Math2000: Solutions for Asmignment #3, Winter 2006

A professor had a file with convergent series and another file with divergent series. Accidently the files were mixed up. Please, help the professor to sort things out.

1. The telescoping technique will help.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)}$

Solution: Observe that

$$\frac{2}{(n+1)(n+2)(n+3)} = \frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}.$$

Then the Nth partial sum becomes after telescoping

$$S_N = \frac{1}{2}\left(\frac{1}{1\cdot 2} - \frac{1}{(N+1)(N+2)}\right)$$

Thus the series converges to $\lim_{N\to\infty} S_N = 1/4$.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

Solution: Observe that

$$\frac{2}{(2n-1)(2n+1)} = \frac{1}{(2n-1)} - \frac{1}{(2n+1)}.$$

Then the Nth partial sum becomes after telescoping

$$S_N = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{(2N+1)}\right)$$

Thus the series converges to $\lim_{N\to\infty} S_N = 1/2$.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Solution: Multiply the nominator and denominator by $\sqrt{n+1}-\sqrt{n}$ to get

$$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n} = \sqrt{n+1}-\sqrt{n}.$$

Then the Nth partial sum becomes after telescoping

$$S_N = \sqrt{N+1}-\sqrt{1}$$

The series diverges since $\lim_{N\to\infty} S_N = \infty$.

2. The comparison tests might be useful. *Solution*: In the majority of problems we will use the Limit Comparison Test which says that if $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite positive number then series $\sum a_n$ and $\sum b_n$ behave the same way i.e. they either both converge or both diverge. The whole idea is to compare a given series to a simple one which convergence is much easier to study. We will often compare to a geometric $\sum_{n=0}^{\infty} r^n$ or p-series $\sum_{n=1}^{\infty} n^{-p}$. Remember that geometric series converges for $|r| < 1$, and p-series converges for $p > 1$.

a) $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ Here $a_n = \frac{1}{2n+3}$. Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} > 0, \text{ and}$$

$\sum_{n=1}^{\infty} b_n$ diverges, therefore $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges.

b) $\sum_{n=1}^{\infty} \frac{1}{2n^2+3}$. Here $a_n = \frac{1}{2n^2+3}$. Let $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{2n^2+3} = \frac{1}{2} \text{ and}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, therefore $\sum_{n=1}^{\infty} \frac{1}{2n^2+3}$ converges.

Another way is to use The Comparison test:

$$\sum_{n=1}^{\infty} \frac{1}{2n^2+3} < \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+3}}$ $a_n = \frac{1}{\sqrt{2n^2+3}}$ Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{2n^2+3}} = \lim_{n\to\infty} \frac{1}{\sqrt{2+\frac{3}{n^2}}} = \frac{1}{\sqrt{2}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+3}}$ diverges.

d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^3+1}}$ $b_n = \frac{1}{n^{\frac{3}{2}}}$ p $= \frac{3}{2} > 1$ converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{\sqrt{2}} \Rightarrow \text{converges.}$$

e) $\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{2n^3+1}} = \sum_{n=1} \frac{n}{\sqrt{2n^3+1}} + \sum_{n=1} \frac{2}{\sqrt{2n^3+1}}$

The second series converges by (d) but the first does NOT.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^3+1}} \ ; \ b_n = \frac{1}{n^{\frac{3}{2}-1}} = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n \cdot n^{\frac{1}{2}}}{\sqrt{2n^3+1}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{2n^3+1}} = \frac{1}{\sqrt{2}}$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^3+1}} \text{ diverges.}$$

f) $\sum_{n=1}^{\infty} (2n^8+n^6+n+1)^{-1/7}$ Let $b_n = \frac{1}{n^{\frac{8}{7}}}$

$$\lim_{n\to\infty} \frac{n^{\frac{8}{7}}}{(2n^8+n^6+n+1)^{\frac{1}{7}}} = \lim_{n\to\infty} \left(\frac{n^8}{2n^8+n^6+n+1}\right)^{\frac{1}{7}} = \frac{1}{2^{\frac{1}{7}}}$$

$$p = \frac{8}{7} > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{\frac{8}{7}}} \text{ converges} \Rightarrow \text{convergent.}$$

g) $\sum_{n=1}^{\infty} \frac{2^n}{3^n+4} < \sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} r^n,$ where $r = \frac{2}{3}$

geometric series with $|r| < 1$ converges. Thus $\sum_{n=1}^{\infty} \frac{2^n}{3^n+4}$ converges.

h) $\sum_{n=1}^{\infty} \frac{3^n}{2^n+4}$, By The divergence test, since $\lim_{n\to\infty} \frac{3^n}{2^n+4} = \infty \neq 0 \Rightarrow$, the series is divergent.

Another way: by the Limit Comparison Test.

Let $b_n = \frac{3^n}{2^n}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n}{2^n+4} \cdot \frac{2^n}{3^n} = \lim_{n\to\infty} \frac{2^n}{2^n+4}$

$\lim_{n\to\infty} \frac{1}{1+\frac{4}{2^n}} = 1$ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ is a geometric series with $r = \frac{3}{2}$.

$r > 1 \Rightarrow$ it diverges. Thus $\sum_{n=1}^{\infty} \frac{3^n}{2^n+4}$ diverges.

i) $\sum_{n=1}^{\infty} \frac{2^n + 3^{n/2} + 2^{2n}}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$

All are geometric. The last one is divergent, thus divergent.

j) $\sum_{n=1}^{\infty} \frac{n!}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{k=2}^{\infty} \frac{1}{k}$ divergent.

k) $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{n+1} - \sum_{n=1}^{\infty} \frac{1}{n+2} = \frac{1}{2}$. Thus converges.
(Prove it by taking limit of partial sums of the telescoping series.)

Another way: by Comparison Test, $b_n = \frac{1}{n^2}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow$convegent.

l) $\sum_{n=1}^{\infty} \frac{1}{n!} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow$ convergent.

P.S. $\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$.

m) $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ Let $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \sin\frac{1}{n} = 1$ b_n diverges $\Rightarrow \sum_{n=1}^{\infty} \sin\frac{1}{n}$ diverges.

n) $\sum_{n=1}^{\infty} \cos(\frac{1}{n})$ By the Divergence Test: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1 \neq 0$

Thus the series is divergent.

o) $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$ Let $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^2 \sin\frac{1}{n^2} = 1$

$$\text{Sequence } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \sin\frac{1}{n^2} \text{ converges}$$

3. Which of the alternating series is convergent?

a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+3}$

$$a_n = \frac{1}{2n+3}. \text{ This sequence decreases, } a_n > 0, \text{ and } \lim_{n\to\infty} a_n \to 0.$$

$$\text{Thus } \sum_{n=1}^{\infty}(-1)^n a_n \text{ converges.}$$

b) $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n}$

First, $a_n = \frac{\ln n}{n} > 0$ for $n > 1$.

Sequence decreases: $f(x) = \frac{\ln x}{x}$, $f' = \frac{1-\ln x}{x^2} < 0$ for all $x \geq e$

$\Rightarrow a_n$ decreases for $n \geq 3$. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$(by L'Hospital rule)$\Rightarrow$ the series converges.

c) $\sum_{n=1}^{\infty}(-1)^n \sin(\frac{1}{n})$

$$a_n = \sin\frac{1}{n} > 0 \text{ for } n \geq 1.$$

Function $\frac{1}{x}$ is decreasing for $x \geq 1$ and takes values in $[0,1]$.

Function $\sin x$ is increasing for $x \in [0,1]$

$$\Rightarrow \sin\frac{1}{x} \text{ is decreasing for } x \in [1,\infty)$$

$$\lim_{n\to\infty} \sin\frac{1}{n} = \sin 0 = 0 \Rightarrow \text{convergent}$$

d) $\sum_{n=1}^{\infty}(-1)^n \cos(\frac{1}{n})$

Again, $\lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$. Thus the series is divergent.

e) $-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n+2}(-1)^n$

$a_n = \frac{n}{n+2}$ $\qquad \lim_{n\to\infty} a_n = 1$ Thus the series is divergent

f) $-\frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n+2}(-1)^n$

$a_n = \frac{1}{n+2} > 0$ $\qquad a_n$ is decreasing

and $\lim_{n\to\infty} a_n = 0$. Thus the series is convergent.