Mathematics 2000: Assignment #5, Winter 2004- Answers

Due Feb 18

1. Find the radius of convergence and the interval of convergence of each power series. Don't forget to check the convergence of the endpoints separately.

   (a) $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1}$ Interval of convergence $(-1, 1]$, radius of convergence 1.

   (b) $\sum_{n=1}^{\infty} \sqrt{n}(x-2)^n$ Interval of convergence $(1, 3)$, radius of convergence 1.

   (c) $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ Interval of convergence $[-3, 3)$, radius of convergence 3.

   (d) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$ Interval of convergence $[-1/3, 5/3)$, radius of convergence 1.

   (e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$ Interval of convergence $(-\infty, \infty]$, radius of convergence ∞.

   (f) $\sum_{n=1}^{\infty} \frac{n(x+4)^n}{2n^2+1}$ Interval of convergence $[-5, -3)$, radius of convergence 1.

   (g) $\sum_{n=2}^{\infty} (-1)^n \frac{(2x+3)^n}{n(\ln n)^2}$ Interval of convergence $[-2, -1]$, radius of convergence $1/2$.

   (h) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^3 8^n}$ Interval of convergence $(-8, 8]$, radius of convergence 8.

   (i) $\sum_{n=1}^{\infty} n^n x^n$ The series converges only at one point $x = 0$, radius of convergence 0.

2. Find a power series representation for the following functions using formula $(1-x)^{-1} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$. Determine their interval of convergence.

   (a) $f(x) = \frac{2}{3-x} = \frac{2}{3}\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

   (b) $f(x) = \frac{x^2}{2x+4} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{2^n}$

   (c) $f(x) = \frac{x}{16-x^4} = \frac{1}{16}\sum_{n=0}^{\infty} \frac{x^{4n+1}}{16^n}$

3. Find the MacLaurin series for $f(x)$ and its radius of convergence given **known** MacLaurin series for $\sin x$, $\cos x$, e^x, $arctanx$ (see lecture notes or your book).

   (a) $f(x) = \begin{cases} x^{-1}\sin(x^3), & x \neq 0, \\ 0, & x = 0 \end{cases}$ $f(x) = \sum_{n=0}^{\infty} \frac{x^{6n+2}}{(2n+1)!}$ Converges for all x.

   (b) $f(x) = x\cos(3x)$ $f(x) = \sum_{n=0}^{\infty} \frac{9^n x^{2n+1}}{(2n)!}$ Converges for all x.

(c) $f(x) = e^x - \cos(x)$ $\qquad f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ Converges for all x.

(d) $f(x) = e^{4x} + \arctan(x/2)$. $\qquad f(x) = \sum_{n=0}^{\infty} \frac{4^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{2n+1}(2n+1)!}$ Converges for $x \in [-2, 2]$.

4. Use the **definition** to find the Taylor series (centered at c) for the functions (calculate just the first three or four nonzero terms):

(a) $f(x) = e^{2x}, c = 1$ $\qquad e^{2x} \approx e^2(1 + 2(x-1) + 2(x-1)^2 + \frac{4}{3}(x-1)^3$

(b) $f(x) = \cos x, c = \frac{\pi}{4}$ $\qquad \cos^x \approx \frac{\sqrt{2}}{2}(1 - (x - \frac{\pi}{4}) - \frac{1}{2}(x - \frac{\pi}{4})^2 + \frac{1}{6}(x - \frac{\pi}{4})^3)$

(c) $f(x) = \cot x, c = \frac{\pi}{2}$ $\qquad \cot x \approx -(x - \frac{\pi}{2}) - \frac{1}{3}(x - \frac{\pi}{2})^3 - \frac{2}{15}(x - \frac{\pi}{2})^5 - \frac{17}{315}(x - \frac{\pi}{2})^7$

(d) $f(x) = \ln x, c = 2$ $\qquad \ln(x) \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{24}(x-2)^3$

(e) $f(x) = x^{-3}, c = -2$ $\qquad x^{-3} \approx -\frac{1}{8} - \frac{3}{16}(x+2) - \frac{3}{16}(x+2)^2 - \frac{5}{32}(x+2)^3$

(f) $f(x) = (1-x)^{-2}, c = 0$ $\qquad (1-x)^{-2} \approx 1 + 2x + 3x^2 + 4x^3$

5. **Bonus Problem.** Find the sum of the series.

Hint: find related power series and the function which it represents.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n n!}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n(2n-1)}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n-1/2}(2n-1)!}$

(d) $\sum_{n=1}^{\infty} \frac{n 2^n}{3^{n-1}}$.