MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Am I ready for Final? **Mathematics 1001** **Winter 2003**

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Final Exam will be on Wed Apr 16 from 3 pm until 5:30 pm in SN 3042

Please, have a picture ID with you.

Exam covers topics (4.1 – 4.5, 4.7 – 4.8, 5.1, 6.1 – 6.4, 7.1 – 7.5, 7.7 – 7.8):

I Applications of integration.

1. Find area between two curves. Sketch the region between two curves given by their equations $y = f(x)$, $y = g(x)$; find points of intersection a, b of the curves by solving $f(x) = g(x)$; set up integral defining the area:

$$A = \int_a^b f(x) - g(x) dx, \quad \text{for } f(x) > g(x) \text{ on the interval } (a, b).$$

2. Find volume of the solid of revolution: disk/washer and shell methods. Sketch the region between two curves given by their equations; sketch solid of revolution of the region about a given line; determine inner r and outer R raduis of the solid (the washer method), set up integral defining the volume:

$$V = \pi \int_a^b (R(x))^2 - (r(x))^2 \, dx, \qquad \text{for horizontal axis of revolution, } y = k,$$

or

$$V = \pi \int_c^d (R(y))^2 - (r(y))^2 \, dx, \qquad \text{for vertical exis of revolution } x = k.$$

Determine radius p and height h of the cylindrical shell (the shell method), set up integral defining the volume

$$V = 2\pi \int_a^b p(x)h(x) \, dx, \qquad \text{for vertical exis of revolution,}$$

or

$$V = 2\pi \int_a^b p(y)h(y) \, dx, \qquad \text{for horizontal axis of revolution.}$$

3*. Find length of a curve given by equation $y = f(x)$, $a \le x \le b$:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$

4*. Find surface area of a solid of revolution (curve $y = f(x)$, $a \le x \le b$ revolves about the x-axis):

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

5*. Find the original function if its rate of change is given. For example, find velocity $v(t)$ if acceleration $a(t)$ is given; find displacement $s(t)$ if the velocity $v(t)$ is given; Find total displacement s if motion with velocity $v(t)$ took place from time $t = a$ until $t = b$:

$$v(t) = \int a(t)\, dt, \quad s(t) = \int v(t)\, dt \quad s = \int_a^b v(t)\, dt.$$

6*. Find the average value $\bar{f}$ of a function $f(x)$ on a segment $[a, b]$:

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

7*. Solve differential equation $y' = f(x)$ or $y' = ky$ i.e. find $y(x)$ if function $f(x)$ or number k is given and $y(0) = y_0$:

$$y(x) = y_0 + \int_0^x f(t)dt \quad \text{or} \quad y(x) = y_0 e^{kx} \quad \text{correspondingly.}$$

II Integration techniques.

Strategy: ginen an integral reduce it to one of the *basic forms* using one of the following methods:

1. Substitution method $u = g(x)$, $du = g'(x)\, dx$ for $\int F(g(x))g'(x)\, dx$

2. Integration by parts $\int u dv = u(x)v(x) - \int v\, du$ (for integrals like $\int x^n \sin(ax)\, dx$, $\int x^n e^{ax}\, dx$, $\int e^{bx} \sin(ax)\, dx$, $\int x^n \ln(ax)\, dx$, $\int x^n \arcsin(ax)\, dx$.)

3. Trigonometric substitution:

   a) for $\sqrt{a^2 - x^2}$ use $x = a\sin\Theta$, $dx = a\cos\Theta d\Theta$

   b) for $\sqrt{a^2 + x^2}$ use $x = a\tan\Theta$, $dx = a\sec^2\Theta d\Theta$

   c) for $\sqrt{x^2 - a^2}$ use $x = a\sec\Theta$, $dx = a\sec\Theta\tan\Theta d\Theta$

   Complete square before doing the substitution, if necessary:

   $x^2 + bx + c = x^2 + 2(b/2)x + (b/2)^2 + c - (b/2)^2 = (x + (b/2))^2 + (c - b^2/4)$

4. Trigonometric integrals:

   a) $\int \sin^m x \cos^n x\, dx$

– if n or m is odd use substitution $u = \sin x$ or $u = \cos x$

– if n and m both are even use $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$.

– if $m = 0$ use reduction formula obtained by integration by parts.

$\int \cos^n x\, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx$

– if $n = 0$ use reduction formula obtained by integration by parts.

$\int \sin^n x\, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$

b) $\int \sec^m x \tan^n x\, dx$

– if m is even use $u = \tan x$, $du = \sec^2 x dx$

– if n is odd use $u = \sec x$, $du = \sec x \tan x dx$

– if $m = 0$ use reduction formula based on $\tan^n x = \tan^{n-2} x\,(1 - \sec^2 x)$

– if $n = 0$ use reduction formula obtained by integration by parts.

$\int \sec^n x\, dx = \frac{1}{n-1} \sec^{n-2} x \tan x - \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$

c) $\int \sin(ax)\sin(bx)dx = \frac{1}{2}(\int \cos((a-b)x) - \cos((a+b)x)\, dx)$

$\int \cos(ax)\cos(bx)dx = \frac{1}{2}(\int \cos((a-b)x) + \cos((a+b)x)\, dx)$

$\int \sin(ax)\cos(bx)dx = \frac{1}{2}(\int \sin((a-b)x) + \sin((a+b)x)\, dx)$

5. Partial Fractions: for $\int \frac{P_n(x)}{P_m(x)}\, dx$, where $P_n(x) = a_n x^n + a_{n-1} x^{n_1} + \cdots + a_1 x + a_0$ is a polynomial of degree n.

Step 1: If $n \geq m$ do long division.

Step 2: If $n \leq m$, factor the denominator $P_m(x)$ into linear $(x-a)$ and **irreducable** quadratic $(ax^2 + bx + c)$ terms

Step 3: for each linear factor $(x-a)^r$, $r \geq 1$ write exactly r partial fractions

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r},$$

for each irreducable quadratic factor $(ax^2 + bx + c)^s$, $s \geq 1$ write exactly s partial fractions

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_s x + B_s}{(ax^2 + bx + c)^s},$$

Set the sum of all the terms equal to $\frac{P_n(x)}{P_m(x)}$, and find unknows coefficients.

Step 4: Integrate each of the partial fractions. Remember that

$$\int \frac{1}{x-a} = \ln|x-a|, \quad \int \frac{1}{(x-a)^r} = \frac{1}{(1-r)(x-a)^{r-1}}, r \neq 1.$$

6. One more useful rule:

$$\text{If} \quad \int f(x)\,dx = g(x) \quad \text{then} \quad \int f(ax)\,dx = \frac{1}{a}g(ax).$$

7. Basic integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1},\; n \neq -1 \qquad \int a^x\,dx = \frac{1}{\ln a}a^x$$

$$\int \sin x\,dx = -\cos x \qquad \int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x \qquad \int \csc^2 x\,dx = -\cot x$$

$$\int \sec x \tan x\,dx = \sec x \qquad \int \csc x \cot x\,dx = -\csc x$$

$$\int \tan x\,dx = -\ln|\cos x| \qquad \int \cot x\,dx = \ln|\sin x|$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| \qquad \int \csc x\,dx = -\ln|\csc x + \cot x|$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin(x/a) \qquad \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan(x/a)$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\,\text{arcsec}\,(|x|/a) \qquad \int \frac{1}{x} = \ln|x|$$

III Definite Integral: definition, propeties and generalization.

1. Definition: for a finite interval $[a, b]$ and a continuous function $f(x)$ on it the Definite Integral is the number obtained by:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \qquad \Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x.$$

Also, remember the following summation formulas:

$$\sum_{i=1}^{n} 1 = n, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

2. Properties:

1) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

2) $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

3) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k =const.

4) If $f(x) \leq g(x)$ on $[a, b]$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

5) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Consequently, $\int_a^a f(x)\,dx = 0$.

6) The Fundamental Theorem (part 1)

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{where } F'(x) = f(x).$$

7) The Fundamental Theorem (part 2)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \quad \text{Therefore} \quad \frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x).$$

8) Mean value theorem (only for continuous functions): There exists a point c on the interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

3. Generalization: Improper Integrals.

1) type 1: infinite interval of integration $[a,b]$ (function $f(x)$ must be continuous on it)

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \quad \text{or} \quad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

2) type 2: infinite discontinuity of the function $f(x)$ at a or at b (function $f(x)$ must be continuous at all other points of $[a,b]$ except only one)

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx \quad \text{(discontinuity at the right end } b\text{)},$$

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx \quad \text{(discontinuity at the left end } a\text{)}.$$

3) mixed type: split the integral into the sum of integrals of either type 1 or type 2, and evaluate each separatly. If one of the limits is equal to infinity, we say that the integral is divergent, otherwise it converges.

4. L'Hospital's Rule for limits:

Let $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ (both limits are zero) **or** $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ (both limits are infinity). Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

To evaluate limit of the form $\lim_{x\to a} f(x)^{g(x)}$ use ln function: $\ln f(x)^{g(x)} = g(x)\ln(f(x))$.

VI Differentiation Rules.

1) Chain Rule $(f(g(x)))' = f'(g(x)) \cdot g'(x)$.

2) Product Rule $(fg)' = f'g + fg'$.

3) Quatient Rule $(\frac{f}{g})' = \frac{f'g-fg'}{g^2}$.

4) The Fundamental Theorem (part 2): see above.

5) Use of ln to find derivative of $h(x) = f(x)^{g(x)}$: $\ln h(x) = g(x)\ln(f(x))$, so

$$\frac{h'(x)}{h(x)} = g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)} \quad \text{and} \quad h'(x) = h(x)\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right).$$

6) Implicit differentiation: find $y'(x)$ if $y(x)$ is given implicitly by algebraic equation $F(x, y) = c$. (Differentiate the equation w.r.t. x treating y as a function of x and using the chain rule; then solve for $y'(x)$).

7) Basic derivatives:

$$\frac{d}{dx} x^n = nx^{n-1} \qquad \frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} a^x = (\ln a)\, a^x$$

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \sec x = \sec x \tan x \qquad \frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \tan x = \sec^2 x \qquad \frac{d}{dx} \cot x = -\cot^2 x$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx} \text{arccot}\, x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx} \text{arcsec}\, x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx} \text{arccsc}\, x = -\frac{1}{|x|\sqrt{x^2-1}}$$

Please, make sure that you understand and **can do** all the problems from test 1 and test 2, as well as from your homework assignments and the last year final exam.

Send me an e-mail if you need something extra or if you have a question : mkondra@math.mun.ca

For more practice try drills on the web site: http://archives.math.utk.edu/visual.calculus/

Good luck!