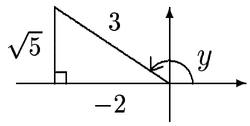
1. (a) $\sin\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$

Let $y = \cos^{-1}\left(-\frac{2}{3}\right)$

$\cos y = -\frac{2}{3}$, find $\sin y$

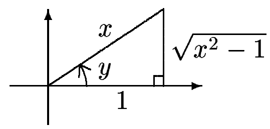


$$\sin\left(\cos^{-1}\left(-\tfrac{2}{3}\right)\right) = \sin y = \frac{\sqrt{5}}{3}$$

(b) $\tan(\sec^{-1} x)$

Let $y = \sec^{-1} x$

$\sec y = x$, find $\tan y$

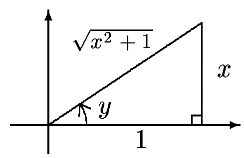


$$\tan(\sec^{-1} x) = \tan y = \sqrt{x^2-1}$$

(c) $\sin(2\tan^{-1} x)$

Let $y = \tan^{-1} x$

$\tan y = x$, find $\sin 2y$

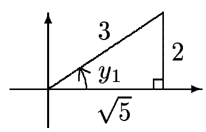


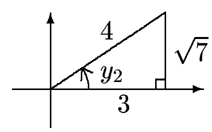
$$\sin(2\tan^{-1} x) = \sin 2y = 2\sin y\cos y = 2\cdot\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} = \frac{2x}{x^2+1}$$

(d) $\cos\left(\sin^{-1}\frac{2}{3} + \cos^{-1}\frac{3}{4}\right)$

Let $y_1 = \sin^{-1}\frac{2}{3}$ and $y_2 = \cos^{-1}\frac{3}{4}$

$\sin y_1 = \frac{2}{3}$ and $\cos y_2 = \frac{3}{4}$, find $\cos(y_1+y_2)$





$$\cos(y_1+y_2) = \cos y_1\cos y_2 - \sin y_1\sin y_2 = \frac{\sqrt{5}}{3}\cdot\frac{3}{4} - \frac{2}{3}\cdot\frac{\sqrt{7}}{4} = \frac{3\sqrt{5}-2\sqrt{7}}{12}$$

2. (a) $f(x) = \dfrac{1}{4}\sin^{-1}\left(\dfrac{4}{x^2}\right)$

$$f'(x) = \frac{1}{4}\cdot\frac{1}{\sqrt{1-(\frac{4}{x^2})^2}}\cdot\frac{d}{dx}\left(\frac{4}{x^2}\right) = \frac{1}{4}\cdot\frac{1}{\sqrt{1-\frac{16}{x^4}}}\cdot\left(-\frac{8}{x^3}\right) = \frac{1}{4}\cdot\frac{x^2}{\sqrt{x^4-16}}\cdot\left(-\frac{8}{x^3}\right) = \frac{-2}{x\sqrt{x^4-16}}$$

(b) $f(x) = \sec^{-1}(\sqrt{x^4+1}\,)$

$$f'(x) = \frac{1}{|\sqrt{x^4+1}|\sqrt{(\sqrt{x^4+1}\,)^2-1}}\cdot\frac{1}{2\sqrt{x^4+1}}\cdot 4x^3 = \frac{1}{\sqrt{x^4+1}\cdot x^2}\cdot\frac{2x^3}{\sqrt{x^4+1}} = \frac{2x}{x^4+1}$$

(c) $f(x) = \tan^{-1}(\sqrt{e^{2x}-1}\,)$

$$f'(x) = \frac{1}{1+(\sqrt{e^{2x}-1})^2}\cdot\frac{1}{2\sqrt{e^{2x}-1}}\cdot 2e^{2x} = \frac{1}{1+e^{2x}-1}\cdot\frac{e^{2x}}{\sqrt{e^{2x}-1}} = \frac{1}{e^{2x}}\cdot\frac{e^{2x}}{\sqrt{e^{2x}-1}} = \frac{1}{\sqrt{e^{2x}-1}}$$

(d) $f(x) = \tan^{-1}\left(\dfrac{x}{a}\right) + \tan^{-1}\left(\dfrac{a}{x}\right)$

$$f'(x) = \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} + \frac{1}{1+\left(\frac{a}{x}\right)^2}\cdot\left(-\frac{a}{x^2}\right) = \frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a} + \frac{1}{1+\frac{a^2}{x^2}}\cdot\left(-\frac{a}{x^2}\right) = \frac{a}{a^2+x^2} - \frac{a}{x^2+a^2} = 0$$

(e) $f(x) = \sin^{-1}\left(\frac{1}{\sqrt{x}}\right) + \sec^{-1}\sqrt{x}$

$$f'(x) = \frac{1}{\sqrt{1-(\frac{1}{\sqrt{x}})^2}} \cdot \frac{d}{dx}(x^{-\frac{1}{2}}) + \frac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2-1}} \cdot \frac{d}{dx}(\sqrt{x}) = \frac{1}{\sqrt{1-\frac{1}{x}}} \cdot \left(\frac{-1}{2x^{\frac{3}{2}}}\right) + \frac{1}{\sqrt{x}\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\sqrt{x}}{\sqrt{x-1}} \cdot \left(\frac{-1}{2x\sqrt{x}}\right) + \frac{1}{2x\sqrt{x-1}} = \frac{-1}{2x\sqrt{x-1}} + \frac{1}{2x\sqrt{x-1}} = 0$$

3. (a) $\int \frac{1}{16x^2+9}\,dx = \int \frac{1}{9+(4x)^2}\,dx = \frac{1}{4}\cdot\frac{1}{3}\tan^{-1}\frac{4x}{3} + C = \frac{1}{12}\tan^{-1}\frac{4x}{3} + C$

(b) $\int_0^{\frac{\sqrt{3}}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx = \int_0^{\frac{\sqrt{3}}{3}} \frac{1}{\sqrt{4-(3x)^2}}\,dx = \frac{1}{3}\sin^{-1}\frac{3x}{2}\Big|_0^{\frac{\sqrt{3}}{3}} = \frac{1}{3}\left(\frac{\pi}{3}-0\right) = \frac{\pi}{9}$

(c) $\int_1^{\sqrt{2}} \frac{1}{x\sqrt{2x^2-1}}\,dx = \int_1^{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}\,x\sqrt{(\sqrt{2}\,x)^2-1}}\,dx$ $\qquad u = \sqrt{2}\,x$, $du = \sqrt{2}\,dx$ $\qquad x = \sqrt{2} \Rightarrow u = 2$, $x = 1 \Rightarrow u = \sqrt{2}$

$$= \int_{\sqrt{2}}^{2} \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u|\,\Big|_{\sqrt{2}}^{2} = \sec^{-1}2 - \sec^{-1}\sqrt{2} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(d) $\int \frac{1}{x(9+4\ln^2 x)}\,dx = \int \frac{1}{x[9+(2\ln x)^2]}\,dx$ $\qquad u = 2\ln x$, $du = \frac{2}{x}dx$

$$= \frac{1}{2}\int \frac{1}{9+u^2}\,du = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\frac{u}{3} + C = \frac{1}{6}\tan^{-1}\left(\frac{2\ln x}{3}\right) + C$$

(e) $\int \frac{e^{2x}}{\sqrt{16-25e^{4x}}}\,dx = \int \frac{e^{2x}}{\sqrt{16-(5e^{2x})^2}}\,dx$ $\qquad u = 5e^{2x}$, $du = 10e^{2x}dx$

$$= \frac{1}{10}\int \frac{1}{\sqrt{16-u^2}}\,du = \frac{1}{10}\sin^{-1}\frac{u}{4} + C = \frac{1}{10}\sin^{-1}\left(\frac{5e^{2x}}{4}\right) + C$$

(f) $\int \frac{4}{\sqrt{x}\,(x+9)}\,dx = \int \frac{4}{\sqrt{x}\,[(\sqrt{x})^2+9]}\,dx$ $\qquad u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$

$$= 4\cdot 2\int \frac{1}{u^2+9}\,du = 8\cdot\frac{1}{3}\tan^{-1}\frac{u}{3} + C = \frac{8}{3}\tan^{-1}\frac{\sqrt{x}}{3} + C$$

(g) $\int \frac{x}{\sqrt{25-16x^4}}\,dx = \int \frac{x}{\sqrt{25-(4x^2)^2}}\,dx$ $\qquad u = 4x^2$, $du = 8x\,dx$

$$= \frac{1}{8}\int \frac{1}{\sqrt{25-u^2}}\,du = \frac{1}{8}\sin^{-1}\frac{u}{5} + C = \frac{1}{8}\sin^{-1}\frac{4x^2}{5} + C$$

(h) $\displaystyle\int \frac{\cos 2x}{\sqrt{9-\sin^2 2x}}\,dx$ $\qquad u = \sin 2x$, $\quad du = 2\cos 2x dx$

$$= \frac{1}{2}\int \frac{1}{\sqrt{9-u^2}}\,du = \frac{1}{2}\sin^{-1}\frac{u}{3} + C = \frac{1}{2}\sin^{-1}\left(\frac{\sin 2x}{3}\right) + C$$

(i) $\displaystyle\int_0^{\frac{\pi}{9}} \frac{\sec^2 3x}{9+\tan^2 3x}\,dx$ $\qquad u = \tan 3x$, $\quad du = 3\sec^2 3x dx$ $\qquad x = \frac{\pi}{9} \Rightarrow u = \sqrt{3}$, $\quad x = 0 \Rightarrow u = 0$

$$= \frac{1}{3}\int_0^{\sqrt{3}} \frac{1}{9+u^2}\,du = \frac{1}{3}\cdot\frac{1}{3}\tan^{-1}\frac{u}{3}\Big|_0^{\sqrt{3}} = \frac{1}{9}\left(\frac{\pi}{6}-0\right) = \frac{\pi}{54}$$

(j) $$\int_0^2 \frac{1}{\sqrt{3+2x-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{4-(x-1)^2}}\,dx = \sin^{-1}\frac{x-1}{2}\Big|_0^2 = \sin^{-1}\tfrac{1}{2} - \sin^{-1}(-\tfrac{1}{2})$$
$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$

(k) $\displaystyle\int \frac{x-4}{4x^2-4x+17}\,dx = \int \frac{x-4}{(2x-1)^2+16}\,dx$ $\qquad u = 2x-1 \;\Rightarrow\; x = \frac{u+1}{2}$, $\quad du = 2dx$

$$= \frac{1}{2}\int \frac{\frac{u+1}{2}-4}{u^2+16}\,du = \frac{1}{4}\int \frac{u-7}{u^2+16}\,du = \frac{1}{4}\int\left(\frac{u}{u^2+16} - \frac{7}{u^2+16}\right)du$$

$$= \frac{1}{4}\left[\frac{1}{2}\ln|u^2+16| - \frac{7}{4}\tan^{-1}\frac{u}{4}\right] + C = \frac{1}{8}\ln(4x^2-4x+17) - \frac{7}{16}\tan^{-1}\frac{2x-1}{4} + C$$

4. (a) $\displaystyle\int \frac{\sqrt{x-1}}{x+3}\,dx$ $\qquad u = \sqrt{x-1}$, $\quad u^2 = x-1 \;\Rightarrow\; x = u^2+1$, $\quad 2udu = dx$

$$= \int \frac{u}{(u^2+1)+3}\cdot 2udu = \int \frac{2u^2}{u^2+4}\,du = \int\left(2 - \frac{8}{u^2+4}\right)du$$

$$= 2u - 8\cdot\frac{1}{2}\tan^{-1}\frac{u}{2} + C = 2\sqrt{x-1} - 4\tan^{-1}\frac{\sqrt{x-1}}{2} + C$$

(b) $\displaystyle\int \frac{1}{x\sqrt{4x-3}}\,dx$ $\qquad u = \sqrt{4x-3}$, $\quad u^2 = 4x-3 \;\Rightarrow\; x = \frac{u^2+3}{4}$, $\quad 2udu = 4dx$

$$= \int \frac{1}{\frac{u^2+3}{4}\cdot u}\cdot\frac{1}{2}u\,du = \int \frac{2}{u^2+3}\,du = 2\cdot\frac{1}{\sqrt{3}}\cdot\tan^{-1}\frac{u}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\tan^{-1}\frac{\sqrt{4x-3}}{\sqrt{3}} + C$$