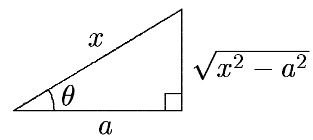
Math 1001 **Solutions** **Assignment #9**

1. (a) $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx$

$$\begin{aligned} x &= a\sec\theta \\ dx &= a\sec\theta\tan\theta d\theta \\ \sqrt{x^2-a^2} &= a\tan\theta \end{aligned}$$

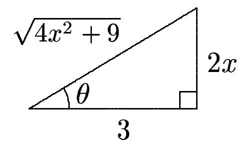


$$= \int \frac{a\tan\theta}{(a\sec\theta)^4}\cdot a\sec\theta\tan\theta\,d\theta = \frac{1}{a^2}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{a^2}\int \sin^2\theta\cos\theta\,d\theta \qquad \begin{aligned} u &= \sin\theta \\ du &= \cos\theta\,d\theta \end{aligned}$$

$$= \frac{1}{a^2}\int u^2\,du = \frac{1}{a^2}\cdot\frac{u^3}{3} + C = \frac{1}{3a^2}\sin^3\theta + C = \frac{1}{3a^2}\left(\frac{\sqrt{x^2-a^2}}{x}\right)^3 + C = \frac{(x^2-a^2)^{\frac{3}{2}}}{3a^2x^3} + C$$

(b) $\displaystyle\int \frac{1}{x\sqrt{4x^2+9}}\,dx$

$$\begin{aligned} 2x &= 3\tan\theta \\ 2dx &= 3\sec^2\theta\,d\theta \\ \sqrt{4x^2+9} &= 3\sec\theta \end{aligned}$$

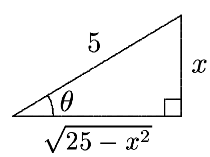


$$= \int \frac{1}{\frac{3}{2}\tan\theta\cdot 3\sec\theta}\cdot\frac{3}{2}\sec^2\,d\theta = \frac{1}{3}\int\frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{3}\int\frac{1}{\sin\theta}\,d\theta = \frac{1}{3}\int\csc\theta\,d\theta$$

$$= -\frac{1}{3}\ln|\csc\theta-\cot\theta| + C = -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}}{2x} - \frac{3}{2x}\right| + C = -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}-3}{2x}\right| + C$$

(c) $\displaystyle\int \frac{x^2}{\sqrt{25-x^2}}\,dx$

$$\begin{aligned} x &= 5\sin\theta \\ dx &= 5\cos\theta\,d\theta \\ \sqrt{25-x^2} &= 5\cos\theta \end{aligned}$$

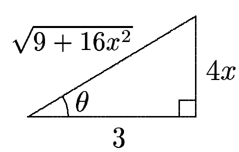


$$= \int\frac{(5\sin\theta)^2}{5\cos\theta}\cdot 5\cos\theta\,d\theta = 25\int\sin^2\theta\,d\theta = 25\int\frac{1-\cos2\theta}{2}\,d\theta = \frac{25}{2}\left(\theta - \frac{1}{2}\sin2\theta\right) + C$$

$$= \frac{25}{2}(\theta - \sin\theta\cos\theta) = \frac{25}{2}\left(\sin^{-1}\frac{x}{5} - \frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5}\right) + C = \frac{25}{2}\sin^{-1}\frac{x}{5} - \frac{1}{2}x\sqrt{25-x^2} + C$$

(d) $\displaystyle\int \frac{1}{x^2\sqrt{9+16x^2}}\,dx$

$$\begin{aligned} 4x &= 3\tan\theta \\ 4dx &= 3\sec^2\theta\,d\theta \\ \sqrt{9+16x^2} &= 3\sec\theta \end{aligned}$$

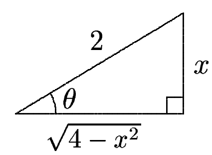


$$= \int\frac{1}{(\frac{3}{4}\tan\theta)^2\cdot 3\sec\theta}\cdot\frac{3}{4}\sec^2\theta\,d\theta = \frac{4}{9}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{4}{9}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta \qquad \begin{aligned} u &= \sin\theta \\ du &= \cos\theta\,d\theta \end{aligned}$$

$$= \frac{4}{9}\int\frac{1}{u^2}\,du = -\frac{4}{9u} + C = -\frac{4}{9\sin\theta} + C = -\frac{4}{9}\csc\theta + C = -\frac{4}{9}\cdot\frac{\sqrt{9+16x^2}}{4x} + C = -\frac{\sqrt{9+16x^2}}{9x} + C$$

(e) $\displaystyle\int_0^2 x^3\sqrt{4-x^2}\,dx$

$$\begin{aligned} x &= 2\sin\theta & x=2 &\Rightarrow \theta=\tfrac{\pi}{2} \\ dx &= 2\cos\theta\,d\theta & x=0 &\Rightarrow \theta=0 \\ \sqrt{4-x^2} &= 2\cos\theta \end{aligned}$$



$$= \int_0^{\frac{\pi}{2}} (2\sin\theta)^3\cdot 2\cos\theta\cdot 2\cos\theta\,d\theta = 32\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,d\theta = 32\int_0^{\frac{\pi}{2}} \sin^2\theta\cdot\cos^2\theta\cdot\sin\theta\,d\theta$$

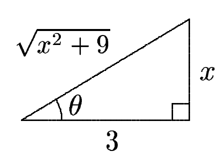
$$= 32\int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\cos^2\theta\cdot\sin\theta\,d\theta$$

$$\begin{aligned} u &= \cos\theta & \theta=\tfrac{\pi}{2} &\Rightarrow u=0 \\ du &= -\sin\theta\,d\theta & \theta=0 &\Rightarrow u=1 \end{aligned}$$

$$= -32\int_1^0 (1-u^2)u^2\,du = -32\int_1^0 (u^2-u^4)\,du = -32\left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Bigg|_1^0 = -32\left[0-\left(\frac{1}{3}-\frac{1}{5}\right)\right] = \frac{64}{15}$$

(f) $\displaystyle\int_0^{3\sqrt{3}} \frac{x^3}{\sqrt{x^2+9}}\,dx$

$$\begin{aligned} x &= 3\tan\theta & x=3\sqrt{3} &\Rightarrow \theta=\tfrac{\pi}{3} \\ dx &= 3\sec^2\theta\,d\theta & x=0 &\Rightarrow \theta=0 \\ \sqrt{x^2+9} &= 3\sec\theta \end{aligned}$$



$$= \int_0^{\frac{\pi}{3}} \frac{(3\tan\theta)^3}{3\sec\theta}\cdot 3\sec^2\theta\,d\theta = 27\int_0^{\frac{\pi}{3}} \tan^3\theta\sec\theta\,d\theta = 27\int_0^{\frac{\pi}{3}} \tan^2\theta\cdot\sec\theta\tan\theta\,d\theta$$

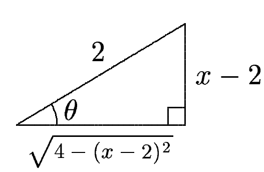
$$= 27\int_0^{\frac{\pi}{3}} (\sec^2\theta-1)\sec\theta\tan\theta\,d\theta$$

$$\begin{aligned} u &= \sec\theta & \theta=\tfrac{\pi}{3} &\Rightarrow u=2 \\ du &= \sec\theta\tan\theta\,d\theta & \theta=0 &\Rightarrow u=1 \end{aligned}$$

$$= 27\int_1^2 (u^2-1)\,du = 27\left(\frac{u^3}{3}-u\right)\Bigg|_1^2 = 27\left[\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right] = 27\left(\frac{4}{3}\right) = 36$$

(g) $\displaystyle\int \frac{x^2}{\sqrt{4x-x^2}}\,dx = \int \frac{x^2}{\sqrt{4-(x-2)^2}}\,dx$

$$\begin{aligned} x-2 &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \\ \sqrt{4-(x-2)^2} &= 2\cos\theta \end{aligned}$$



$$= \int \frac{(2+2\sin\theta)^2}{2\cos\theta}\cdot 2\cos\theta\,d\theta = \int (2+2\sin\theta)^2\,d\theta = \int (4+8\sin\theta+4\sin^2\theta)\,d\theta$$

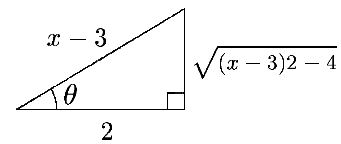
$$= \int [4+8\sin\theta+2(1-\cos 2\theta)]\,d\theta = \int (6+8\sin\theta-2\cos 2\theta)\,d\theta = 6\theta-8\cos\theta-\sin 2\theta+C$$

$$= 6\theta-8\cos\theta-2\sin\theta\cos\theta+C = 6\cdot\sin^{-1}\frac{x-2}{2}-8\cdot\frac{\sqrt{4x-x^2}}{2}-2\cdot\frac{x-2}{2}\cdot\frac{\sqrt{4x-x^2}}{2}+C$$

$$= 6\sin^{-1}\frac{x-2}{2}-4\sqrt{4x-x^2}-\frac{1}{2}(x-2)\sqrt{4x-x^2}+C$$

(h) $\displaystyle\int \frac{x}{\sqrt{x^2-6x+5}}\,dx = \int \frac{x}{\sqrt{(x-3)^2-4}}\,dx$

$$\begin{aligned} x-3 &= 2\sec\theta \\ dx &= 2\sec\theta\tan\theta\,d\theta \\ \sqrt{(x-3)^2-4} &= 2\tan\theta \end{aligned}$$



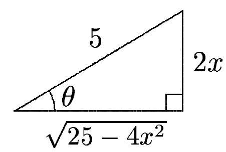
$$= \int \frac{3+2\sec\theta}{2\tan\theta}\cdot 2\sec\theta\tan\theta\,d\theta = \int (3\sec\theta + 2\sec^2\theta)\,d\theta$$

$$= 3\ln\left|\sec\theta+\tan\theta\right| + 2\tan\theta + C = 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{x^2-6x+5}}{2}\right| + 2\cdot\frac{\sqrt{x^2-6x+5}}{2} + C$$

$$= 3\ln\left|\frac{x-3+\sqrt{x^2-6x+5}}{2}\right| + \sqrt{x^2-6x+5} + C$$

(i) $\displaystyle\int \frac{1}{(25-4x^2)^{\frac{3}{2}}}\,dx$

$$\begin{aligned} 2x &= 5\sin\theta \\ 2dx &= 5\cos\theta\,d\theta \\ \sqrt{25-4x^2} &= 5\cos\theta \end{aligned}$$

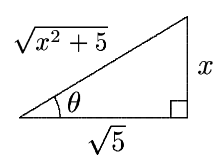


$$= \int \frac{1}{(5\cos\theta)^3}\cdot\frac{5}{2}\cos\theta\,d\theta = \frac{1}{50}\int\frac{1}{\cos^2\theta}\,d\theta = \frac{1}{50}\int\sec^2\theta\,d\theta = \frac{1}{50}\tan\theta + C$$

$$= \frac{1}{50}\cdot\frac{2x}{\sqrt{25-4x^2}} + C = \frac{x}{25\sqrt{25-4x^2}} + C$$

(j) $\displaystyle\int \frac{1}{x^4+10x^2+25}\,dx = \int \frac{1}{(x^2+5)^2}\,dx$

$$\begin{aligned} x &= \sqrt{5}\,\tan\theta \\ dx &= \sqrt{5}\sec^2\theta\,d\theta \\ \sqrt{x^2+5} &= \sqrt{5}\,\sec\theta \end{aligned}$$



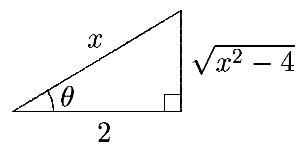
$$= \int \frac{1}{(\sqrt{5}\,\sec\theta)^4}\cdot\sqrt{5}\,\sec^2\theta\,d\theta = \frac{\sqrt{5}}{25}\int\frac{1}{\sec^2\theta}\,d\theta = \frac{\sqrt{5}}{25}\int\cos^2\theta\,d\theta = \frac{\sqrt{5}}{25}\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{\sqrt{5}}{25}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C = \frac{\sqrt{5}}{25}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C$$

$$= \frac{\sqrt{5}}{25}\left(\frac{1}{2}\tan^{-1}\frac{x}{\sqrt{5}} + \frac{1}{2}\cdot\frac{x}{\sqrt{x^2+5}}\cdot\frac{\sqrt{5}}{\sqrt{x^2+5}}\right) + C = \frac{\sqrt{5}}{50}\tan^{-1}\frac{x}{\sqrt{5}} + \frac{x}{10(x^2+5)} + C$$

2. (a) $\int \frac{x^3}{\sqrt{x^2-4}}\,dx$

$$\begin{aligned} x &= 2\sec\theta \\ dx &= 2\sec\theta\tan\theta d\theta \\ \sqrt{x^2-4} &= 2\tan\theta \end{aligned}$$



$$= \int \frac{(2\sec\theta)^3}{2\tan\theta}\cdot 2\sec\theta\tan\theta\,d\theta = 8\int \sec^4\theta\,d\theta = 8\int \sec^2\theta\sec^2\theta\,d\theta$$

$$= 8\int (1+\tan^2\theta)\sec^2\theta\,d\theta \qquad \begin{aligned} u &= \tan\theta \\ du &= \sec^2\theta\,d\theta \end{aligned}$$

$$= 8\int (1+u^2)du = 8\left(u+\frac{u^3}{3}\right)+C = 8\left(\tan\theta+\frac{\tan^3\theta}{3}\right)+C = 8\left[\frac{\sqrt{x^2-4}}{2}+\frac{1}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)^3\right]+\mathrm{C}$$

$$= 4\sqrt{x^2-4}+\frac{1}{3}(x^2-4)^{3/2}+C$$

(b) $\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{x^2}{\sqrt{x^2-4}}\cdot x\,dx$

$$\begin{aligned} u &= x^2-4 \quad \Rightarrow \quad x^2 = u+4 \\ du &= 2x\,dx \end{aligned}$$

$$= \frac{1}{2}\int \frac{u+4}{\sqrt{u}}\,du = \frac{1}{2}\int \left(u^{\frac{1}{2}}+4u^{-\frac{1}{2}}\right)\,du = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+\frac{4u^{\frac{1}{2}}}{\frac{1}{2}}\right]+C = \frac{1}{3}u^{\frac{3}{2}}+4u^{\frac{1}{2}}+C$$

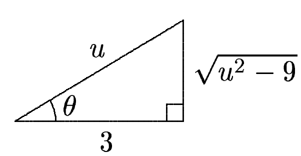
$$= \frac{1}{3}(x^2-4)^{\frac{3}{2}}+4(x^2-4)^{\frac{1}{2}}+C$$

3. $\int \sqrt{e^{2t}-9}\,dt = \int \sqrt{(e^t)^2-9}\,dt$

$$\begin{aligned} u &= e^t \\ du &= e^t dt \quad \Rightarrow \quad dt = \frac{1}{e^t}\,du = \frac{1}{u}\,du \end{aligned}$$

$$= \int \sqrt{u^2-9}\cdot\frac{1}{u}\,du = \int \frac{\sqrt{u^2-9}}{u}\,du$$

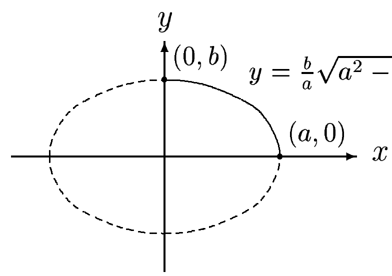
$$\begin{aligned} u &= 3\sec\theta \\ du &= 3\sec\theta\tan\theta d\theta \\ \sqrt{u^2-9} &= 3\tan\theta \end{aligned}$$



$$= \int \frac{3\tan\theta}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta = 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta-1)\,d\theta = 3(\tan\theta-\theta)+C$$

$$= 3\cdot\frac{\sqrt{u^2-9}}{3}-3\sec^{-1}\frac{u}{3}+C = \sqrt{e^{2t}-9}-3\sec^{-1}\frac{e^t}{3}+C$$

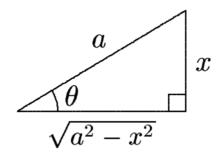
4. We solve for y to get the top half of the ellipse as $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$.



The area is four times the area in the first quadrant. So

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$\begin{aligned} x &= a\sin\theta \\ dx &= a\cos\theta\,d\theta \\ \sqrt{a^2 - x^2} &= a\cos\theta \end{aligned} \qquad \begin{aligned} x = a &\Rightarrow \theta = \tfrac{\pi}{2} \\ x = 0 &\Rightarrow \theta = 0 \end{aligned}$$



$$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\,d\theta = 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 4ab\int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\,d\theta$$

$$= 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]\Bigg|_0^{\frac{\pi}{2}} = 2ab\left[\left(\frac{\pi}{2} + 0\right) - 0\right] = \pi ab$$