MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 **MATHEMATICS 1001** **March 24, 2003**

NAME:	**Lab Section:**

Marks

1. A bottle of good Canadian beer is placed into a fridge. The temperature inside the fridge is +3 C. After one hour the beer has temperature +11 C, and after two hours it has temperature +7 C and is consumed with a great pleasure. What was the initial temperature of the beer if it cools down according to the Newton's law $T(t) = T_m + (T_0 - T_m)e^{kt}$.

[5] (a) Set up equations

Solution. The temperature in the fridge $T_m = 3$. After 1 hour the temperature of the beer is

$$T(1) = 3 + (x-3)e^k = 11,$$

where the initial temperature $T_0 = x$ is unknown. Atter two hours we have

$$T(2) = 3 + (x-3)e^{2k} = 7.$$

Therefore we have 2 equations and 2 unknowns x and k.

[5] (b) solve the equations

Solution. From the first equation we obtain $e^k = 8/(x-3)$. So, $e^{2k} = 64/(x-3)^2$. Substituting this into the second equation we get $64/(x-3) = 4$. Therefore $x - 3 = 16$, and $x = 19$.

Answer: The initial temperature of the beer was 19 C. No wonder it was not consumed at that time.

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2. Evaluate **any three** of the following four integrals. Check your answer by differentiation.

[10] (a) $\int \ln(1000x)\,dx$

Solution. By parts. $u = \ln(1000x)$, $u' = 1/x$, $v' = 1$, $v = x$.

$$\int \ln(1000x)\,dx = x\ln(1000x) - \int x \cdot \frac{1}{x}\,dx = x\ln(1000x) - x + C$$

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[10] (b) $\int \frac{x-3}{(x-1)^2+4}\,dx$

Solution. Substitution $u = x - 1, du = dx$ leads to

$$\int \frac{u-2}{u^2+4}\,dx = \int \frac{u}{u^2+4}\,dx - 2\int \frac{1}{u^2+4}\,dx$$

The second integral gives us $\int \frac{1}{u^2+4}\,dx = \frac{1}{2}\arctan\frac{u}{2}$. One more substitution for the first integral $v = u^2+4, dv = 2du$ gives $\int \frac{u}{u^2+4}\,dx = \frac{1}{2}\int \frac{1}{v}dv = \ln|v| = \ln|u^2+4|$. After back substitution we finaly get

$$\int \frac{x-3}{(x-1)^2+4}\,dx = \frac{1}{2}\ln|(x-1)^2+4| - \arctan\frac{x-1}{2} + C$$

[10] (c) $\int x^2 \cos(2x)\, dx$

Solution. By parts twice. First, $u = x^2, u' = 2x, v' = \cos(2x), v = \frac{1}{2}\sin(2x)$. It gives

$$\int x^2 \cos(2x)\, dx = \frac{x^2 \sin(2x)}{2} - \int x \sin(2x)\, dx$$

Now take $u = x, u' = 1, v' = \sin(2x), v = -\frac{1}{2}\cos(2x)$. Then

$$\int x \sin(2x)\, dx = -\frac{x\cos(2x)}{2} + \frac{1}{2}\int \cos(2x)\, dx = -\frac{x\cos(2x)}{2} + \frac{1}{4}\sin(2x)$$

Finaly we obtain

$$\int x^2 \cos(2x)\, dx = \frac{x^2 \sin(2x)}{2} + \frac{x\cos(2x)}{2} - \frac{1}{4}\sin(2x) + C$$

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[10] (d) $\int \frac{1+4x^2}{x\sqrt{x^2-1}}\, dx$

Solution. Rewrite as a sum of two integrals

$$\int \frac{1+4x^2}{x\sqrt{x^2-1}}\, dx = \int \frac{1}{x\sqrt{x^2-1}}\, dx + 4\int \frac{x^2}{x\sqrt{x^2-1}}\, dx$$

The first integral gives us $\int \frac{1}{x\sqrt{x^2-1}}\, dx = \operatorname{arcsec}|x|$. The second, after cancelation x and substitution $u = x^2 - 1$ becomes

$$\int \frac{x^2}{x\sqrt{x^2-1}}\, dx = \int \frac{x}{\sqrt{x^2-1}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = u^{1/2} = \sqrt{x^2-1}.$$

Finaly,

$$\int \frac{1+4x^2}{x\sqrt{x^2-1}}\, dx = \operatorname{arcsec}|x| + 4\sqrt{x^2-1}.$$

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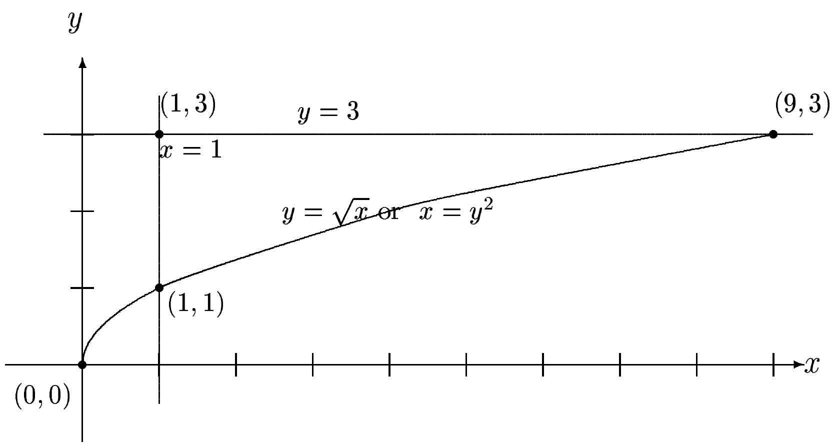
[5] 3. (a) Sketch the region bounded by $y = \sqrt{x}$, $x = 1$ and $y = 3$.
Mark all curves and points of intersection. Double check yourself because it is critical for the whole problem.



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[10] (b) Find the area of the region by two different ways: (1) integrating w.r.t x; (2) integrating w.r.t y
Solution.
(1) $V = \int_1^9 3 - \sqrt{x}\, dx = 20/3$.
(2) $V = \int_1^3 y^2 - 1\, dy = 20/3$.

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[10] (c) Find the volume of the solid generated by revolving the region about x-axis using the disk/washer method
Solution. Outer radius $R = 3$. Inner radius $r = \sqrt{x}$.

$$V = \pi \int_1^9 (9 - x)\, dx = 32\pi$$

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[10] (d) Find the volume of the solid generated by revolving the region about x-axis using the cylindrical shell method. Compare to your result in (c).
Solution. Radius $p(y) = y$, hight $h(y) = y^2 - 1$.

$$V = 2\pi \int_1^3 y(y^2 - 1)\, dy = 32\pi.$$

Bonus Problem (optional) Show that

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|)$$

Solution. Differeniate the RHS using the Product rule and the Chain rule.

$$\frac{d}{dx}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|) = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} + \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

now simplify the last summand

$$= \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} + \frac{1}{x+\sqrt{1+x^2}} \cdot \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right) = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}$$

and now make common denominator and cancel like terms to get

$$= \frac{(1+x^2)+x^2+1}{\sqrt{1+x^2}} = \frac{2(1+x^2)}{\sqrt{1+x^2}} = \sqrt{1+x^2}.$$

We have shown that

$$\frac{d}{dx}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|) = \sqrt{1+x^2}.$$

Therefore

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|).$$

P.S. This formula ulternatively could be derived using integration by trig substitution $x = \tan\Theta, dx = \sec^2\Theta d\Theta$. Try it.