MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Am I ready for Final? **Mathematics 2000** **Fall 2003**

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Final Exam will be on Wed Dec 17 from 3 pm until 5:30 pm in SN 4083

Please, have a picture ID with you.

There two major topics covered in the course and will be tested on final exam:

I. Sequences and series.

II. Functions of several variables.

I Sequences and series.

I.1 Sequences.

1. Sequence is a list of numbers $\{a_n\}$. The main question we ask about an infinite sequence is whether it is convergent or divergent. By definition a sequence is convergent if the limit of a_n as $n \to \infty$ is a number. For example, $\lim(1/n) = 0$. A sequence is divergent if the limit as $n \to \infty$ is $\pm\infty$ or does not exist. Examples are $a_n = n$ and $a_n = (-1)^n$.

2. If a sequence is increasing i.e. $a_{n+1} \geq a_n$ for all n, and it is bounded from above i.e. $a_n \leq M$ for all n for some number M then the sequence has limit $L \leq M$.

3. Likewise, If a sequence in decreasing i.e. $a_{n+1} \leq a_n$ for all n, and it is bounded from below i.e. $a_n \geq M$ for all n for some number M then the sequence has limit $L \geq M$.

4. To check if a sequence has some properties like being monotonic or bounded it could be useful to find a function $f(x)$ such that $a_n = f(n)$ first, and then check properties of the function. For example, $f(x) = 1/x$ is decreasing for $x \geq 1$ because its derivative is negative on this interval. Thus $a_n = 1/n$, $n \geq 1$ is decreasing as well.

5. If general term of a sequence is a fraction e.g. $a_n = \dfrac{n^2}{n^3+1}$ of type ∞/∞ as $n \to \infty$ then try one of the techniques to find the limit : divide top and bottom by the highest power of n or use L'Hospital rule.

I.2 Number series.

1. If you sum up all terms of a sequence you will get a series $\sum_{n=1}^{\infty} a_n$. Sometimes the result of the summation is a number. Then we say that the series is convergent. By definition, the sum is the limit (as $N \to \infty$) of the sequence of partial sums $s_N = \sum_{n=1}^{N} a_n$.

2. An important family of series are geometric series. $\sum_{k=0}^{\infty} r^k$. If $-1 < r < 1$ the result of the summation is $1/(1-r)$. Otherwise it diverges.

3. Another important family is p-series $\sum_{n=1}^{\infty} n^{-p}$. If $p > 1$ then the series is convergent. For $p \leq 1$ the series is divergent.

Remember: when $p = 1$, series $\sum 1/n$ is divergent.

4. If limit as $n \to \infty$ of a sequence a_n is not 0 then corresponding series $\sum a_n$ is divergent.

5. If limit as $n \to \infty$ of a sequence a_n is 0 then corresponding series $\sum a_n$ maybe convergent, but maybe divergent. For example $\sum 1/n$ is divergent, but $\sum 1/n^2$ is convergent.

There are several tests to find out if a series is convergent.

6. You can try to compare given sequence to a geometric or p-series, i.e. use straight comparison or limit comparison test. The second one is used more often. It says that if $\lim_{n\to\infty} a_n/b_n$ is a number then series $\sum a_n$ and $\sum b_n$ either both converges or both diverges.

7. Another useful tests are the Ratio test and the Root test. In the Ratio test you calculate $L = \lim_{n\to\infty} |a_{n+1}/a_n|$. In the Root test you calculate $L = \lim_{n\to\infty} |a_n|^{1/n}$. In both tests $L < 1$ would mean convergence of the series, and $L > 1$ means divergence. Case $L = 1$ is inconclusive. You must try another test.

8. Integral test gives you positive answer for $\sum a_n$, where $a_n = f(n)$, if corresponding improper integral $\int_1^\infty f(x)dx$ is convergent. If you use this test do not forget to make sure that $f(x)$ is decreasing, positive and continuous on the interval of integration.

9. For an alternating series $\sum(-1)^n b_n$, where $b_n > 0$ try the A.S.T. To prove convergence you must show two things: b_n is decreasing and $b_n \to 0$ as $n \to \infty$.

10. Sometimes the series itself $\sum a_n$ is convergent, but the series of the absolute values $\sum |a_n|$ is divergent. Then we say that the series is conditionally convergent. All alternating p-series $\sum(-1)^n n^{-p}$ are conditionally convergent for $0 < p \leq 1$. They are absolutely convergent for $p > 1$.

I.3 Power series. Representation of a function as a power series.

1. Power series has a form $\sum_{n=0}^{\infty} a_n(x-c)^n$. Here c is a number called the center of the power series. If you fix value of x you will obtain a number series. Therefore, for each x you can say whether it is convergent or divergent.

2. It turns out that a power series always converges on an interval $x \in (A, B)$. The half of the length of the interval is called the radius of convergence $R = (B - A)/2$. The smallest possible interval of convergence has radius $R = 0$. It consists of only one point $x = c$, the center of the power series where it always converges, and the sum of the series is a_0.

The largest possible interval has radius $R = \infty$. It means that the power series converges for all values of x.

Often the interval of convergence is finite. You must take a special care about its end-points $x = A$ and $x = B$. One of the tests for number series would be applicable at each of the end points.

3. To find the interval of convergence you should apply the Ratio test to the power series and find L as a function of x. Then condition $L(x) < 1$ gives you restrictions for x.

4. Taylor gave recipes for finding the power series representation for functions. This gives us a powerful tool for approximations. Taylor series for $f(x)$ is a power series with coefficients $a_0 = f(c)$, $a_1 = f'(c)$, $a_2 = f''(c)/2!$, $a_3 = f'''(c)/3!$, etc

5. You can use Taylor polynomials to approximately find value of a function at a point (providing that the series converges at such point). There are techniques to estimate the error bound of the approximation (e.g. Taylor inequality).

Example: Taylor polinimial for $f(x) = \sin x$ centered at $c = \pi/3$ is

$$\sin(x) \approx \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \pi/3) - \frac{\sqrt{3}}{4}(x - \pi/3)^2.$$

This is first 3 terms approximation. You can use it to find an approximate value of $sin(\pi/5)$ by calculating $\sin(\pi/5) \approx \sqrt{3}/2 + (\pi/5 - \pi/3)/2 - \sqrt{3}(\pi/5 - \pi/3)^2/4$.

6. You can differentiate (integrate) a Taylor series term by term. As a result you will obtain new Taylor series which represents derivative (anti-derivative) of the function given by the initial Taylor series.

7. Taylor series centered at $c = 0$ are called Maclaurin series. Some important Maclaurin series are $e^x = \sum x^k/k!$, $\sin x = \sum(-1)^n x^{2n+1}/(2n+1)!$, $\cos x = \sum(-1)^n x^{2n}/(2n)!$, $\ln|1 + x| = \sum(-1)^{k+1}x^k/k$. First tree have infinite radius of convergence. The last one converges only for $-1 < x \leq 1$.

II. Functions of several variables and double integrals..

II.1 Functions of several variables

1. We will be mostly talking about functions of two variables $z = f(x, y)$. It is a rule to calculate number z given two numbers x and y. We will be refering to our knowledge about functions of one variables all the time.

2. Domain of $f(x, y)$ is the set of all pairs (x, y) for which the function makes sense. Geometrically, the domain is a region in the xy-plane.

For example, to evaluate $f(x, y) = (x - y)^{-1}$ we must avoid division by zero. Thus $x \neq y$. The domain is whole xy-plne excluding one line $y = x$.

3. Graph of $z = f(x, y)$ is a surface in 3D space. Some simple examples are:

$f(x, y) = 1$ is a horizontal plane $z = 1$.

$f(x, y) = x^2 + y^2$ is a paraboloid opened up with vertex at the origin.

$f(x, y) = x^2 - y^2$ is a hyperboloid, it has saddle point at the origin.

$f(x, y) = \sqrt{x^2 + y^2}$ is a cone opened up with vertex at the origin.

$f(x, y) = \sqrt{R^2 - x^2 - y^2}$ is the upper hemisphere with radius R and center at the origin.

4. Function $f(x, y)$ is continuous at a point (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. The limit must be the same along any possible path in the plane arrived at (a, b). If the limit depends upon the path we say that it does not exist. For example, $f(x, y) = \frac{x^3y^6}{x^6+y^{12}}$ does not exist (try $x = y^2$ and $x = 0$).

5. Partial derivatives $f_x(a, b)$ and $f_y(a, b)$ show the rate of change of $f(x, y)$ at point(a, b) in x and y directions respectively.

6. To find the rate of change of $f(x, y)$ along a curve $x = x(t)$, $y = y(t)$ in the plane you must use the chain rule $\frac{dF}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$. The result will depend only on t.

7. If a function is given implicitly by a relation $F(x, y, z) = 0$ you still can calculate partial derivatives z_x and z_y. For this differentiate the relation thinking that $z = z(x, y)$ and use the chain rule. You will obtain $F_x + F_z z_x = 0$ and $F_y + F_z z_y$.

Thus $z_x = -\frac{F_x}{F_z}$ and $z_y = -\frac{F_y}{F_z}$.

8. Function $f(x, y)$ may have critical points where both partial derivatives are equal to 0 of are undefined. A critical point may be a (local) maximum, a (local) minimum or a saddle point. To classify it you need to find second partial derivatives f_{xx}, f_{xy}, f_{yy} and evaluate them at the critical point. Then calculate $D = f_{xx}f_{yy} - f_{xy}^2$ at the critical point.

Maximum correspond to $D > 0$ and $f_{xx} < 0$,

Minimum correspond to $D > 0$ and $f_{xx} > 0$,

Saddle point correspond to $D < 0$.

If $D = 0$ the test is inconclusive.

II.2 Double integrals

1. Double integral of a function $f(x, y)$ over region D in the plane represents volume between the surface $z = f(x, y)$ restricted over region D and the xy-plane.

2. A region D in the xy-plane can be described in terms of inequalities. For example, a

rectangular region is given by $a \le x \le b$, $c \le y \le d$. An example of non-rectangular region would be $0 \le x \le 1$, $0 \le y \le \sqrt{x}$. Note that the same region can be given by a different couple of inequalities : $0 \le y \le 1$, $y^2 \le x \le 1$. If you consider a function $f(x, y)$ defined over this region, there are two ways to set up the double integral:

$$\int_0^1 \int_0^{\sqrt{x}} f(x, y) dy dx = \int_0^1 \int_{y^2}^1 f(x, y) dx dy.$$

The numerical result will be the same. Depending on the integrant $f(x, y)$ one integral can be easier to evaluate than another. You must take an advantage of this fact.

3. Polar coordinates (r, θ) are an alternative to the Rectangular coordinates (x, y). They are related via formulas $x = r \cos\theta$, $y = r \sin\theta$.

Polar coordinates are very useful to describe circular regions. For example, $0 \le r \le 2$, $0 \le \theta \le 2\pi$ describe the interior of circle of radius 2 with its centre at the origin, while $0 \le r \le 2$, $0 \le \theta \le \pi$ describe its half.

4. A function $f(x, y)$ given in rectangular coordinates can be rewritten in polar coordinates as $F(r, \theta) = f(r \cos\theta, r \sin\theta)$.

5. If you do change of variables from rectangular to polar in the double integral do not forget about EXTRA FACTOR r, called Jacobian. Example :

$$\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy^2 dx dy = \int_0^{2\pi} \int_0^2 (r \cos\theta)(r \sin\theta)^2 \, r \, dr d\theta$$

6. If 3D body can be described in terms of inequalities: $f(x, y) \le z \le g(x, y)$, where $(x, y) \in D$ then its volume can be calculated as double integral over region D.

$$V = \int \int_D g(x, y) - f(x, y) \, dA$$

Example:

$$\int_0^{2\pi} \int_0^1 (\sqrt{1 - r^2} - (r^2 - 1)) \, r \, dr d\theta$$

This integral represents the volume between the upper hemisphere $z = \sqrt{1 - r^2}$ or radius 1 and a paraboloid $z = r^2 - 1$ expressed in polar coordinates.

Please, make sure that you understand and **can do** all the problems from test 1 and test 2, as well as from your homework assignments and the last year final exam.

Send me an e-mail if you need something extra or if you have a question : mkondra@math.mun.ca

For integration and differentiation techniques also see my handout for Math-1001.

Good luck!