Math 2000: Assignment #4 Solutions, Winter 2006

1.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3\cdot 6\cdot 9\cdots(3n)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3\cdot 6\cdot 9\cdots(3n)(3n+3)}\frac{3\cdot 6\cdot 9\cdots(3n)}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{3n+3}\right| = \frac{1}{3} \Rightarrow \text{convergent.}$$

b) $\displaystyle\frac{2}{5} + \frac{2\cdot 6}{5\cdot 8} + \frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11} + \frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14} + \cdots = \sum_{n=1}^{\infty}\frac{2\cdot 6\cdot 10\cdots(-2+4n)}{5\cdot 8\cdot 11\cdots(2+3n)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 6\cdot 10\cdots(-2+4n)\cdot(-2+4(n+1))}{5\cdot 8\cdot 11\cdots(2+3n)(2+3(n+1))}\cdot\frac{5\cdot 8\cdot 11\cdots(2+3n)}{2\cdot 6\cdot 10\cdots(-2+4n)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-2+4(n+1)}{2+3(n+1)}\right| = \lim_{n\to\infty}\left|\frac{2+4n}{5+3n}\right| = \frac{4}{3} \Rightarrow \text{divergent.}$$

2.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right| = \frac{1}{2} < 1$$

$\Rightarrow$ absolute convergence.

b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(\arctan n)^n}$ Use the root test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{1}{\arctan n} = \frac{2}{\pi} < 1$$

$\Rightarrow$ absolute convergence.

c) $\sum_{n=1}^{\infty} e^{-n}n!$ Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{-(n+1)}(n+1)!}{e^{-n}n!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{e}\right|$$

$= \infty \quad \Rightarrow$ diverges.

d) $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ Direct comparison:

$$0 \leq \left|\frac{\sin(4n)}{4^n}\right| \leq \frac{1}{4^n}$$

$\sum_{n=1}^{\infty}(\frac{1}{4})^n$ is a convergent geometric series$\Rightarrow$ absolute convergence

e) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2+1}$ Use the alternating series test:

$$a_n = \frac{n}{n^2+1} > 0, \lim_{n\to\infty}\frac{n}{n^2+1} = 0, \frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{-(x^2-1)}{(x^2+1)^2} < 0$$

for $x > 1$ $\Rightarrow$the sequence is decreasing $\Rightarrow$ the series converges.

However, by the limit comparison test (with $\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent)

$$\lim_{n\to\infty}\frac{\left(\frac{n}{n^2+1}\right)}{\frac{1}{n}} = 1 \Rightarrow \sum_{n=1}^{\infty}\frac{n}{n^2+1} \text{ diverges}$$

$\Rightarrow$conditional convergence.

f) $\sum_{n=1}^{\infty} \frac{(-1)^n 5^{2n}}{3^{3n}}$ Note $\frac{5^{2n}}{3^{3n}} = \frac{25^n}{27^n} = \left(\frac{25}{27}\right)^n$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{25}{27}\right)^n \qquad \text{is a convergent geometric series.}$$

$\Rightarrow$ absolute convergence.

g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^3}$ Use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)^3} \frac{n^3}{3^n}\right| = 3$$

$\Rightarrow$ diverges.

h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ Use the root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 \Rightarrow \text{absolutely convergent.}$$

\# 3.

a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$ Use the alternating series test:

$$a_n = \frac{n}{n^2+25},\ i)\lim_{n\to\infty} \frac{n}{n^2+25} = 0, ii)\frac{d}{dx}\left(\frac{x}{x^2+25}\right)$$

$$= \frac{-(x^2-25)}{(x^2+25)^2} < 0, \text{ for } x > 5 \Rightarrow \text{ converges.}$$

b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)$ Use the divergence test:

$$\lim_{n\to\infty} \left(\frac{3n}{1+8n}\right) = \frac{3}{8} \neq 0 \Rightarrow \text{diverges.}$$

c) $\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!} = \sum_{n=1}^{\infty} \frac{2^n}{(n+2)(n+1)}$ Use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+3)(n+2)} \frac{(n+2)(n+1)}{2^n}\right| = 2 > 1 \Rightarrow \text{diverges.}$$

Another way:
$a_n = \frac{2^n n!}{(n+2)!} = \frac{2^n}{(n+1)(n+2)} = \frac{2^n}{(n^2+3n+2)}$

Calculate limit by L'Hostital rule
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{(n^2+3n+2)} = \lim_{n\to\infty} \frac{2^n \ln 2}{(2n+3)} = \lim_{n\to\infty} \frac{2^n (\ln 2)^2}{2} = (\ln 2)^2 \lim_{n\to\infty} 2^{n-1} = \infty$

Thus the series is divergent by the Divergence Test.

d) $\sum_{n=1}^{\infty} \sin n$ Use the divergence test:

$$\lim_{n\to\infty} \sin n \quad \text{doesn't exist} \Rightarrow \text{diverges}$$

e) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ Use the alternating series test:

$$a_n = \frac{1}{\sqrt{n}-1} \qquad i) \lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$$

$$ii) \frac{d}{dx}\left(\frac{1}{\sqrt{x}-1}\right) = \frac{-1}{2\sqrt{x}(\sqrt{x}-1)^2} < 0, \quad x > 0 \quad \Rightarrow \text{the series converges.}$$

f) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ Use the limit comparison test, comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$(convergent)

$$\lim_{n\to\infty} \left(\frac{\sqrt{n^2-1}}{n^3+2n^2+5} \cdot n^2\right) = \lim_{n\to\infty} \frac{n^2\sqrt{n^2-1}}{n^3+2n^2+5} = \lim_{n\to\infty} \frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}} = 1 \Rightarrow \text{convergent.}$$

g) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ $\quad 0 \le e^{\frac{1}{n}} \le e \Rightarrow 0 \le \frac{e^{\frac{1}{n}}}{n^2} \le \frac{e}{n^2}$

$$\sum_{n=1}^{\infty} \frac{e}{n^2} \qquad \text{is a convergent p-series} \Rightarrow \text{convergent by direct comparison.}$$

h) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}} \qquad \frac{(2n)^n}{n^{2n}} = \frac{2^n n^n}{n^{2n}} = \frac{2^n}{n^n} = \left(\frac{2}{n}\right)^n$

Use the ratio test $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(\frac{2}{n+1}\right)^{n+1} \left(\frac{n}{2}\right)^n\right|$

$$= \lim_{n\to\infty} \left|\frac{2}{n+1} \left(\frac{n}{n+1}\right)^n\right| = \lim_{n\to\infty} \left|\frac{2}{n+1} \left(\frac{1}{1+\frac{1}{n}}\right)^n\right| = \frac{1}{e} \lim_{n\to\infty} \frac{2}{n+1} = 0 \Rightarrow \text{convergent.}$$

Simplier way: $\displaystyle a_n = \left(\frac{2n}{n^2}\right)^n \qquad L = \sqrt[n]{|a_n|} = \frac{2n}{n^2} = \frac{2}{n}$

As $n \to \infty \qquad \dfrac{2}{n} \to 0 \qquad L = 0 \Rightarrow$converges by root test.

i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n} \qquad \frac{1}{n + n\cos^2 n} = \frac{1}{n(1+\cos^2 n)} \geq \frac{1}{2n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent, so by direct comparison $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$ diverges.

j) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right) \qquad$ Limit compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\sqrt[n]{2} - 1}{\left(\frac{1}{n}\right)} \longrightarrow \frac{0}{0} \qquad \lim_{n\to\infty} \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}}$$

(Using L'Hospital's rule) $\displaystyle= \lim_{n\to\infty} \frac{\frac{-1}{n^2}(\ln 2)2^{\frac{1}{n}}}{\frac{-1}{n^2}} = \lim_{n\to\infty} (\ln 2)2^{\frac{1}{n}} = \ln 2 \Rightarrow$divergent.