline{c_1}+\overline{b_2}\,\overline{c_0})\overline{z_0}^{\,2}+\cdots$$

$$= \left(\sum_{k=0}^{0}\overline{b_k}\,\overline{c_{0-k}}\right) + \left(\sum_{k=0}^{1}\overline{b_k}\,\overline{c_{1-k}}\right)\overline{z_0} + \left(\sum_{k=0}^{2}\overline{b_k}\,\overline{c_{2-k}}\right)\overline{z_0}^{\,2}+\cdots$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\overline{b_k}\,\overline{c_{n-k}}\right)\overline{z_0}^{\,n}$$

i.e. the sum is the Cauchy product of the two series represented by $\overline{S}$ and $\overline{T}$.

Thus the sum is equal to $\overline{(ST)}$.

(3) $$f(z) = \begin{cases} \dfrac{\sin z}{z^2-\pi^2} & \text{for } z \neq \pm\pi \\ -\dfrac{1}{2\pi} & \text{for } z = \pm\pi \end{cases}$$

We need to check whether it is entire function.

First, we check if it is continuous at $z=\pm\pi$.

$$\lim_{z\to\pi}\frac{\sin z}{z^2-\pi^2} = -\frac{1}{2\pi}$$ (Can use L'Hospital Rule or series expansion at $z_0=\pi$).

$$\lim_{z\to-\pi}\frac{\sin z}{z^2-\pi} = \frac{1}{2\pi} \neq -\frac{1}{2\pi}$$ thus the function

is NOT continuous at $z=-\pi$ and

is NOT entire.

4a) $f(z) = \frac{z^2}{2}\sum_{n=0}^{\infty}\frac{(-z^6)^n}{2^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-z^6}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n z^{6n+2}}{2^{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)z^{6n}}{2^{n+1}}$

This is a power series valid for $|z| < 2^{1/6}$

$$f(z) = \frac{z^2+1}{z^6+2} = \frac{z^2+1}{z^6(1+(\frac{2}{z^6}))} = \frac{z^2+1}{z^6}\cdot\frac{1}{1+(2/z^6)}$$

$$= \frac{z^2+1}{z^6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z^6}\right)^n = \sum_{n=0}^{\infty}(-2)^n z^{-6n-4} + \sum_{n=0}^{\infty}(-2)^n z^{-6n-6}$$

This is not a power series and has domain of convergence $|z| > 2^{1/6}$.

b) $\cos z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$, $|z| < \infty$

and, $\cosh z = \cos(iz) = \sum_{n=0}^{\infty}(-1)^n\frac{(iz)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(i)^{2n}z^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$, $|z| < \infty$

So, $z^5\cosh(z^{-4}) = z^5\sum_{n=0}^{\infty}\frac{(z^{-4})^{2n}}{(2n)!}$ $|z^{-4}| < \infty$

$= \sum_{n=0}^{\infty}\frac{z^{5-8n}}{(2n)!}$ $|z| > 0$ ✓ This is not a power series. ✓

c) $\sin(z) = \sin(z - \frac{\pi}{2} + \frac{\pi}{2}) = \cos(z - \frac{\pi}{2})$

$\sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^n (z-\frac{\pi}{2})^{2n}}{(2n)!}$ for all $z \in \mathbb{C}$.

This is a power series.

d) $\sinh(z) = -i\cosh(z + \frac{i\pi}{2}) = -i\sum_{n=0}^{\infty}\frac{(z+\frac{i\pi}{2})^{2n}}{(2n)!}$

for all $z \in \mathbb{C}$. This is a power series.

e) $f(z) = \frac{1}{5z+z^2} = \frac{1}{5z}\cdot\frac{1}{1+\frac{z}{5}} = \frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{5}\right)^n$ for $\left|\frac{z}{5}\right| < 1$

$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}z^{n-1}$, $0 < |z| < 5$ This is not a power series (it contains $\frac{1}{5}z^{-1}$). ✓

$f(z) = \frac{1}{z^2}\cdot\frac{1}{\frac{5}{z}+1} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{z^n} = \sum_{n=0}^{\infty}(-1)^n 5^n z^{-(n+2)}$

for $|z| > 5$. This is not a power series.

could use a simpler form

⑤ $f(z) = \dfrac{z^2+8z+2}{z^2+7z+6} = \dfrac{z^2+8z+2}{(z+6)(z+1)} = \dfrac{z+8}{z+6} - \dfrac{1}{z+1} = 2 + \dfrac{2}{z+6} - \dfrac{1}{z+1}$

Case 1 $|z| < 1$

$$f(z) = \frac{z+8}{6\left(1-\left(-\frac{z}{6}\right)\right)} - \frac{1}{1-(-z)} = \frac{z+8}{6}\sum_{n=0}^{\infty}\left(\frac{-z}{6}\right)^n - \sum_{n=0}^{\infty}(-z)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{6^{n+1}}z^{n+1} + 8\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{6^{n+1}} - \sum_{n=0}^{\infty}(-1)^n z^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{6^{n+1}}z^{n+1} + \sum_{n=0}^{\infty}\left(\frac{8}{6^{n+1}}-1\right)(-1)^n z^n$$

Case 2 $1 < |z| < 6$

$$f(z) = \frac{z+8}{6\left(1-\left(-\frac{z}{6}\right)\right)} - \frac{1}{z\left(1-\left(-\frac{1}{z}\right)\right)} = \frac{z+8}{6}\sum_{n=0}^{\infty}\left(\frac{-z}{6}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{6^{n+1}} + 8\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{6^{n+1}} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{z^{n+1}}$$ ✓

Case 3 $|z| > 6$

$$f(z) = \frac{z+8}{z\left(1-\left(\frac{-6}{z}\right)\right)} - \frac{1}{z\left(1-\left(-\frac{1}{z}\right)\right)} = \frac{z+8}{z}\sum_{n=0}^{\infty}\left(\frac{-6}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 6^n}{z^n} + 8\sum_{n=0}^{\infty}\frac{(-1)^n 6^n}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n 6^n}{z^n} + \sum_{n=0}^{\infty}\left(8\cdot 6^n - 1\right)(-1)^n\frac{1}{z^{n+1}}$$ ✓

⑥ $f(z) = \begin{cases} \dfrac{\mathrm{Log}(z)}{z-1}, & z \neq 1 \\ 1, & z = 1 \end{cases}$

Note that $\dfrac{\mathrm{Log}(z)}{z-1} = \dfrac{1}{z-1}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^{n+1}}{n+1}$ for $-\pi < \mathrm{Arg}(z) < \pi$ and $0 < |z| < \infty$, $z \neq 1$

$= 1 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(z-1)^n}{n+1}$ for $-\pi < \mathrm{Arg}(z) < \pi$ and $0 < |z| < \infty$ (even $z = 1$)

Thus, $f(z)$ can be represented as a power series that converges to 1 at $z=1$ and $\dfrac{\mathrm{Log}(z)}{z-1}$ for all other z $(z \neq 0)$, $-\pi < \mathrm{Arg}\,z < \pi$.

i.e. $f(z)$ is analytic on the given domain

⑦ $J_n(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int+ia\sin t}\,dt,\ n \in \mathbb{Z}$

C is given by $z = e^{it},\ -\pi < t < \pi$ and $f(z) = e^{b\left(\frac{z^2-1}{z}\right)} = \exp\left(bz - bz^{-1}\right)$

$$C_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi \cancel{i}}\int_{-\pi}^{\pi} \frac{\exp\left[be^{it} - be^{-it}\right]}{\left(e^{it}\right)^{n+1}} \cdot \cancel{i}\, e^{it}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp\left[b\cos t + ib\sin t - b\cos t + ib\sin t\right]}{e^{int}}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int+2ib\sin t}\,dt = J_n(2b)$$

And so $f(z) = \sum_{n=-\infty}^{\infty} C_n z^n = \sum_{n=-\infty}^{\infty} J_n(2b)\, z^n$ for $0 < |z| < \infty$