Math 2000: Answers for Assignment #9, Winter 2006

1. Find the double integral by identifying it as a volume of a solid

a) $\int\int_R 8 \; dA, \qquad R = \{(x,y)|1 \leq x \leq 2, -1 \leq y \leq 1\}$

This is a rectangular box with sides 1,2, and 8. Thus, volume is 16.

b) $\int\int_R 8 - x \; dA, \qquad R = \{(x,y)|1 \leq x \leq 2, -1 \leq y \leq 1\}$

This is a "box" with trapezoidal side. Area at the trapezoid is (7+6)/2.

Thus volume is 7+6=13.

c) $\int\int_R 8 - y \; dA, \qquad R = \{(x,y)|1 \leq x \leq 2, -1 \leq y \leq 1\}$

This is a "box" with trapezoidal side. Volume is $(9+7)/2 \cdot 2 \cdot = 16$

d) $\int\int_R \sqrt{4 - x^2 - y^2} \; dA, \qquad R = \{(x,y)|x^2 + y^2 \leq 4\}$

This is a hemisphere of radius 2. Volume of a sphere is $\frac{4}{3}\pi R^3$. Thus, the answer is

$\frac{1}{2} \cdot \frac{4}{3}\pi 2^3 = \frac{16}{3}\pi.$

2. Calculate the iterated integral

a) $\int_1^2 \int_0^4 1 + xy \, dxdy = \int_1^2 (x + \frac{x^2}{2}y)|_0^4 dy = \int_1^2 (4 + 8y) dy = 4(y + y^2)|_1^2 = 16$

b) $\int_1^2 \int_0^4 1 + xy \, dydx = \int_1^2 (y + x \cdot \frac{y^2}{2})|_0^4 dx = \int_1^2 (4 + 8x) dy = 4(x + x^2)|_1^2 = 16$

c) $\int_1^2 \int_0^{\pi/2} x \sin y \, dydx \int_1^2 x(-\cos y)|_0^{\frac{\pi}{2}}$

$= \int_1^2 x dx = \frac{x^2}{2}|_1^2 = \frac{3}{2}$

d) $\int_0^{\ln 2}\int_0^{\ln 3} e^{2x-y}\, dxdy = \int_0^{\ln 2}\int_0^{\ln 3} e^{2x}e^{-y} dxdy$

$$= \frac{1}{2}e^{2x}|_0^{\ln 3} \cdot (-1)e^{-y}|_0^{\ln 2} = \frac{1}{2}(e^{2\ln 3} - 1)(1 - e^{-\ln 2}) = 2$$

e) $\int_1^2\int_2^4 \frac{x}{y} - \frac{y}{x}\, dxdy = \int_1^2 (\frac{x^2}{2y} - y\ln x)|_2^4 dy$

$$= \int_1^2 (\frac{6}{y} - y\ln 2)dy = (6\ln y - \frac{y^2}{2}\cdot \ln 2)|_1^2 = \frac{9}{2}\ln 2$$

f) $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+10}}\, dydx$, substitution $u = y^2$

$$\int_0^1\int_0^1 \frac{1}{2}\frac{x}{\sqrt{x^2+u+10}} dudx = \int_0^1 (\sqrt{x^2+u+10}|_0^1 xdx$$

$$= \int_0^1 (\sqrt{x^2+11} - \sqrt{x^2+10})xdx, \text{ substitution } v = x^2$$

$$\int_0^1 \frac{1}{2}(\sqrt{v+11} - \sqrt{v+10})dv$$

$$= \frac{1}{2}\left(\frac{2}{3}(v+11)^{\frac{3}{2}} - \frac{2}{3}(v+10)^{\frac{3}{2}}\right)|_0^1$$

$$= \frac{1}{3}(12^{\frac{3}{2}} - 11^{\frac{3}{2}} - 11^{\frac{3}{2}} + 10^{\frac{3}{2}}) = 8\sqrt{8} - \frac{22}{3}\sqrt{11} + \frac{10}{3}\sqrt{10}$$

g) $\int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\, dxdy = \int_0^1 \frac{x+\frac{x^3}{3}}{1+y^2}|_0^1$

$$\frac{4}{3}\int_0^1 \frac{1}{1+y^2}dy = \frac{4}{3}\arctan y|_0^1 = \frac{4}{3}\cdot\frac{\pi}{4} = \frac{\pi}{3}$$

h) $\int_0^{\pi}\int_0^{\pi/2} \cos(x+2y)\,dydx = \int_0^{\pi} \frac{1}{2}\sin(x+2y)|_0^{\frac{\pi}{2}}\,dx$

$$= \frac{1}{2}\int_0^{\pi} \sin(x+\pi) - \sin x = \frac{1}{2}(-\cos(x+\pi) + \cos x)|_0^{\pi} = \frac{1}{2}(-1-1-1-1) = -2$$

3. Calculate the double integral over general region. Sketch the region.

a) $\int\int_D x^2y\,dA, \qquad D = \{(x,y)|1 \le x \le 2, -x \le y \le x\}$

$$\int_1^2\int_{-x}^{x} x^2ydydx = 0$$

b) $\int\int_D \sqrt{x}\,dAD = \{(x,y)|y \le x \le e^y, 0 \le y \le 1\}$

$$= \int_0^1\int_y^{e^y} \sqrt{x}dxdy = \int_0^1 \left(\frac{2x^{\frac{3}{2}}}{3}\right)|_y^{e^y}dy$$

$$= \frac{2}{3}\int_0^1 e^{\frac{3y}{2}} - y^{\frac{3}{2}}dy = \frac{2}{3}\left(\frac{2}{3}e^{\frac{3y}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)|_0^1$$

$$= \frac{4}{9}e^{\frac{3}{2}} - \frac{32}{45}$$

c) $\int\int_D x+y\,dA, \qquad D$ is bounded by $y=\sqrt{x}$ and $y=x^2$

$$= \int_0^1\int_{x^2}^{\sqrt{x}} (x+y)dydx = \int_0^1 (xy + \frac{y^2}{2})|_{x^2}^{\sqrt{x}}dx$$

$$= \int_0^1 x^{\frac{3}{2}} = \frac{x}{2} - x^3\frac{x^4}{2}dx = \frac{3}{10}$$

d) $\int\int_D xy\,dA, \qquad D$ is the triangular region with vertices $(0,2)$, $(4,0)$, $(0,0)$.

$$= \int_0^4\int_0^{-\frac{1}{2}x+2} xydydx = \int_0^4 x\cdot\frac{(-\frac{1}{2}x+2)^2}{2}dx = \frac{8}{3}$$

P.S. Line via point (0,2) and (4,0) has equation $y = -\frac{1}{2}x + 2$.

It becomes the upper limit in the integeral w.r. to y.

e) $\int\int_D x\sqrt{y^2 - x^2}\, dA \qquad D = \{(x, y)|0 \le x \le y, 0 \le y \le 1\}$

$= \int_0^1 \int_0^y x\sqrt{y^2 - x^2} dxdy$ use substitution $u = -x^2$, $u \in [-y^2, 0]$

$= \int_0^1 \int_{-y^2}^0 \frac{1}{2}(y^2 + u)^{\frac{1}{2}} dudy = \int_0^1 \left[\frac{1}{2}\frac{2}{3}(y^2 + u)^{\frac{3}{2}}|_{-y^2}^0\right] dy$

$= \int_0^1 \frac{1}{3}y^3 dy = \frac{1}{12}y^4|_0^1 = \frac{1}{12}$

f) $\int_0^1 \int_0^v \sqrt{1 - v^2}\, dudv = \int_0^1 v\sqrt{1 - v^2} \cdot dv$, substitution $u = -v^2, u \in [-1, 0]$

$= \frac{1}{2}\int_{-1}^0 \sqrt{1 + u} du = \frac{1}{3}(1 + u)^{\frac{3}{2}}|_{-1}^0 = \frac{1}{3}$