Due March 17 2004

Mathematics 2000: Assignment #7, Winter 2004

1. For the following functions find domain, sketch contor map, sketch the graph of the function, name it.

   (a) $F(x,y) = 1 - \sqrt{x^2 + 4y^2}$
   Domain R^2 (whole plane); level curves are ellipses; graph is a *cone* open down, lifted up by 1.

   (b) $F(x,y) = 1 + \sqrt{16 - x^2 - y^2}$
   Domain $16 - x^2 - y^2 \geq 0$ inside of circle of radius 4; level curves are circles, graph is the upper part of the *sphere* of radius 4, lifted up by 1.

   (c) $F(x,y) = 4x^2 + y^2 - 2$
   Domain R^2 (whole plane); level curves are ellipses; graph is a *(elliptic) parabolid* open up, shifted down by 2.

   (d) $F(x,y) = -\sqrt{1 - x^2 - 4y^2}$
   Domain $1 - x^2 - 4y^2 \geq 0$ inside of ellipse; level curves are ellipses, graph is the lower part of the *ellipsoid.* (ellipsoid is a sphere elongated in one direction).

   (e) $F(x,y) = 1 + x$
   Domain R^2 (whole plane); level curves are straight vertical lines $x = const$, graph is a *plane.*

   (f) $F(x,y) = \frac{x^2}{4} - 4y^2$
   Domain R^2 (whole plane); level curves are two lines crossing at the origin and hyporbolas for which the lines are asymptotas, graph is *hyperbolic paraboloid* (the one which has saddle point at the origin.

   P.S. In the lecture I accidently called it hyperboloid which is incorrect, its name is hyperbolic paraboloid. Sorry about this.

2. If the limit exists, find it. Otherwise show that it doesn't exist.

   (a) $\lim_{(x,y)\to(0,0)} 2\cos(x^3) - 3e^x = -1$

   (b) $\lim_{(x,y)\to(1,1)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \frac{\sin 2}{2}$

   (c) $\lim_{(x,y)\to(0,0)} \frac{\sin(2x^2+3y^2)}{2x^2+3y^2} = 1$

   (d) $\lim_{(x,y)\to(0,0)} \frac{2x^2-3y^2}{2x^2+3y^2}$
   The limit does not exist since it depends upon the path. Path $x = 0$ $y \to 0$ gives -1. Path $y = 0$ $x \to 0$ gives 1.

   (e) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$
   The limit does not exist since it depends upon the path. Path $x = 0$ $y \to 0$ gives 0. Path $y = x^2 \to 0$ gives 1/2.

(f) $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{\sqrt{x^2+y^2}} = 0$

Let $\sqrt{x^2+y^2} < \delta$. Then $|\dfrac{2xy}{\sqrt{x^2+y^2}} - 0| \leq \sqrt{x^2+y^2} < \delta = \epsilon$.

Thus for every $\epsilon > 0$ there is $\delta = \epsilon$ such that if $|\sqrt{x^2+y^2}| < \delta$ then $|f(x,y) - 0| < \epsilon$. So by definition the limit exists and is equal to 0.

3. Calculate the indicated partial derivatives.

(a) $u = x\sin(x+2y)$; u_x, u_{xy}, u_{xx}

$u_x = \sin(x+2y) + x\cos(x+2y)$,

$u_{xy} = -2x\sin(x+2y) + 2\cos(x+2y)$,

$u_{xx} = -x\sin(x+2y) + 2\cos(x+2y)$

(b) $z = \tan(x^2y) + \cot(xy^2)$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

$\dfrac{\partial z}{\partial x} = 2xy\sec^2(x^2y) - y^2\csc^2(xy^2)$;

$\dfrac{\partial z}{\partial y} = x^2\sec^2(x^2y) - 2xy\csc^2(xy^2)$

(c) $f = \arcsin(x^2y^{-3})$; f_x, f_{xx}, f_y

$f_x = \dfrac{2x}{y^3\sqrt{1-x^4/y^6}} = \dfrac{2x}{\sqrt{y^6-x^4}}$

$f_y = \dfrac{-3x^2}{y^4\sqrt{1-x^4/y^6}} = \dfrac{-3x^2}{y\sqrt{y^6-x^4}}$

(d) $z = \ln(\cos(x-2y))$; z_{yxx}

$z_{xxy} = 4\tan(x-2y)\sec^2(x-2y)$

4. Find all first partial derivatives of the functions given below.

(a) $f(x,y) = x^7y^9 + x^5y^{-5} + 100x$

$f_x = 7x^6y^9 + 5x^4y^{-5} + 100$, $f_y = 9x^7y^8 - 5x^5y^{-6}$

(b) $f(x,y) = \dfrac{x+y}{x-y}$

$f_x = \dfrac{-2y}{(x-y)^2}$

$f_y = \dfrac{2x}{(x-y)^2}$

(c) $f(x,y) = \ln(x^2y) = 2\ln x + \ln y$

$f_x = 2/x$, $f_y = 1/y$

(d) $f(x,y) = x^y$

$f_x = yx^{y-1}$, $f_y = x^y \ln x$

(e) $f(x,y,z) = x^{xz}$

$f_x = z(\ln x + 1)x^{xz}$, $f_y = 0$, $f_z = x^{1+xz}\ln x$

(f) $f(x,y,z,t) = x^{-1}t^3z^6\sqrt{y}$
$f_x = -t^3z^6y^{1/2}x^{-2}$
$f_y = (1/2)t^3z^6y^{-1/2}x^{-1}$
$f_z = 6t^3z^5y^{1/2}x^{-1}$
$f_t = 3t^2z^6y^{1/2}x^{-1}$

5. Confirm that Clairaut's theorem holds for the following functions. That is, calculate f_{xy} and f_{yx} and show that they are equal.

(a) $f(x,y) = \exp(x^4y^2 + x^3y^4)$
$f_{xy} = f_{yx} = (8x^3y + 12x^2y^3 + (2x^4y + 4y^2x^3)(4x^3y^2 + 3x^2y^4))\exp(x^4y^2 + x^3y^4)$

(b) $f(x,y) = \cos(xy^2 + x^3)$
$f_{xy} = f_{yx} = -2y\sin(xy^2 + x^3) - (2xy^3 + 6x^3y)\cos(xy^2 + x^3)$

6. Which of the following are solutions of Heat equation : $\dfrac{\partial u}{\partial x} - \dfrac{\partial^2 u}{\partial y^2} = 0$?

(a) $u = x^3 + y^4/4$ No
(b) $u = x^3 - y^4/4$ No
(c) $u = x^2 + 3xy^2$ No
(d) $u = \ln\sqrt{4x + 2y}$ No
(e) $u = e^{-2x}\cos y - e^{-y}\cos 2x$ No
(f) $u = \exp(x + y)$ Yes.

7. Bonus. Find f_{yy} of a function $f(x,y)$ such that $f_{xx} = x + 4y$ and $f_{xy} = 4x - y$.