Math 1001 **Solutions** **Assignment #11**

1. (a) $\lim_{x\to 0}\dfrac{6^x-2^x}{x}=\lim_{x\to 0}\dfrac{6^x\ln 6-2^x\ln 2}{1}=\ln 6-\ln 2=\ln 3$

(b) $\lim_{x\to 0}\dfrac{\sin mx}{\sin nx}=\lim_{x\to 0}\dfrac{m\cos mx}{n\cos nx}=\dfrac{m(1)}{n(1)}=\dfrac{m}{n}$

(c) $\lim_{x\to\frac{\pi}{2}^-}(\sec 7x\cos 3x)=\lim_{x\to\frac{\pi}{2}^-}\dfrac{\cos 3x}{\cos 7x}=\lim_{x\to\frac{\pi}{2}^-}\dfrac{-3\sin 3x}{-7\sin 7x}=\dfrac{-3(-1)}{-7(-2)}=\dfrac{3}{7}$

(d) $\lim_{x\to 0}\dfrac{e^{3x}-e^{-3x}}{2x}=\lim_{x\to 0}\dfrac{3e^{3x}+3e^{-3x}}{2}=\dfrac{3+3}{2}=\dfrac{6}{2}=3$

(e) $\lim_{x\to\frac{\pi}{2}}(\tan x-\sec x)=\lim_{x\to\frac{\pi}{2}}\left(\dfrac{\sin x}{\cos x}-\dfrac{1}{\cos x}\right)=\lim_{x\to\frac{\pi}{2}}\dfrac{\sin x-1}{\cos x}=\lim_{x\to\frac{\pi}{2}}\dfrac{\cos x}{-\sin x}=\dfrac{0}{-1}=0$

(f) $\lim_{x\to 0}\dfrac{\cos mx-\cos nx}{x^2}=\lim_{x\to 0}\dfrac{-m\sin mx+n\sin nx}{2x}=\lim_{x\to 0}\dfrac{-m^2\cos mx+n^2\cos nx}{2}=\dfrac{n^2-m^2}{2}$

(g) $\lim_{x\to\infty}\dfrac{\ln^3 x}{x^2}=\lim_{x\to\infty}\dfrac{3\ln^2 x\cdot\frac{1}{x}}{2x}=\lim_{x\to\infty}\dfrac{3\ln^2 x}{2x^2}=\lim_{x\to\infty}\dfrac{6\ln x\cdot\frac{1}{x}}{4x}=\lim_{x\to\infty}\dfrac{3\ln x}{2x^2}=\lim_{x\to\infty}\dfrac{3\cdot\frac{1}{x}}{4x}$

$=\lim_{x\to\infty}\dfrac{3}{4x^2}=0$

(h) $\lim_{x\to\infty}\dfrac{x^2+1}{x\ln x}=\lim_{x\to\infty}\dfrac{2x}{x\cdot\frac{1}{x}+\ln x}=\lim_{x\to\infty}\dfrac{2x}{1+\ln x}=\lim_{x\to\infty}\dfrac{2}{\frac{1}{x}}=\lim_{x\to\infty}2x=\infty$

(i) $\lim_{x\to 0^+}\dfrac{1-\cos\sqrt{x}}{x}=\lim_{x\to 0^+}\dfrac{\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}}{1}=\lim_{x\to 0^+}\dfrac{\sin\sqrt{x}}{2\sqrt{x}}=\lim_{x\to 0^+}\dfrac{\cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}}{\frac{1}{\sqrt{x}}}=\lim_{x\to 0^+}\dfrac{\cos\sqrt{x}}{2}=\dfrac{1}{2}$

(j) $\lim_{x\to\infty}\dfrac{\ln(1+e^{2x})}{x}=\lim_{x\to\infty}\dfrac{\frac{1}{1+e^{2x}}\cdot 2e^{2x}}{1}=\lim_{x\to\infty}\dfrac{2e^{2x}}{1+e^{2x}}=\lim_{x\to\infty}\dfrac{4e^{2x}}{2e^{2x}}=\lim_{x\to\infty}2=2$

(k) $\lim_{x\to 0}\dfrac{x-\sin x}{x^3}=\lim_{x\to 0}\dfrac{1-\cos x}{3x^2}=\lim_{x\to 0}\dfrac{\sin x}{6x}=\lim_{x\to 0}\dfrac{\cos x}{6}=\dfrac{1}{6}$

(l) $\lim_{t\to 0}\dfrac{\ln(\cos 2t)}{t^2}=\lim_{t\to 0}\dfrac{\frac{1}{\cos 2t}\cdot(-2\sin 2t)}{2t}=\lim_{t\to 0}\dfrac{-\tan 2t}{t}=\lim_{t\to 0}\dfrac{-2\sec^2 2t}{1}=-2$

(m) $\lim_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)=\lim_{x\to 1}\dfrac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\dfrac{1-\frac{1}{x}}{(x-1)\cdot\frac{1}{x}+\ln x}=\lim_{x\to 1}\dfrac{x-1}{x-1+x\ln x}$

$=\lim_{x\to 1}\dfrac{1}{1+x\cdot\frac{1}{x}+\ln x}=\lim_{x\to 1}\dfrac{1}{2+\ln x}=\dfrac{1}{2}$

(n) $$\lim_{x\to\infty}\frac{\ln(x^2+1)}{\ln x}=\lim_{x\to\infty}\frac{\frac{1}{x^2+1}\cdot 2x}{\frac{1}{x}}=\lim_{x\to\infty}\frac{2x^2}{x^2+1}=2$$

(o) $\lim_{x\to 0^+}(\sin x)^{\tan x}$ (0^0 type) Let $y=\lim_{x\to 0^+}(\sin x)^{\tan x}$

$$\ln y=\lim_{x\to 0^+}\tan x\ln(\sin x)=\lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x}=\lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cdot\cos x}{-\csc^2 x}=\lim_{x\to 0^+}(-\sin x\cos x)=0$$

So $y=e^0=1$. Thus $\lim_{x\to 0^+}(\sin x)^{\tan x}=1$.

(p) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$ (1^∞ type) Let $y=\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

$$\ln y=\lim_{x\to\infty}bx\ln\left(1+\frac{a}{x}\right)=\lim_{x\to\infty}\frac{b\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{b}{1+\frac{a}{x}}\cdot\left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{ab}{1+\frac{a}{x}}=ab$$

So $y=e^{ab}$. Thus $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}$

(q) $\lim_{x\to\infty}(x+e^x)^{\frac{1}{x}}$ (∞^0 type) Let $y=\lim_{x\to\infty}(x+e^x)^{\frac{1}{x}}$

$$\ln y=\lim_{x\to\infty}\frac{1}{x}\ln(x+e^x)=\lim_{x\to\infty}\frac{\ln(x+e^x)}{x}=\lim_{x\to\infty}\frac{\frac{1}{x+e^x}\cdot(1+e^x)}{1}=\lim_{x\to\infty}\frac{1+e^x}{x+e^x}=\lim_{x\to\infty}\frac{e^x}{1+e^x}$$

$$=\lim_{x\to\infty}\frac{e^x}{e^x}=\lim_{x\to\infty}1=1$$

So $y=e^1=e$. Thus $\lim_{x\to\infty}(x+e^x)^{\frac{1}{x}}=e$

(r) $\lim_{x\to 0}(\cos 3x)^{\frac{5}{x}}$ (1^∞ type) Let $y=\lim_{x\to 0}(\cos 3x)^{\frac{5}{x}}$

$$\ln y=\lim_{x\to 0}\frac{5}{x}\ln(\cos 3x)=\lim_{x\to 0}\frac{5\ln(\cos 3x)}{x}=\lim_{x\to 0}\frac{5\cdot\frac{1}{\cos 3x}\cdot(-3\sin 3x)}{1}=\lim_{x\to 0}(-15\tan 3x)=0$$

So $y=e^0=1$. Thus $\lim_{x\to 0}(\cos 3x)^{\frac{5}{x}}=1$

2. (a) $$\int_2^\infty\frac{1}{\sqrt{4x+1}}\,dx=\lim_{t\to\infty}\int_2^\infty\frac{1}{\sqrt{4x+1}}\,dx=\lim_{t\to\infty}\left[\frac{1}{2}\sqrt{4x+1}\,\Big|_2^t\right]=\lim_{t\to\infty}\left(\frac{1}{2}\sqrt{4t+1}-\frac{3}{2}\right)=\infty$$

So the integral diverges.

(b) $$\int_0^4\frac{1}{\sqrt{4-x}}\,dx=\lim_{t\to 4^-}\int_0^t\frac{1}{\sqrt{4-x}}\,dx=\lim_{t\to 4^-}\left[-2\sqrt{4-x}\,\Big|_0^t\right]=\lim_{t\to 4^-}(-2\sqrt{4-t}+4)=4$$

So the integral converges.

(c) $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx = \lim_{t\to-\infty}\int_{t}^{0} e^{3x}\,dx = \lim_{t\to-\infty}\left[\frac{1}{3}e^{3x}\Big|_{t}^{0}\right] = \lim_{t\to-\infty}\left(\frac{1}{3}-\frac{1}{3}e^{3t}\right) = \frac{1}{3}-0 = \frac{1}{3}$

So the integral converges.

(d) $\displaystyle\int_{1}^{\infty}\frac{1}{(x+3)^{\frac{3}{2}}}\,dx = \lim_{t\to\infty}\int_{1}^{t}(x+3)^{-\frac{3}{2}}\,dx = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x+3}}\Big|_{1}^{t}\right] = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t+3}}+1\right) = -0+1 = 1$

So the integral converges.

(e) $\displaystyle\int_{0}^{\infty}\frac{x}{x^4+1}\,dx = \lim_{t\to\infty}\int_{0}^{t}\frac{x}{1+(x^2)^2}\,dx \overset{u=x^2}{=} \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}x^2\Big|_{0}^{t}\right] = \lim_{t\to\infty}\left(\frac{1}{2}\tan^{-1}t^2-0\right) = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

So the integral converges.

(f) $\displaystyle\int_{-\infty}^{0}\frac{e^x}{1+e^x}\,dx = \lim_{t\to-\infty}\int_{t}^{0}\frac{e^x}{1+e^x}\,dx \overset{u=1+e^x}{=} \lim_{t\to-\infty}\left[\ln(1+e^x)\Big|_{t}^{0}\right] = \lim_{t\to-\infty}\left[\ln 2-\ln(1+e^t)\right]$

$\displaystyle = \ln 2-\ln(1+0) = \ln 2$

So the integral converges.

(g) $\displaystyle\int_{0}^{3}\frac{1}{\sqrt{9-x^2}}\,dx = \lim_{t\to3^-}\int_{0}^{t}\frac{1}{\sqrt{9-x^2}}\,dx = \lim_{t\to3^-}\left[\sin^{-1}\frac{x}{3}\Big|_{0}^{t}\right] = \lim_{x\to3^-}\left(\sin^{-1}\frac{t}{3}-\sin^{-1}0\right)$

$\displaystyle = \frac{\pi}{2}-0 = \frac{\pi}{2}$

So the integral converges.

(h) $\displaystyle\int_{0}^{3}\frac{x}{\sqrt{9-x^2}}\,dx = \lim_{t\to3^-}\int_{0}^{t}\frac{x}{\sqrt{9-x^2}}\,dx \overset{u=9-x^2}{=} \lim_{t\to3^-}\left[-\sqrt{9-x^2}\Big|_{0}^{t}\right] = \lim_{t\to3^-}\left(-\sqrt{9-t^2}+3\right) = 0+3 = 3$

So the integral converges.

(i) $\displaystyle\int_{e}^{\infty}\frac{1}{x\ln^2 x}\,dx = \lim_{t\to\infty}\int_{e}^{t}\frac{1}{x\ln^2 x}\,dx \overset{u=\ln x}{=} \lim_{t\to\infty}\left[-\frac{1}{\ln x}\Big|_{e}^{t}\right] = \lim_{t\to\infty}\left(-\frac{1}{\ln t}+1\right) = -0+1 = 1$

So the integral converges.

(j) $\displaystyle\int_{-\infty}^{\frac{3}{2}}\frac{1}{4x^2+9}\,dx = \lim_{t\to-\infty}\int_{t}^{\frac{3}{2}}\frac{1}{(2x)^2+9}\,dx = \lim_{t\to-\infty}\left[\frac{1}{6}\tan^{-1}\frac{2x}{3}\Big|_{t}^{\frac{3}{2}}\right] = \lim_{t\to-\infty}\left(\frac{1}{6}\cdot\frac{\pi}{4}-\frac{1}{6}\tan^{-1}\frac{2t}{3}\right)$

$\displaystyle = \frac{\pi}{24}-\frac{1}{6}\left(-\frac{\pi}{2}\right) = \frac{\pi}{24}+\frac{\pi}{12} = \frac{\pi}{8}$

So the integral converges.

(k) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+9x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+9x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+9x^2}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+9x^2}\,dx + \lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+9x^2}\,dx$

$$= \lim_{t\to-\infty}\left[\frac{1}{3}\tan^{-1}3x\Big|_t^0\right] + \lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}3x\Big|_0^t\right] = \lim_{t\to-\infty}\left[0-\frac{1}{3}\tan^{-1}3t\right] + \lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}3t - 0\right]$$

$$= -\frac{1}{3}\left(-\frac{\pi}{2}\right) + \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

So the integral converges.

(l) $\displaystyle\int_0^{\infty} xe^{-x}dx$ $\qquad u = x\,,\quad v' = e^{-x}$; $\quad u' = 1\,,\quad v = -e^{-x}$

$$= \lim_{t\to\infty}\int_0^t xe^{-x}dx = \lim_{t\to\infty}\left[-xe^{-x}\Big|_0^t + \int_0^t e^{-x}dx\right] = \lim_{t\to\infty}\left[\left(-xe^{-x} - e^{-x}\right)\Big|_0^t\right] = \lim_{t\to\infty}\left(-te^{-t} - e^{-t} + 1\right)$$

$$= \underbrace{\lim_{t\to\infty}\frac{-t}{e^t}}_{\text{L'Hospital's Rule}} - \lim_{t\to\infty}e^{-t} + \lim_{t\to\infty}1 = \lim_{t\to\infty}\frac{-1}{e^t} - 0 + 1 = 0 - 0 + 1 = 1$$

So the integral converges.

(m) $\displaystyle\int_1^{\infty} \frac{\ln x}{x\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x\sqrt{x}}\,dx$ $\qquad u = \ln x\,,\quad v' = x^{-\frac{3}{2}}$; $\quad u' = \frac{1}{x}\,,\quad v = -\frac{2}{\sqrt{x}}$

$$= \lim_{t\to\infty}\left(-\frac{2\ln x}{\sqrt{x}}\Big|_1^t + \int_1^t \frac{2}{x\sqrt{x}}\,dx\right) = \lim_{t\to\infty}\left(-\frac{2\ln x}{\sqrt{x}} - \frac{4}{\sqrt{x}}\right)\Big|_1^t = \lim_{t\to\infty}\left(-\frac{2\ln t}{\sqrt{t}} - \frac{4}{\sqrt{t}} + 4\right)$$

$$= \underbrace{\lim_{t\to\infty}\frac{-2\ln t}{\sqrt{t}}}_{\text{L'Hospital's Rule}} - \lim_{t\to\infty}\frac{4}{\sqrt{t}} + \lim_{t\to\infty}4 = \lim_{t\to\infty}\frac{-\frac{2}{t}}{\frac{1}{2\sqrt{t}}} - 0 + 4 = \lim_{t\to\infty}\frac{-4}{\sqrt{t}} + 4 = -0 + 4 = 4$$

So the integral converges.

3. (a) $y = \sqrt{4-x^2}\,,\quad y' = \dfrac{1}{2\sqrt{4-x^2}}\cdot(-2x) = \dfrac{-x}{\sqrt{4-x^2}}$

$$s = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1+(y')^2}\,dx = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1+\frac{x^2}{4+x^2}}\,dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{2}{\sqrt{4-x^2}}\,dx = 2\sin^{-1}\frac{x}{2}\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 2\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{4\pi}{3}$$

(b) $y = \dfrac{1}{3}(x^2+2)^{\frac{3}{2}}\,,\quad y' = \dfrac{1}{2}(x^2+2)^{\frac{1}{2}}\cdot 2x = x\sqrt{x^2+2}$

$$s = \int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \sqrt{1+x^2(x^2+2)}\,dx = \int_0^1 \sqrt{x^4+2x^2+1}\,dx = \int_0^1 \sqrt{(x^2+1)^2}\,dx$$

$$= \int_0^1 (x^2+1)dx = \left(\frac{x^3}{3}+x\right)\Big|_0^1 = \frac{1}{3}+1 = \frac{4}{3}$$

(c) $y = 2x\sqrt{x} = 2x^{\frac{3}{2}}$, $y' = 3x^{\frac{1}{2}} = 3\sqrt{x}$

$$s = \int_0^{11} \sqrt{1+(y')^2}\,dx = \int_0^{11} \sqrt{1+9x}\,dx = \frac{1}{9}\,\frac{(1+9x)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{11} = \frac{2}{27}\,(1+9x)^{\frac{3}{2}}\Bigg|_0^{11}$$

$$= \frac{2}{27}(1000-1) = 74$$

(d) $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, $y' = x - \dfrac{1}{4x} = \dfrac{4x^2-1}{4x}$

$$s = \int_2^{4} \sqrt{1+(y')^2}\,dx = \int_2^{4} \sqrt{1+\frac{16x^4-8x^2+1}{16x^2}}\,dx = \int_2^{4} \sqrt{\frac{16x^4+8x^2+1}{16x^2}}\,dx$$

$$= \int_0^{4} \sqrt{\frac{(4x^2+1)^2}{16x^2}}\,dx = \int_2^{4} \frac{4x^2+1}{4x}\,dx = \int_2^{4} \left(x + \frac{1}{4x}\right)dx = \left(\frac{x^2}{2} + \frac{1}{4}\ln|x|\right)\Bigg|_2^{4}$$

$$= \left(8 + \frac{1}{4}\ln 4\right) - \left(2 + \frac{1}{4}\ln 2\right) = 6 + \frac{1}{4}\ln 2$$

(e) $y = \ln(\sin x)$, $y' = \dfrac{1}{\sin x}\cdot\cos x = \cot x$

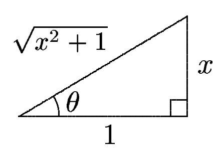
$$s = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1+(y')^2}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1+\cot^2 x}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{\csc^2 x}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \csc x\,dx$$

$$= \ln|\csc x - \cot x|\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \ln|1-0| - \ln\left|\frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\right| = 0 - \ln\frac{\sqrt{3}}{3} = \ln\sqrt{3} = \frac{1}{2}\ln 3$$

(f) $y = \ln x$, $y' = \dfrac{1}{x}$

$$s = \int_1^{\sqrt{3}} \sqrt{1+(y')^2}\,dx = \int_1^{\sqrt{3}} \sqrt{1+\frac{1}{x^2}}\,dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x}\,dx$$

$$x = \tan\theta \qquad x = \sqrt{3} \Rightarrow \theta = \tfrac{\pi}{3}$$
$$dx = \sec^2\theta\,d\theta \qquad x = 1 \Rightarrow \theta = \tfrac{\pi}{4}$$
$$\sqrt{x^2+1} = \sec\theta$$



$$s = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan\theta}\cdot\sec^2\theta\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^3\theta}{\tan\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1+\tan^2\theta)\sec\theta}{\tan\theta}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\sec\theta}{\tan\theta} + \sec\theta\tan\theta\right)d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\csc\theta + \sec\theta\tan\theta)d\theta = (\ln|\csc\theta - \cot\theta| + \sec\theta)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\ln\left|\frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\right| + 2\right) - \left(\ln|\sqrt{2}-1| + \sqrt{2}\right) = \ln\frac{\sqrt{3}}{3} + 2 - \ln(\sqrt{2}-1) - \sqrt{2}$$