Math 2000: Solutions for Assignment #7, Winter 2006

1. If the limit exists, find it. Otherwise show that it doesn't exist.

   (a) $\lim_{(x,y)\to(-3,4)} (x^3 + 3x^2y^2 - 2x + 3) = 414$

   (b) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y^2} = 1$

   Since after introduction of variable $u = x^2 + y^2$ we obtain

   $$\lim_{u\to 0} \frac{\sin(u)}{u} = 1.$$

   (c) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^6}{x^2+y^6} \to$ Limit doesn't exist

   If you approach $(0,0)$ along $x = 0, \lim_{y=0} \frac{-y^6}{y^6} = -1$ but, along $(y = 0)$, $\lim_{x=0} \frac{x^2}{x^2} = 1$

   (d) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$. To prove this note that $x^2 \le x^2 + y^2 \Rightarrow$

   $|x| \le \sqrt{x^2+y^2} \Rightarrow \frac{|x|}{\sqrt{x^2+y^2}} \le 1 \Rightarrow \lim_{(x,y)\to(0,0)} \left| \frac{xy}{\sqrt{x^2+y^2}} \right|$

   $= \left| \lim_{(x,y)\to(0,0)} y \left(\frac{x}{\sqrt{x^2+y^2}} \right) \right| \le \lim_{(x,y)\to(0,0)} |y| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

   (e) $\lim_{(x,y)\to(0,0)} \frac{2x^6y}{x^9+y^3}$ If you approach $(0,0)$ along $x = 0$, or along $y = 0$, the limit is 0. but, along $(y = x^3)$, $\lim_{x=0} \frac{2x^9}{x^9+x^9} = 1$
   Thus the limit is path-dependent and does not exist.

   (f) $\lim_{(x,y)\to(0,0)} (1 + 2x^2y^2)^{\frac{1}{x^2y^2}}$ Introduce new variable $u = (xy)^2$. Then we have

   $$\lim_{u\to 0} (1+u)^{1/u} = \lim_{n\to\infty} (1 + 1/n)^n = e.$$

2. Calculate the indicated partial derivatives.

   (a) $u = xy\sec(xy)$; u_x, u_{xy}, u_{xx}

   $$\frac{\partial u}{\partial x} = y\sec(xy) + xy^2\sec(xy)\tan(xy) = y\sec(xy)\,(1 + xy\tan(xy))$$

$$\frac{\partial^2 u}{\partial x \partial y} = \sec(xy) + xy\sec(xy)\tan(xy) + 2xy\sec(xy)\tan(xy) + xy^2\left(x\sec(xy)\tan^2(xy) + x\sec^3(xy)\right)$$

$= \sec(xy)\left(1 + 3xy\tan(xy) + x^2y^2\tan^2(xy) + x^2y^2\sec^2(xy)\right)$
$= \sec(xy)\left(1 + 3xy\tan(xy) + 2x^2y^2\tan^2(xy) + x^2y^2\right)$ (using $1 + \tan^2 z = \sec^2 z$) either form o.k.

$$\frac{\partial^2 u}{\partial x^2} = y^2\sec(xy)\tan(xy) + y^2\sec(xy)\tan(xy) + xy^3\sec(xy)\tan^2(xy) + xy^3\sec^3 9xy)$$

$$= y^2\sec(xy)\left(2\tan(xy) + xy\tan^2(xy) + xy\sec^2(xy)\right)$$

$= y^2\sec(xy)\left(2\tan(xy) + 2xy\tan^2(xy) + xy\right)$ either form is o.k.

(b) $z = \frac{x}{y} + \frac{y}{x}$; $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ $\quad \Rightarrow \frac{\partial z}{\partial x} = \frac{1}{y} - \frac{y}{x^2} \quad \frac{\partial z}{\partial y} = \frac{-x}{y^2} + \frac{1}{x}$

(c) $xy + yz = zx$; $x_y, y_z, z_x \Rightarrow x = \frac{yz}{z-y} \Rightarrow \frac{\partial x}{\partial y} = \frac{z(z-y)+yz}{(z-y)^2} = \frac{z^2}{(z-y)^2}$

$$\Rightarrow y = \frac{xz}{x+z} \Rightarrow \frac{\partial y}{\partial z} = \frac{x(x+z)-xz}{(x+z)^2} = \frac{x^2}{(x+z)^2}$$

$$z = \frac{xy}{x-y} \Rightarrow \frac{\partial z}{\partial x} = \frac{y(x-y)-xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

(d) $z = \ln\left(\sin(x-y)\right)$; z_{yxx}

$$z = \ln\left(\sin(x-y)\right) \Rightarrow \frac{\partial z}{\partial x} = \frac{\cos(x-y)}{\sin(x-y)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} \Rightarrow \frac{-\sin^2(x-y) - \cos^2(x-y)}{\sin^2(x-y)} = \frac{-1}{\sin^2(x-y)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial x \partial y} = \frac{\partial}{\partial y}\left(-\sin^{-2}(x-y)\right) = \frac{-2\cos(x-y)}{\sin^3(x-y)}$$

(e) $f(x,y,z) = x^5 + x^4y^4z^3 + y^2x$; $\frac{\partial^3 f}{\partial x \partial y \partial z}$

$$\Rightarrow \frac{\partial f}{\partial z} = 3x^4y^4x^2$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial z} = 12x^4y^3z^2$$

$$\Rightarrow \frac{\partial^3 f}{\partial x \partial y \partial z} = 48x^3y^3z^2$$

(f) $f(x,y,z) = xyz$; $f_x(0,1,2)$

$$\Rightarrow \frac{\partial f}{\partial x} = yz \Rightarrow \frac{\partial f}{\partial x}|_{(0.1.2)} = 2$$

3. Find all first partial derivatives of the functions given below.

(a) $f(x,y) = x^3y^5 - 2x^3y^2 + 4x \Rightarrow \frac{\partial f}{\partial x} = 3x^2y^5 - 6x^2y^2 + 4$ $\qquad \frac{\partial f}{\partial y} = 5x^3y^4 - 4x^3y$

(b) $f(x,y) = \frac{x-y}{x+y} \Rightarrow \frac{\partial f}{\partial x} = \frac{2y}{(x+y)^2} \rightarrow \frac{\partial f}{\partial y} = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$

(c) $f(x,y) = \ln(x^2+y^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}$ $\qquad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$

(d) $f(x,t) = e^{\sin(t/x)} \Rightarrow \frac{\partial f}{\partial x} = \frac{-t}{x^2}\cos(\frac{t}{x})e^{\sin(\frac{t}{x}}$ $\qquad \frac{\partial f}{\partial t} = \frac{1}{x}\cos(\frac{t}{x})e^{\sin(\frac{t}{x})}$

(e) $f(x,y,z) = x^{yz} \Rightarrow \frac{\partial f}{\partial x} = yzx^{yz-1}$ $\qquad \frac{\partial f}{\partial y} = z(\ln x)x^{yz}$ $\qquad \frac{\partial f}{\partial z} =$ $y(\ln x)x^{yz}$

(f) $f(x,y,z,t) = xy^3z^2\sqrt{t} \Rightarrow \frac{\partial f}{\partial x} = y^3z^2\sqrt{t}$ $\qquad \frac{\partial f}{\partial y} = 3xy^2z^2\sqrt{t}$ $\qquad \frac{\partial f}{\partial z} = 2xy^3z\sqrt{t}$ $\qquad \frac{\partial f}{\partial t} = \frac{xy^3z^2}{2\sqrt{t}}$

4. Confirm that Clairaut's theorem holds for the following functions. That is, calculate f_{xy} and f_{yx} and show that they are equal.

(a) $f(x,y) = x^4y^2 + x^3y^4 + xy^2 + x + y^2$

$$\Rightarrow \frac{\partial f}{\partial x} = 4x^3y^2 + 3x^2y^4 + y^2 + 1 \Rightarrow \frac{\partial^2 f}{\partial y \partial x} = 8x^3y + 12x^2y^3 + 2y$$

$$\frac{\partial f}{\partial y} = 2x^4y + 4x^3y^3 + 2xy + 2y \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 8x^3y + 12x^2y^3 + 2y$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ as required}$$

(b) $f(x, y) = \sin(x^2 y + x^2)$

$\Rightarrow \dfrac{\partial f}{\partial x} = (2xy+2x)\cos(x^2y+x^2) \Rightarrow \dfrac{\partial^2 f}{\partial y \partial x} = 2x\cos(x^2y+x^2) - x^2(2xy+2x)\sin(x^2y+x^2)$

$\dfrac{\partial f}{\partial y} = x^2\cos(x^2y+x^2) \Rightarrow \dfrac{\partial^2 f}{\partial x \partial y} = 2x\cos(x^2y+x^2) - x^2(2xy+2x)\sin(x^2y+x^2)$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ as required}$$

5. Which of the following are solutions of Laplace's equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$?

(a) $u = x^2 + y^2 \Rightarrow U_{xx} = 2, U_{yy} = 2 \Rightarrow U_{xx} + U_{yy} = 4 \neq 0$ Not a solution

(b) $u = x^2 - y^2 \Rightarrow U_{xx} = 2, U_{yy} = -2 \Rightarrow U_{xx} + U_{yy} = 0 = 0$ A solution

(c) $u = x^3 + 3xy^2 \Rightarrow U_{xx} = 6x, U_{yy} = 6x \Rightarrow U_{xx} + U_{yy} = 12x \neq 0$ Not a solution

(d) $u = \ln\sqrt{x^2+y^2} \Rightarrow u_x = \frac{x}{x^2+y^2} \Rightarrow u_{xx} = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$

$u_y = \dfrac{y}{x^2+y^2} \Rightarrow u_{yy} = \dfrac{-(y^2-x^2)}{(x^2+y^2)^2} \Rightarrow U_{xx} + U_{yy} = 0 = 0$ A solution

(e) $u = \sin x \cosh y + \cos x \sinh y \Rightarrow u_x = \cos x \cosh y - \sin x \sinh y$

$\Rightarrow u_{xx} = -\sin x \cosh y - \cos x \sinh y$

$\Rightarrow u_y = \sin x \sinh y + \cos x \cosh y \Rightarrow u_{yy} = \sin x \cosh y + \cos x \cosh y \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow$ A solution

(f) $u = e^{-x}\cos y - e^{-y}\cos x \Rightarrow u_{xx} = e^{-x}\cos y + e^{-y}\cos x$

$\Rightarrow u_{yy} = -e^{-x}\cos y - e^{-y}\cos x \Rightarrow u_{xx} = u_{yy} = 0 \Rightarrow$ A solution