Mathematics 2000: Assignment #6, Winter 2004- Answers

Due March 3

1. Find the MacLaurin series for $f(x)$ and their radius of convergence. Use Binomial Theorem and/or an appropriate known series.

(a) $f(x) = (x+2)^{10} = 2^{10}\sum_{n=0}^{10}\binom{10}{n}\frac{x^n}{2^n} = 1024 + 5120x + 11520x^2 + 15360x^3 + 13440x^4 + 8064x^5 + 3360x^6 + 960x^7 + 180x^8 + 20x^9 + x^{10}$, Radius of convergence $R = \infty$.

(b) $f(x) = (2x-3)^6 = 3^6\sum_{n=0}^{6}\binom{6}{n}\frac{-2^n x^n}{3^n} = 729 - 2916x + 4860x^2 - 4320x^3 + 2160x^4 - 576x^6 + 64x^6$, Radius of convergence $R = \infty$.

(c) $f(x) = (x^2+a)^5 = a^5 + 5a^4x^2 + 10a^3x^4 + 10a^2x^6 + 5ax^8 + x^{10}$, Radius of convergence $R = \infty$.

(d) $f(x) = (x-1)^{-4} = \sum_{n=0}^{\infty}\binom{-4}{n}(-x)^n = 1 + 4x + 10x^2 + 20x^3 + ...,\ R = 1$

(e) $f(x) = (x+1)^{-5} = \sum_{n=0}^{\infty}\binom{-5}{n}x^n = 1 - 5x + 15x^2 - 35x^3 + ...,\ R = 1$

(f) $f(x) = (2x-1)^{-3} = -\sum_{n=0}^{\infty}\binom{-3}{n}(-2x)^n = -1 - 6x - 24x^2 - 80x^3 + ...,\ R = 1/2$

(g) $f(x) = (16+x^4)^{-2} = 16^{-2}\sum_{n=0}^{\infty}\binom{-2}{n}\frac{(x^4)^n}{16^n} = \frac{1}{256} - \frac{x^4}{2048} + \frac{3x^8}{65536} + ...,\ R = 2$

(h) $f(x) = (1-x)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-x)^n = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + ...,\ R = 1$

(i) $f(x) = (1-x)^{-3/2} = \sum_{n=0}^{\infty}\binom{-3/2}{n}(-x)^n = 1 + \frac{3x}{2} + \frac{15x^2}{8} + \frac{35x^3}{16} + ...,\ R = 1$

(j) $f(x) = (x+9)^{-1/2} = \frac{1}{3}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^n}{9^n} = \frac{1}{3} - \frac{x}{54} + \frac{x^2}{648} - \frac{5x^3}{34992} + ...,\ R = 9$

2. a) MacLaurin series for

$$f(x) = \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-x^2)^n = 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + ...,\ R = 1.$$

b) Use your resalt in (a) to find MacLaurin series for $f(x) = \arcsin(x)$

$$\arcsin(x) = \int \frac{1}{\sqrt{1-x^2}}\,dx = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n\frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + ...,$$

3. a) Find four first terms of MacLaurin series for function $f(x) = (1+x)^{1/3}$

$$f(x) = (1+x)^{1/3} \approx 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81}$$

b) Use your result in (a) to find approximate value of $(1.01)^{1/3}$

$$(1.01)^{1/3} \approx 1 + \frac{.01}{3} - \frac{.01^2}{9} + \frac{5(.01)^3}{81} = 1.003322284$$

This is a good approximation since $x = 0.01$ is close to 0.

c) Use your result in (a) to find approximate value of $(0.09)^{1/3}$

$$(0.09)^{1/3} = (1-0.91)^{1/3} \approx 1 + \frac{-.91}{3} - \frac{.91^2}{9} + \frac{5(-.91)^3}{81} = 0.5581388272$$

This approximation is bad since the value $x = -0.91$ is not close to 0. More terms of the series are required to get better approximation.

4. Use Alternating Series Estimation Theorem to estimate the range of **positive** values of x for which the given approximation is accurate within the error.

a) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}$, $\quad |error| < 10^{-5}$. **Answer**: $0 \leq x \leq 0.439$

a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$, $\quad |error| < 10^{-4}$ **Answer**: $0 \leq x \leq 0.141$

5. **Bonus Problem.** Find MacLaurin series $\sum a_n x^n$ for function $\frac{x}{1-x-x^2}$. The coefficients a_n form a sequence. Do you know its name? What else do you know about the sequence.