Math 2000: Solutions for Assignment #5, Winter 2006

1. Find the radius of convergence and the interval of convergence of each power series. Don't forget to check the convergence of the endpoints seperately.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

If $a_n = \dfrac{(-1)^n x^n}{n+1} \Rightarrow L = \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n}\right| = |x|$

By the Ratio Test the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$.

Where $x = -1 \Rightarrow$ series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series,

which converges by the AST, thus $I = (-1, 1]$.

(b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} x^n$

$a_n = \sqrt{n} x^n$, so we need $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n+1}|x|^{n+1}}{\sqrt{n}|x|^n} =$

$\displaystyle\lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} \cdot |x| = |x| < 1$ for convergence(by the Ratio Test)$R = 1$. When

$x = 1, x = -1, \lim |a_n| = \displaystyle\lim_{n\to\infty} \sqrt{n} = \infty \Rightarrow$ divergent $\Rightarrow I = (-1, 1)$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$a_n = (-1)^n \dfrac{x^{2n}}{(2n)!}$, so $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}} =$

$\displaystyle\lim_{n\to\infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0$. Thus, by the Ratio Test, the series converges for all real x, so

$R = \infty \Rightarrow I = (-\infty, \infty)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$

$a_n = \dfrac{n(x-4)^n}{n^3+1}$, so $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x-4|^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{n|x-4|^n} =$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}|x-4| = |x-4|$. By the Ratio Test, the series converges when $|x-4|<1$ Thus Radius $R=1$.

$\Rightarrow -1 < x-4 < 1 \Leftrightarrow 3 < x < 5$. When $|x-4|=1$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n}{n^3+1}$, which converges by comparison

with the convergent p-series $\sum_{n=0}^{\infty} \frac{1}{n^2}$ $\quad (p=2>1) \Rightarrow I=[3,5]$.

(e) $\sum_{n=2}^{\infty} (-1)^n \frac{(2x+3)^n}{n \ln n}$

$a_n = (-1)^n \frac{(2x+3)^n}{n\ln n}$, thus we need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| =$

$|2x+3| \lim_{n\to\infty} \frac{n\ln n}{(n+1)\ln(n+1)} = |2x+3| < 1$ for convergence. Thus $-2 < x < -1 \Rightarrow R = \frac{1}{2}$

When $x=-1$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{n\ln n}$ (by the Integral Test, diverges).

When $x=1$ $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n\ln n}$ converges (AST)$\Rightarrow I=(-2,-1]$.

(f) $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} x^n$

$a_n = \frac{2\cdot 4\cdot 6 \ldots (2n)x^n}{1\cdot 3\cdot 5\ldots(2n-1)}$. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| =$

$\lim_{n\to\infty} |x| \left(\frac{2n+2}{2n+1}\right) = |x| < 1$ for convergence, so $R=1$. If $x=\pm 1$, $|a_n| =$

$\frac{2\cdot 4\cdot 6\ldots(2n)}{1\cdot 3\cdot 5\ldots(2n-1)} > 1$, for all n because each integer in the numerator is larger

than the corresponding one in the denominator, so $\sum_{n=0}^{\infty} a_n$ diverges in both cases by the Test for divergence and $I=(-1,1)$.

(g) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 5^{n+1}}\cdot\frac{n^2 5^n}{x^n}\right| =$

$\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^2}\cdot\frac{|x|}{5} = \frac{|x|}{5}$, so by the Ratio Test, the series converges when

$\frac{|x|}{5} < 1 \Leftrightarrow |x| < 5$; $R = 5$. When $x = -5 \Rightarrow \sum_{n=0}^{\infty}\frac{1}{n^2}$(p-series, p=2, convergent)

when $x = 5 \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$ converges by the AST. So, $I = [-5, 5]$.

(h) $\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{(2+n)!}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n(x-2)^n}\right| =$

$\lim_{n\to\infty}\frac{2}{n+3}\cdot|x-2| = 0 < 1 \Rightarrow$ the series converges for all x. $R = \infty$; $I = (-\infty, \infty)$.

2. Use the definition to find the Taylor series (centered at c) for the functions:

(a) $f(x) = e^{3x}, c = 0$

For $c = 0$, we have, $f(x) = e^{3x}$; $f^{(n)}(x) = 3^n e^{3x} \Rightarrow f^{(n)}(x) = 3^n$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \cdots = \sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$$

(b) $f(x) = \sin x, c = \frac{\pi}{4}$

For $c = \frac{\pi}{4}$ we have

$f(x) = \sin x$; $f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$; $f'(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$; $f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$; $f'''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f''''(x) = \sin x$; $f''(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

and so on. Therefore we have:

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{(n)}(\frac{\pi}{4})[x-(\frac{\pi}{4})]^n}{n!} =$$

$$\frac{\sqrt{2}}{2}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{[x-\frac{\pi}{4}]^2}{2!}-\frac{[x-\frac{\pi}{4}]^3}{3!}+\frac{[x-\frac{\pi}{4}]^4}{4!}+\cdots\right]=$$

$$\frac{\sqrt{2}}{2}\left[\sum_{n=0}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}[x-(\frac{\pi}{4})]^{n+1}}{(n+1)!}+1\right].$$

(c) $f(x)=\tan x, c=0$ (calculate just the first three nonzero terms)

For $c=0$ we have:

$f(x)=\tan x;\ f'(x)=\sec^2 x;\ f''(x)=2\sec^2 x\tan x;$

$f'''(x)=2[\sec^4 x+2\sec^2 x\tan^2 x];\ f''''(x)=8[2\sec^4 x\tan x+\sec^2 x\tan^3 x]$

$f'''''(x)=88\sec^4 x\tan^2 x+16\sec^6 x+16\sec^2 x\tan^4 x$

$f(0)=0;\ f'(0)=1;\ f''(0)=0;\ f'''(0)=2;\ f''''(0)=0;\ f'''''(0)=16.$

$$\tan x=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}=x+\frac{2x^3}{3!}+\frac{16x^5}{5!}+...=x+\frac{x^3}{3}+\frac{2}{15}+...$$

3. Find the MacLaurin series for $f(x)$ and its radius of convergence. You may use either the definition of a Maclaurin series or start with a known series such as a geometric series or the MacLaurin series for e^x, $\sin x$, and $\tan^{-1} x$.

(a) $f(x)=\arctan(x^2)$

$\tan^{-1}(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1,1]$, so ,

$$\tan^{-1}(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$$

which converges when $x^2\in[-1,1]\Leftrightarrow x\in[-1,1]$, $R=1$.

(b) $f(x)=xe^{2x}$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}\Rightarrow xe^{2x}=x\sum_{n=0}^{\infty}\frac{2^nx^n}{n!}=$$

$$\sum_{n=0}^{\infty}\frac{2^nx^{n+1}}{n!}\ ;\ R=\infty.$$

(c) $f(x) = (1-3x)^{-5}$.

$(1-3x)^{-5} = \sum_{n=0}^{\infty}(\frac{-5}{n})(-3x)^n = 1+(-5)(-3x)+\frac{(-5)(-6)}{2!}(-3x)^2+\frac{(-5)(-6)(-7)}{3!}(-3x)^3+$
$... =$

$1+\sum_{n=0}^{\infty}\frac{5\cdot 6\cdot 7...(n+4)3^n x^n}{n!}$ for $|-3x|<1 \Leftrightarrow |x|<\frac{1}{3}$; $R=\frac{1}{3}$.

4. Find a power series representation for the following functions and determine their interval of convergence.
(a)$f(x) = \dfrac{x}{4x+1}$

$$f(x) = \frac{x}{4x+1} = x\cdot\frac{1}{1-(-4x)} = x\sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1};$$

$$|-4x|<1;\ I = (-\frac{1}{4},\frac{1}{4})$$

(b)$f(x) = \dfrac{x^2}{a^3-x^3}, (a\neq x)$.

$$f(x) = \frac{x^2}{a^3-x^3} = \frac{x^2}{a^3}\cdot\frac{1}{1-\frac{x^3}{a^3}} = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n =$$

$\sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}$. Convegent for $\left|\frac{x^3}{a^3}\right| < 1 \Leftrightarrow |x^3| < |a^3| \Leftrightarrow$

$|x| < |a| \Rightarrow R = |a|;\ I = (-|a|, |a|)$.

5. Express $f(x) = \dfrac{7x-1}{3x^2+2x-1}$ as a power series by first using partial fractions. Find the interval of convergence.

$$f(x) = \frac{7x-1}{3x^2+2x-1} = \frac{7x-1}{(3x-1)(x+1)} = \frac{A}{3x-1}+\frac{B}{x+1} = \frac{1}{3x-1}+\frac{2}{x+1} =$$

$$2\cdot\frac{1}{1-(-x)} - \frac{1}{1-3x} = 2\sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}[2(-1)^n - 3^n]x^n$$

The series $\sum_{n=0}^{\infty}(-x)^n$ converges for $x\in(-1,1)$ and the series $\sum_{n=0}^{\infty}(3x)^n$ converges for $x\in(-\frac{1}{3},\frac{1}{3}) \Rightarrow$ their sum converges for $x\in(-\frac{1}{3},\frac{1}{3}) = I$.

6. Find a power series represetation for $f(x) = \ln(1+x)$. What is the radius of convergence? Use the above result to find a power series for $f(x) = x\ln(1+x)$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n \text{ (Geometric series } R = 1)$$

$$f(x) = \ln(1+x) = \int_{x=0}^{\infty} \frac{dx}{1+x} = \int_{x=0}^{\infty} \left[\sum_{n=0}^{\infty}(-1)^n x^n\right] dx =$$

$$c + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \qquad (c = 0 \text{ since } f(0) = \ln 1 = 0);\ R = 1.$$

b.)

$$f(x) = x\ln(x+1) = x\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n+1} =$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1};\ R = 1.$$

7. If $f(x) = \dfrac{3}{x+2}$, find the power series for

(a) $f(x)$ centered at 0

$$f(x) = \frac{3}{2(1+\frac{x}{2})} = \frac{3}{2} \cdot \frac{1}{1+\frac{x}{2}} = \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \qquad |x| < 2.$$

(b) $f(x)$ centered 1

$$f(x) = \frac{3}{3+x-1} = \frac{3}{3(1+\frac{x-1}{3})} = \frac{1}{1+\frac{x-1}{3}} = \sum_{n=0}^{\infty}\left(-\frac{x-1}{3}\right)^n \qquad |x-1| < 3.$$

(c) $\frac{df}{dx}$ centered at 0

$$f'(x) = \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n nx^{n-1} = \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n nx^{n-1}}{2^n};\ |x| < 2.$$

(d) $\frac{df}{dx}$ centered at 1

$$f'(x) = \sum_{n=0}^{\infty}(-1)^n \frac{n}{3}\left(\frac{x-1}{3}\right)^{n-1} \qquad |x-1| < 3.$$

$-2 < x \leq 2.$

8. Find the sum of the power series:

(a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\sum_{n=0}^{\infty} nx^n = x\sum_{n=0}^{\infty} nx^{n-1} = x\left[\sum_{n=0}^{\infty} \frac{d}{dx}x^n\right] = x\left[\frac{d}{dx}[\sum_{n=0}^{\infty} x^n]\right] =$$

$$x\left[\frac{d}{dx}[\frac{1}{1-x}]\right] = x\left(\frac{1}{(1-x)^2}\right) = \frac{x}{(1-x)^2};\ |x| < 1.$$

put $x = \frac{1}{2}$ above $\Rightarrow \sum_{n=0}^{\infty} \frac{n}{2^n} = \sum_{n=0}^{\infty} n(\frac{1}{2})^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})} = 2.$

(b) $\sum_{n=1}^{\infty} (-1)^n nx^{n-1}$

Set $S_1(x) = \sum_{n=0}^{\infty} (-1)^n nx^{n-1} \Rightarrow \int_0^x S_1(x)dx = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$

$\Rightarrow S_1(x) = \left(\frac{1}{1+x}\right)' = \frac{-1}{(1+x)^2};\ |x| < 1.$

9. Expand $f(x) = x^{-2}$ as a Taylor series around $c = 1$.

$f(x) = x^{-2}$ $\qquad f(1) = 1$

$f'(x) = -2x^{-3}$ $\qquad f'(1) = -2$

$f''(x) = 6x^{-4}$ $\qquad f''(1) = 6$

$f'''(x) = -24x^{-5}$ $\qquad f'''(1) = -24$

$f''''(x) = 120x^{-6}$ $\qquad f''''(1) = 120$

$$x^{-2} = 1 - 2(x-1) + 6\frac{(x-1)^2}{2!} - 24\frac{(x-1)^3}{3!} + 120\frac{(x-1)^4}{4!} + ...$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 + ...$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$