Math 2000: Assignment #6, Winter 2006

1. A Professor lost his calculator. Help him to find an approximate value of $\ln(1.5)$ and $\ln(0.5)$ using the Taylor expansion.

$$\ln|1+x| = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}.$$

How many terms do you need in each case to find 3 correct digits of the numbers?

We use $\ln|1+x| = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. First set $x = 0.5$ to get

$$\ln(1.5) = \ln(1+\frac{1}{2}) = \sum_{k=1}^{\infty}(-1)^{k+1}(\frac{1}{2})^k \cdot \frac{1}{k}$$

$$\approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} \cdots \approx 0.401\ldots \text{ (exact value} = 0.405464\cdots)$$

Now set $x = -0.5$ to obtain $\ln(0.5) = \ln(1-\frac{1}{2}) = \sum_{k=1}^{\infty}(-1)^{k+1}(-\frac{1}{2})^k \cdot \frac{1}{k}$

$$= (-1)\sum_{k=1}^{\infty}(-1)^k(-1)^k(\frac{1}{2})^k \cdot \frac{1}{k} = -\sum_{k=1}^{\infty}(\frac{1}{2})^k \cdot \frac{1}{k}$$

$$\approx -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} - \frac{1}{64} - \cdots = -0.682\ldots \text{(exact value } -0.69314\ldots)$$

How many terms are needed to find 3 correct digits? We see that four terms in not enough yet!

For $\ln(1.5)$ we have alternating series. Thus the error $\left|\ln(1.5) - \sum_{k=1}^{n}(-1)^{k+1}(\frac{1}{2})^k\frac{1}{k}\right|$

is less the absolute value of the $(n+1)$-th term which is $(\frac{1}{2})^{n+1}\frac{1}{n+1}$. We need 3 correct

digits. Thus set the error $(\frac{1}{2})^{n+1}\frac{1}{n+1} \leq 0.0001$ and find that $n = 9$ is the smallest number that works.

So, 9 terms are enough to get 3 correct digits for $\ln(1.5)$. For $\ln(0.5)$ the story is more

complicated. Here we have to use the Taylor inequality $|R_n(x_0)| < \frac{M}{(n+1)!}|x_0 - a|^{n+1}$,

$M = \max_{[2a-x_0,x_0]}|f^{(n+1)}(x)|$. Here $a = 0$, $x_0 = -\frac{1}{2}$, $M = \max_{[-\frac{1}{2},\frac{1}{2}]}\frac{n!}{(1+x)^{n+1}} = 2^{n+1}\cdot n!$

Thus $|R_n(x_0)| < \frac{2^{n+1}n!}{(n+1)!\cdot 2^{n+1}} = \frac{1}{n+1}$. This must be equal to 0.0001

So, to make error less than $\frac{1}{10000}$ *it is enough* $n = 9.999$ terms. This estimate is too rough!!!

In reality 10 terms gives 3 correct digits.

2. Estimate the range of values of x for which the approximation $\sin x \approx x - \frac{x^3}{6}$ is accurate to within the error 0.0001? Graph both functions on the same Figure to support you statement.

$\sin x \approx x - \frac{x^3}{6}$

For $x > 0$ $\sin x$ is represented by an alternating series. Thus $|\sin x - (x - \frac{x^3}{6})| < \frac{|x^5|}{5!}$.

Set the error $\frac{x^5}{5!} \leq 0.0001$

So $|x|^5 < \frac{5!}{10000}$, $|x| < \sqrt[5]{\frac{5!}{10,000}}$, so $|x| < \sqrt[5]{.012} \approx 0.41289$.

Check, for example, $\sin(0.4) = 0.38941$

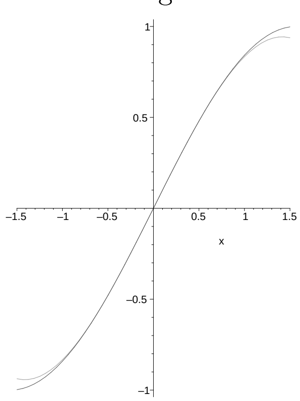
$0.4 - \frac{(0.4)^3}{6} = 0.38933$ o.k.

For $x < 0$ the series is no longer alternating. We will use the Taylor inequality. Here $a = 0$, $x_0 = x$, $n = 4$ and $M = 1$.

Thus the error is $|\sin x - (x - \frac{x^3}{6})| < \frac{|x^5|}{5!}$, the same again.

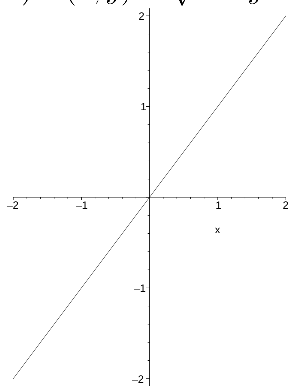
So we obtain from both cases, $x > 0$ and $x < 0$, that the answer is $|x| < \sqrt[5]{.012} \approx 0.41289$.

Next we plot functions $\sin x$ and $x - x^3/6$ to show that their graphs are almost indistinguishable when the argument x is near zero.

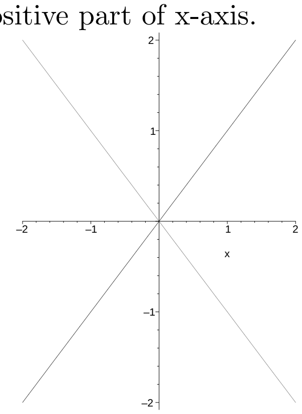


Problem 3. Find domain of the function of two variables.

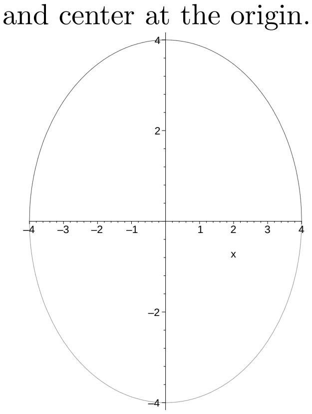
a) $F(x, y) = \sqrt{x - y}$. Domain is $x \geq y$. It is the area below the line $y = x$.



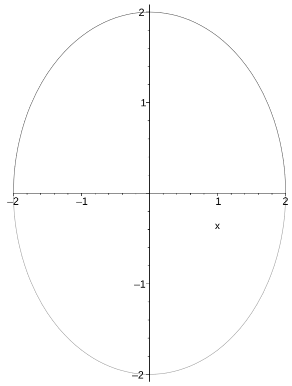
b) $F(x, y) = \sqrt{x - y}\ln(x + y)$. Domain is $x \geq y$ and $x \geq -y$. It is the quater which contains positive part of x-axis.



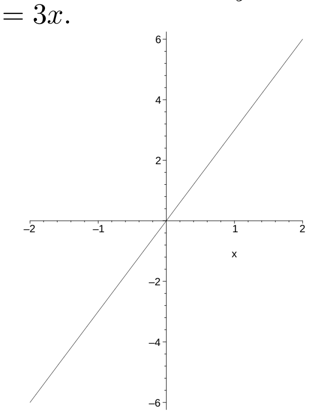
c) $F(x, y) = \sqrt{x^2 + y^2 - 16}$. Domain $x^2 + y^2 - 16 \geq 0$. It is the exterior of the circle of radius 4 and center at the origin.



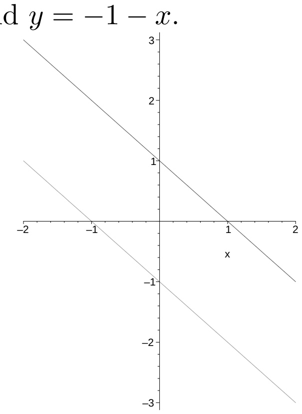
d) $F(x, y) = \sqrt{4 - x^2 - y^2}$. Domain $4 - x^2 - y^2 \geq 0$. It is the interior of the circle of radius 2 and center at the origin.



e) $F(x,y) = \frac{3x+y}{3x-y}$ Domain exclude points $3x - y = 0$. This is the whole plane except the line $y = 3x$.



f) $f(x,y) = \arcsin(x+y)$.Domain is $|x+y| \leq 1$. This is the strip betweeen the lines $y = 1 - x$ and $y = -1 - x$.



4. For given function sketch the level curve $F(x, y) = 1$, the counter map of the function and the graph of the function. Give the name or a word description of the surface (like "This is a plane", "This is a paraboloid ", or "That is bizarre..." etc).

a) $F(x, y) = 4x^2 + y^2 + 1$
The level curve consists of just one point — the origin.
The counter map is a collection of ellipses $4x^2 + y^2 = k$, where $k \geq 0$ is a constant.
This is an elliptic paraboloid opened up and shifted up by 1.

b) $F(x, y) = 1 - x - y$
The level curve is the line $x = -y$.
The counter map is a collection of lines with slope -1.
This is a plane.

c) $F(x, y) = -2$
The level curve does not exist for any level other then $F = -2$.
For $F = -2$ the whole xy-plane is the "level curve".
The surface is a plane parallel to xy-plane.

d) $F(x, y) = y$
The level curve is the line $y = 1$.
The counter map is a collection of lines $y = k$, where k is a constant.
This is a plane.

e) $F(x, y) = 1 + \cos x$
The level curve is the set of lines line $x = \pi/2 + \pi k,\ k = 0, \pm 1, \pm 2, \ldots$.
The surface looks like a wave.

f) $F(x, y) = -4x^2 - y^2 + 2$
The level curve is an ellipse $4x^2 + y^2 = 1$.
The counter map is a collection of ellipses $4x^2 + y^2 = k$, where $k \geq 0$ is a constant.
This is an elliptic paraboloid opened down and shifted up by 2.