Group gradings on simple Lie and related algebras

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Gradings and (semi)group gradings

Let $\mathcal{A}$ be a nonassociative algebra over a field $\mathbb{F}$.

Definition (Grading on an algebra)

A *grading* on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ such that, whenever $\mathcal{A}_x\mathcal{A}_y \neq 0$, there exists a unique $z \in S$ such that $\mathcal{A}_x\mathcal{A}_y \subseteq \mathcal{A}_z$. This gives a partially defined operation on S: $x * y := z$.

Definition (G-graded algebra)

Let G be a (semi)group, written multiplicatively.

- A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ such that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$.
- $(\mathcal{A}, \Gamma)$ is said to be a *G-graded algebra*, and $\mathcal{A}_g$ is its *homogeneous component* of degree g.

A grading $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ is a *(semi)group grading* if there exists a (semi)group G such that $S \subseteq G$ and $*$ is a restriction of the operation of G, i.e., $\mathcal{A}_x\mathcal{A}_y \neq 0 \;\Rightarrow\; x * y = xy$.

Universal grading groups

Example (Gradings from matrix units)

$M_n(\mathbb{F}) = \bigoplus_{1\le i,j\le n} \mathbb{F}E_{ij}$ is a semigroup grading, but not a group grading.
$M_n(\mathbb{F}) = \mathrm{Span}\{E_{11},\dots,E_{nn}\} \oplus \bigoplus_{1\le i\ne j\le n} \mathbb{F}E_{ij}$ is an ab. group grading.

The *support* of a G-grading Γ is the set $\mathrm{Supp}\,\Gamma := \{g\in G \mid \mathcal{A}_g \ne 0\}$.

Fact: For any semigroup grading on a simple Lie algebra, the support generates an abelian group.

Question: Are there non-semigroup gradings on simple Lie algebras?

Definition (Universal group and universal abelian group)

The *universal (abelian) group* of $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$, where all $\mathcal{A}_s \ne 0$, is the (abelian) group $U(\Gamma)$ with generating set S and defining relations $xy = z$ whenever $0 \ne \mathcal{A}_x\mathcal{A}_y \subseteq \mathcal{A}_z$ (i.e., $xy = x * y$ whenever defined).

$S \hookrightarrow U(\Gamma) \;\Leftrightarrow\; \Gamma$ is an (ab.) group grading. Then Γ is a $U(\Gamma)$-grading, and if it is a G-grading then $\exists!$ homom. $U(\Gamma) \to G$ that restricts to id_S.

Examples of abelian group gradings

Example

The following is a $\mathbb{Z}$-grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where

$$\mathfrak{g}_{-1} = \mathrm{Span}\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\},\ \mathfrak{g}_0 = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\},\ \mathfrak{g}_1 = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}.$$

This can also be regarded as a $\mathbb{Z}_m$-grading for any $m > 2$, but the universal group is $\mathbb{Z}$.

Example (Cartan grading)

Let $\mathfrak{g}$ be a s.s. Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\Big)$$

can be viewed as a grading by the root lattice $G = \langle\Phi\rangle$.
$\mathrm{Supp}\,\Gamma = \{0\} \cup \Phi$; $U(\Gamma) = G \cong \mathbb{Z}^r$ where $r = \dim \mathfrak{h}$.

Examples continued

Example (Pauli grading)

There is a grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Namely, $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ where $\mathbb{Z}_2^2 = \{e, a, b, c\}$ and

$$\mathfrak{g}_a = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \ \mathfrak{g}_b = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}, \ \mathfrak{g}_c = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}.$$

$\mathrm{Supp}\,\Gamma = \{a, b, c\}$; $U(\Gamma) = \mathbb{Z}_2^2$.

Given $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, a group homomorphism $\alpha\colon G \to H$ induces ${}^{\alpha}\Gamma : \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}_h$ where $\mathcal{A}_h = \bigoplus_{g\in\alpha^{-1}(h)} \mathcal{A}_g$.

Example (Gradings induced by group homomorphisms)

$\mathbb{F}[x_1, \ldots, x_n] = \bigoplus_{h\in H} \mathcal{A}_h$ where $\mathcal{A}_h = \mathrm{Span}\{x_1^{k_1} \cdots x_n^{k_n} \mid h_1^{k_1} \cdots h_n^{k_n} = h\}$.
$\alpha \in \mathrm{Aut}(\mathbb{Z}_2^2)$ induce permutations of a, b, c in the Pauli grading.

Isomorphism and equivalence

Definition

- Two G-gradings on $\mathcal{A}$, $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}'_g$, are *isomorphic* if $\exists$ an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$ (i.e., $(\mathcal{A}, \Gamma) \cong (\mathcal{A}, \Gamma')$ as G-graded alg.)
- A G-grading $\mathcal{A} = \bigoplus_{g \in S \subseteq G} \mathcal{A}_g$ and an H-grading $\mathcal{A} = \bigoplus_{h \in S' \subseteq H} \mathcal{A}'_h$ are *equivalent* if $\exists$ an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ and a bijection $\alpha \colon S \to S'$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_{\alpha(g)}$ for all $g \in S$.

If $\mathbb{F}$ contains a primitive n-th root of unity ε, then there is a grading on $\mathcal{R} = M_n(\mathbb{F})$ by $G = \mathbb{Z}_n^2$ associated to the *generalized Pauli matrices*

$$X = \begin{bmatrix} \varepsilon^{n-1} & 0 & \dots & 0 & 0 \\ 0 & \varepsilon^{n-2} & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & \varepsilon & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Namely, choose generators a and b of G and set $\mathcal{R}_{a^i b^j} = \mathbb{F} X^i Y^j$.

All such gradings are equivalent, but how many are non-isomorphic?

Refinements, coarsenings, and fine group gradings

Definition

Consider a G-grading $\Gamma : \mathcal{A} = \bigoplus_{g \in S \subseteq G} \mathcal{A}_g$ and an H-grading $\Gamma' : \mathcal{A} = \bigoplus_{h \in S' \subseteq H} \mathcal{A}'_h$. We say that Γ' is a *coarsening* of Γ (or Γ is a *refinement* of Γ') if for any $g \in G$ there exists $h \in H$ such that $\mathcal{A}_g \subseteq \mathcal{A}'_h$. If we have $\neq$ for some $g \in S = \operatorname{Supp} \Gamma$, then Γ a *proper* refinement of Γ'. A grading is *fine* if it does not have proper refinements.

Example

$\mathfrak{sl}_2(\mathbb{C}) = \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\} \oplus \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ is a $\mathbb{Z}_2$-grading that is a proper coarsening of the Cartan grading and also of the Pauli grading. Up to equivalence, there are exacly 2 fine (ab.) group gradings on $\mathfrak{sl}_2(\mathbb{F})$, $\operatorname{char} \mathbb{F} \neq 2$: the Cartan grading and the Pauli grading.

Example

The group grading $M_n(\mathbb{F}) = \operatorname{Span}\{E_{11}, \ldots, E_{nn}\} \oplus \bigoplus_{1 \leq i \neq j \leq n} \mathbb{F} E_{ij}$ is fine. (It has a proper refinement that is not a group grading.)

Classification problems

Given a "nice" algebra $\mathcal{A}$, classify
- fine (abelian) group gradings on $\mathcal{A}$ up to equivalence;
- all G-gradings on $\mathcal{A}$ up to isomorphism, for a fixed (ab.) group G.

If we classified G-gradings on $\mathcal{A}$ for any G, it is still not trivial to determine which of them are fine and which of them are equivalent to each other (even for fine gradings).

If $\dim \mathcal{A} < \infty$ then for any G-grading Γ on $\mathcal{A}$, $\exists$ a fine grading Δ on $\mathcal{A}$ and a homom. $\alpha : U(\Delta) \to G$ such that $\Gamma = {}^{\alpha}\Delta$, but it is often hard to determine which of the induced gradings are isomorphic to each other.

From now on, we assume that $\dim \mathcal{A} < \infty$ and G is abelian and f.g.

Remark (Reformulation over an a.c. field of characteristic 0)

The (equivalence classes of) fine abelian group gradings on $\mathcal{A}$ $\leftrightarrow$ (conjugacy classes of) maximal quasitori in the algebraic group $\mathrm{Aut}(\mathcal{A})$.
The (isomorphism classes of) G-gradings on $\mathcal{A}$ $\leftrightarrow$ (conjugacy classes of) algebraic group homomorphisms $\widehat{G} \to \mathrm{Aut}(\mathcal{A})$.

Definition of the Weyl group of a grading

Let $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a G-grading on an algebra $\mathcal{A}$.

Definition (Patera–Zassenhaus 1989)

- $\mathrm{Aut}(\Gamma) = \{\psi \in \mathrm{Aut}(\mathcal{A}) \mid \forall g \in G\ \exists h \in G \quad \psi(\mathcal{A}_g) = \mathcal{A}_h\}$
- $\mathrm{Stab}(\Gamma) = \{\psi \in \mathrm{Aut}(\mathcal{A}) \mid \forall g \in G \quad \psi(\mathcal{A}_g) = \mathcal{A}_g\}$
- $\mathrm{Diag}(\Gamma) = \{\psi \in \mathrm{Stab}(\Gamma) \mid \forall g \in G \quad \psi|_{\mathcal{A}_g}$ is scalar$\}$

If $G = U(\Gamma)$, then each $\psi \in \mathrm{Aut}(\Gamma)$ defines $\alpha \in \mathrm{Aut}(G)$: $\psi(\mathcal{A}_g) = \mathcal{A}_{\alpha(g)}$ for all $g \in G$. Hence we get a homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(G)$ whose kernel is $\mathrm{Stab}(\Gamma)$.

Definition

$W(\Gamma) := \mathrm{Aut}(\Gamma)/\mathrm{Stab}(\Gamma)$ is called the *Weyl group* of Γ.

By def, $W(\Gamma)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$. Its index is the number of non-isomorphic gradings with universal group G that are equivalent to Γ.

Examples of Weyl groups

Remark

If $\mathbb{F}$ is a.c., $\operatorname{char}\mathbb{F} = 0$, then any ab. group grading Γ on $\mathcal{A}$ is the eigenspace decomposition w.r.t. a quasitorus $Q \subseteq \operatorname{Aut}(\mathcal{A})$.
We can take $Q = \operatorname{Diag}(\Gamma)$. Then $U(\Gamma) = \mathfrak{X}(Q)$, $\operatorname{Aut}(\Gamma) = N(Q)$, $\operatorname{Diag}(\Gamma) = C(Q)$ and hence $W(\Gamma) = N(Q)/C(Q)$.

Example

If $\mathfrak{g}$ a s.s. Lie algebra with root system Φ and Γ is a Cartan grading on $\mathfrak{g}$, then $W(\Gamma) = \operatorname{Aut}\Phi$, i.e., the classical *extended Weyl group* of Φ.

Example (Havlíček–Patera–Pelantová–Tolar 2002)

If Γ is a Pauli grading on $M_n(\mathbb{C})$ by $\mathbb{Z}_n^2$, then $W(\Gamma) \cong \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

It follows that there are $\phi(n)$ (Euler function) non-isomorphic Pauli gradings on $M_n(\mathbb{C})$. (Hence $\frac{1}{2}\phi(n)$ on $\mathfrak{sl}_n(\mathbb{C})$ for $n > 2$.)

Graded-simple associative algebras

$\mathcal{D}$ is a *graded-division* algebra if all nonzero homogeneous elements are invertible ($\Rightarrow$ graded $\mathcal{D}$-modules have a graded basis).

Theorem ("Graded Wedderburn Theorem")

Let $\mathcal{R}$ be a G-graded algebra (or ring). Then $\mathcal{R}$ is graded-simple and satisfies d.c.c. on graded one-sided ideals $\Leftrightarrow$ there exists a graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$ of finite rank such that $\mathcal{R} \cong \mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as G-graded algebras.

Select a graded $\mathcal{D}$-basis $\{v_1, \ldots, v_k\}$ of $\mathcal{V}$, and let $\deg v_i = g_i$.
$\mathcal{R} \cong M_k(\mathbb{F}) \otimes \mathcal{D}$, where $\deg(E_{ij} \otimes d) = g_i(\deg d)g_j^{-1}$ for homog. $d \in \mathcal{D}$.

Definition (Multisets)

A *multiset* is a pair (A, κ) where A is a set and $\kappa : A \to \mathbb{Z}_{>0}$. For $A \subseteq X$, where X is fixed, we consider $\kappa : X \to \mathbb{Z}_{\geq 0}$ with $A = \{x \mid \kappa(x) \neq 0\}$.

$T := \mathrm{Supp}\,\mathcal{D}$ is a subgroup of G, and the isomorphism class of $\mathcal{V}$ is determined by the multiset $\Xi = \{g_1 T, \ldots, g_k T\}$ in G/T.

G-gradings on $M_n(\mathbb{F})$

$\mathcal{R} = M_n(\mathbb{F}) \;\Rightarrow\; \mathcal{D} \cong M_\ell(\mathbb{F})$ with a *division grading*, $k\ell = n$.
If $\mathbb{F}$ is a.c. then $\mathcal{D}_e = \mathbb{F}$, hence, with any G-grading on $M_n(\mathbb{F})$, we have $M_n(\mathbb{F}) \cong M_k(\mathbb{F}) \otimes M_\ell(\mathbb{F})$ where all homog. components of $M_\ell(\mathbb{F})$ are 1-dim (Bahturin–Sehgal–Zaicev 2001).

Theorem (HPP 1998,BSZ 2001 for $\operatorname{char}\mathbb{F} = 0$; BZ 2003)

Let T be an ab. group and $\mathbb{F}$ an a.c. field. Then, for any division grading on $\mathcal{D} = M_\ell(\mathbb{F})$ with support T, there exists a decomposition $T = H_1 \times \cdots \times H_r$ such that $H_i \cong \mathbb{Z}_{\ell_i}^2$ and $\mathcal{D} \cong M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ where $M_{\ell_i}(\mathbb{F})$ has a Pauli grading by H_i.

$\mathcal{D}_t = \mathbb{F}X_t$ for some invertible X_t ($\Rightarrow \mathcal{D}$ is a *twisted group algebra* of T), and if T is abelian, we have $X_sX_t = \beta(s,t)X_tX_s$, where $\beta : T \times T \to \mathbb{F}^\times$ is a nondegenerate alternating bicharacter.

For abelian G and a.c. $\mathbb{F}$, the isomorphism classes of G-gradings on $M_n(\mathbb{F})$ are parametrized by (T, β, Ξ) where Ξ is a finite multiset in G/T, determined up to shift, and $|\Xi|\sqrt{|T|} = n$ (Bahturin–K 2010).

Fine gradings on $M_n(\mathbb{F})$ and their Weyl groups

Let k be a divisor of n. Set $\ell = n/k$, identify $M_n(\mathbb{F}) = M_k(\mathbb{F}) \otimes M_\ell(\mathbb{F})$,

- give $M_k(\mathbb{F})$ an elementary $\mathbb{Z}^k$-grading: $\deg E_{ij} = \widetilde{g}_i \widetilde{g}_j^{-1}$, where $\{\widetilde{g}_1, \ldots, \widetilde{g}_k\}$ is the standard basis of $\mathbb{Z}^k$,
- give $M_\ell(\mathbb{F})$ a division grading with support $T = \widehat{B} \times B$, where B is an ab. group, $|B| = \ell$, and $\beta((\chi, b), (\chi', b')) = \chi(b')/\chi'(b)$,

so $M_n(\mathbb{F})$ gets a $\mathbb{Z}^k \times T$-grading, which we denote by $\Gamma_M(T, k)$.

This grading is fine and its universal (ab.) group is $\mathbb{Z}^{k-1} \times T$.

Remark

Explicitly, the division grading on $M_\ell(\mathbb{F})$ is obtained by taking a vector space with basis $\{e_b \mid b \in B\}$, and letting $X_{(\chi,b)} e_{b'} = \chi(bb') e_{bb'}$.

Any fine ab. group grading on $M_n(\mathbb{F})$ is equivalent to some $\Gamma_M(T, k)$. $\Gamma_M(T_1, k_1)$ and $\Gamma_M(T_2, k_2)$ are equivalent $\Leftrightarrow$ $T_1 \cong T_2$ and $k_1 = k_2$.

Theorem (Elduque–K 2012)

Let $\Gamma = \Gamma_M(T, k)$. Then $W(\Gamma) \cong T^{k-1} \rtimes (\mathrm{Sym}(k) \times \mathrm{Aut}(T, \beta))$.

Fine φ-gradings on $M_n(\mathbb{F})$

Definition (Elduque 2010)

Let G be an abelian group. Let $\mathcal{A}$ be an algebra and let φ be an anti-automorphism of $\mathcal{A}$. A grading $\Gamma:\ \mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g$ is said to be a *φ-grading* if $\varphi(\mathcal{A}_g)=\mathcal{A}_g$ for all $g\in G$ and $\varphi^2\in\mathrm{Diag}(\Gamma)$.

Let $\mathbb{F}$ be a.c., $\mathrm{char}\,\mathbb{F}\neq 2$. Then the classification of fine gradings on the simple Lie algebras of types A, B, C, D reduces to matrix algebras:

- B_r, resp. D_r ($r\neq 4$): pairs (Γ,φ), up to equivalence, where Γ is a fine φ-grading and φ is an orthogonal involution on $M_{2r+1}(\mathbb{F})$, resp. $M_{2r}(\mathbb{F})$;
- C_r: pairs (Γ,φ), up to equivalence, where Γ is a fine φ-grading and φ is a symplectic involution on $M_{2r}(\mathbb{F})$.

For A_r ($r>1$), there are two types of gradings:

- Type I: fine gradings on $M_{r+1}(\mathbb{F})$, up to equivalence or anti-equivalence.
- Type II: fine φ-gradings on $M_{r+1}(\mathbb{F})$, up to weak equivalence.

Construction of fine φ-gradings on $M_n(\mathbb{F})$

Let T be an elementary abelian 2-group of even rank. It is a vector space over the field $\mathbb{Z}_2$, so $T \cong \mathbb{Z}_2^{2m} = \mathbb{Z}_2^m \times \mathbb{Z}_2^m$. Let $\mathcal{D}$ be $M_{2^m}(\mathbb{F})$ that has a division grading with support T and

$$\beta(u,v) = (-1)^{\sum_{j=1}^m u_j v_{m+j} + \sum_{j=1}^m v_j u_{m+j}}, \quad u,v \in \mathbb{Z}_2^{2m}.$$

Then matrix transposition is an involution of $\mathcal{D}$ as a graded algebra, given by $X_u \mapsto \eta(u) X_u$ where

$$\eta(u) = (-1)^{\sum_{j=1}^m u_j u_{m+j}}, \quad u \in \mathbb{Z}_2^{2m}.$$

Let $q \geq 0$ and $s \geq 0$ be two integers. Let $\tau = (t_1, \ldots, t_q) \in T^q$.
Denote by $\widetilde{G} = \widetilde{G}(T, q, s, \tau)$ the abelian group generated by T and the symbols $\widetilde{g}_1, \ldots, \widetilde{g}_{q+2s}$ with defining relations

$$\widetilde{g}_1^2 t_1 = \ldots = \widetilde{g}_q^2 t_q = \widetilde{g}_{q+1}\widetilde{g}_{q+2} = \ldots = \widetilde{g}_{q+2s-1}\widetilde{g}_{q+2s}.$$

Let $n = (q+2s)2^m$, identify $M_n(\mathbb{F})$ with $M_{q+2s} \otimes \mathcal{D}$ via Kronecker product and define a $\widetilde{G}$-grading $\Gamma = \Gamma_M(T, q, s, \tau)$ on $M_n(\mathbb{F})$: $\deg(E_{ij} \otimes X_t) = \widetilde{g}_i \widetilde{g}_j^{-1} t$.

Construction of fine φ-gradings on $M_n(\mathbb{F})$ continued

Let $\Gamma = \Gamma_M(T, q, s, \tau)$, $T \cong \mathbb{Z}_2^{2m}$, $\tau = (t_1, \ldots, t_q) \in T^q$, $n = (q + 2s)2^m$.

Theorem (Elduque 2010)

Let $\mu = (\mu_1, \ldots, \mu_s)$ where $\mu_i \in \mathbb{F}^\times$. Let $\varphi = \varphi_{\tau,\mu}$ be the anti-automorphism of $M_n(\mathbb{F})$ defined by $\varphi(X) = \Phi^{-1} X^\top \Phi$, $X \in M_n(\mathbb{F})$, where Φ is the block-diagonal matrix given by

$$\Phi = \operatorname{diag}\left(X_{t_1}, \ldots, X_{t_q}, \begin{bmatrix} 0 & I \\ \mu_1 I & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & I \\ \mu_s I & 0 \end{bmatrix} \right)$$

and $I = X_e$ is the identity element of $\mathcal{D}$. Then Γ is a fine φ-grading unless $q = 2$, $s = 0$, and $t_1 = t_2$. Any fine φ-grading on $M_n(\mathbb{F})$ is equivalent to one of these.

Let T_0 be the subgroup of T generated by the elements $t_i t_{i+1}$, $i = 1, \ldots, q-1$. Then

$$U(\Gamma) \cong \mathbb{Z}_2^{2m-2\dim T_0 + \max(0,q-1)} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s.$$

Classification of fine φ-gradings on $M_n(\mathbb{F})$

$T \cong \mathbb{Z}_2^{2m} \;\Rightarrow\; \mathrm{Aut}(T,\beta) \cong \mathrm{Sp}_{2m}(2)$. Let $\mathrm{ASp}_{2m}(2) = \mathbb{Z}_2^{2m} \rtimes \mathrm{Sp}_{2m}(2)$. Then $\mathrm{Sp}_{2m}(2)$ and $\mathrm{ASp}_{2m}(2)$ act naturally on T. For $\mathrm{Sp}_{2m}(2)$, we will need the following *twisted action* on T.

Let $Q(T,\beta)$ be the set of quadratic forms on T whose polarization is β. We identify $Q(T,0)$ with linear forms on T, and the latter with T via β. Hence, $Q(T,\beta)$ is a *T-torsor*, i.e., a set on which T acts simply transitively and in a way compatible with the $\mathrm{Aut}(T,\beta)$-action on both.

Using $\eta \in Q(T,\beta)$ as a base point, we can identify $Q(T,\beta)$ with T, but the corresponding $\mathrm{Aut}(T,\beta)$-action on T is the following:

$\alpha \cdot t = \alpha(t)t_\alpha$ where $t_\alpha \in T$ is def. by $\beta(t_\alpha, u) = \eta(\alpha^{-1}(u))\eta(u) \quad \forall u \in T$.

$\varphi_{\tau,\mu}$ is an orthogonal (resp., symplectic) involution if and only if $\eta(t_1) = \ldots = \eta(t_q) = \mu_1 = \ldots = \mu_s = 1$ (resp., -1).
Denote $T_+ = \{t \in T \mid \eta(t) = 1\}$ and $T_- = \{t \in T \mid \eta(t) = -1\}$.

The actions of $\mathrm{Sp}_{2m}(2)$ and $\mathrm{ASp}_{2m}(2)$ on T induce actions on multisets in T. Denote by $\Sigma(\tau)$ the multiset $\{t_1, \ldots, t_q\}$.

Fine gradings on the simple Lie algebra of type B_r

Here $n = 2r + 1$ is odd, so $T = \{e\}$. Consider the grading $\Gamma_M(\{e\}, q, s, \tau)$ on $\mathcal{R} = M_n(\mathbb{F})$ and the involution $\varphi(X) = \Phi^{-1} X^\top \Phi$ where

$$\Phi = \operatorname{diag}\left(1, \ldots, 1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

Restricting to $\mathcal{K}(\mathcal{R}, \varphi) \cong \mathfrak{so}_n(\mathbb{F})$, we obtain a fine grading $\Gamma_B(q, s)$ with universal group $\mathbb{Z}_2^{q-1} \times \mathbb{Z}^s$.

Theorem (HPP 1998 for $\operatorname{char} \mathbb{F} = 0$; EK 2012 for $\operatorname{char} \mathbb{F} \neq 2$)

Let $\mathbb{F}$ be a.c., $\operatorname{char} \mathbb{F} \neq 2$. Let $n \geq 5$ be odd. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_B(q, s)$ where $q + 2s = n$. Also, $\Gamma_B(q_1, s_1)$ and $\Gamma_B(q_2, s_2)$ are equivalent if and only if $q_1 = q_2$ and $s_1 = s_2$.

Thus, there are exactly $r + 1$ equivalence classes of fine gradings on the simple Lie algebra of type B_r. For $\Gamma = \Gamma_B(q, s)$, we have

$$W(\Gamma) \cong \operatorname{Sym}(q) \times (\mathbb{Z}_2^s \rtimes \operatorname{Sym}(s)).$$

Fine gradings on simple Lie algebra of types C_r and D_r

Here $n = 2r$. Consider the grading $\Gamma_M(T, q, s, \tau)$ on $\mathcal{R} = M_n(\mathbb{F})$, where $t_1 \neq t_2$ if $q = 2$ and $s = 0$, and the involution $\varphi(X) = \Phi^{-1} X^\top \Phi$ where

$$\Phi = \operatorname{diag}\left(X_{t_1}, \ldots, X_{t_q}, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix} \right),$$

with $t_i \in T_+$ and $\delta = 1$ for type D, resp. $t_i \in T_-$ and $\delta = -1$ for type C. Restricting to $\mathcal{K}(\mathcal{R}, \varphi)$, we obtain a fine grading $\Gamma_D(T, q, s, \tau)$, resp. $\Gamma_C(T, q, s, \tau)$, on the simple Lie algebra of type D_r, resp. C_r.

Theorem (Elduque–K 2012)

Let $\mathbb{F}$ be a.c., $\operatorname{char}\mathbb{F} \neq 2$. Let $n \geq 4$ be even. Then any fine grading on $\mathfrak{sp}_n(\mathbb{F})$ is equivalent to $\Gamma_C(T, q, s, \tau)$ where $T = \mathbb{Z}_2^{2m}$ and $(q + 2s)2^m = n$. Moreover, $\Gamma_C(T_1, q_1, s_1, \tau_1)$ and $\Gamma_C(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of $\mathrm{Sp}_{2m}(2)$.

The same holds for $\mathfrak{so}_n(\mathbb{F})$ where $n \geq 6$ and $n \neq 8$.

Fine gradings on simple Lie algebra of types A_r

I Restricting $\Gamma_M(T,k)$, where $k \geq 3$ if T is an elementary 2-group, we obtain a fine grading $\Gamma_A^{(\mathrm{I})}(T,k)$ on $\mathcal{L} := [\mathcal{R},\mathcal{R}]/(Z(\mathcal{R}) \cap [\mathcal{R},\mathcal{R}])$.

II Refining $\Gamma_M(T,q,s,\tau)$ by φ, where T is an elementary 2-group and $t_1 \neq t_2$ if $q=2$ and $s=0$, we obtain a grading on $\mathcal{R}^{(-)}$, which restricts to a fine grading $\Gamma_A^{(\mathrm{II})}(T,q,s,\tau)$ on $\mathcal{L}$.

Theorem (Elduque–K 2012)

Let $\mathbb{F}$ be a.c., $\operatorname{char}\mathbb{F} \neq 2$. Let $n \geq 3$ if $\operatorname{char}\mathbb{F} \neq 3$ and $n \geq 4$ if $\operatorname{char}\mathbb{F} = 3$. Then any fine grading on $\mathfrak{psl}_n(\mathbb{F})$ is equivalent $\Gamma_A^{(\mathrm{I})}(T,k)$ where $k\sqrt{|T|} = n$ or to $\Gamma_A^{(\mathrm{II})}(T,q,s,\tau)$ where $T = \mathbb{Z}_2^{2m}$, $(q+2s)2^m = n$. Gradings belonging to different types are not equivalent. Also,

- *$\Gamma_A^{(\mathrm{I})}(T_1,k_1)$ and $\Gamma_A^{(\mathrm{I})}(T_2,k_2)$ are equivalent iff $T_1 \cong T_2$ and $k_1 = k_2$;*
- *$\Gamma_A^{(\mathrm{II})}(T_1,q_1,s_1,\tau_1)$ and $\Gamma_A^{(\mathrm{II})}(T_2,q_2,s_2,\tau_2)$ are equivalent iff $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$, and $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the natural action of $\mathrm{ASp}_{2m}(2)$.*

Fine gradings for type G_2

Theorem (Elduque 1998)

Let $\mathbb{F}$ be a.c. and let $\mathcal{C}$ be the Cayley algebra over $\mathbb{F}$. Any fine grading on $\mathcal{C}$ is equivalent to either the Cartan grading (universal group $\mathbb{Z}^2$) or the Cayley–Dickson grading ($\mathrm{char}\,\mathbb{F} \neq 2$; universal group $\mathbb{Z}_2^3$).

As a corollary, there are exactly 2 fine gradings on $\mathcal{L} = \mathrm{Der}(\mathcal{C})$, which is the simple Lie algebra of type G_2 ($\mathrm{char}\,\mathbb{F} \neq 2, 3$).

The Weyl groups of the fine gradings on $\mathcal{C}$ (and hence on $\mathcal{L}$):

- $\mathbb{Z}^2$ (Cartan): the classical Weyl group of type G_2;
- $\mathbb{Z}_2^3$ (division): $\mathrm{Aut}(\mathbb{Z}_2^3) \cong \mathrm{GL}_3(2)$.

The latter corresponds to the fact that there is only one division grading on $\mathcal{C}$ up to isomorphism.

Remark

The simple Lie algebra of type D_4 is the *triality algebra* of $\mathcal{C}$. It has 17 fine gradings, 3 of which are exceptional (Elduque 2010, EK 2015).

Fine gradings for type F_4

Theorem (Draper–Martín 2009 for $\operatorname{char}\mathbb{F} = 0$; EK for $\operatorname{char}\mathbb{F} \neq 2$)

Let $\mathbb{F}$ be a.c., $\operatorname{char}\mathbb{F} \neq 2$, and let $\mathcal{A} = \mathcal{H}_3(\mathcal{C})$ be the Albert algebra over $\mathbb{F}$. Any fine grading on $\mathcal{A}$ is equivalent to exactly one of the following: the $\mathbb{Z}^4$-grading (Cartan), the $\mathbb{Z}_2^5$-grading, the $\mathbb{Z}_2^3 \times \mathbb{Z}$-grading or the $\mathbb{Z}_3^3$-grading ($\operatorname{char}\mathbb{F} \neq 3$ for the last one).

The same holds for $\mathcal{L} = \operatorname{Der}(\mathcal{A})$, which is the simple Lie algebra of type F_4 ($\operatorname{char}\mathbb{F} \neq 2$).

The Weyl groups of the fine gradings on $\mathcal{A}$ (and hence on $\mathcal{L}$):

- $\mathbb{Z}^4$ (Cartan): the classical Weyl group of type F_4;
- $\mathbb{Z}_2^5$-grading: the stabilizer of $\mathbb{Z}_2^3$ in $\operatorname{Aut}(\mathbb{Z}_2^5)$, where $\mathbb{Z}_2^3$ is the support of the division grading on $\mathcal{C}$;
- $\mathbb{Z}_2^3 \times \mathbb{Z}$-grading: $\operatorname{Aut}(\mathbb{Z}_2^3 \times \mathbb{Z})$ (which stabilizes $\mathbb{Z}_2^3$);
- $\mathbb{Z}_3^3$ (division): the commutator subgroup of $\operatorname{Aut}(\mathbb{Z}_3^3)$, which is isomorphic to $\mathrm{SL}_3(3)$.

There are two non-isomorphic division gradings on $\mathcal{A}$!

Fine gradings for “series” E, infinite universal group

The ground field $\mathbb{F}$ is assumed a.c., $\mathrm{char}\,\mathbb{F} \neq 2, 3$.

E_6		E_7		E_8	
Universal group	Model	Universal group	Model	Universal group	Model
$\mathbb{Z}^6$	Cartan	$\mathbb{Z}^7$	Cartan	$\mathbb{Z}^8$	Cartan
$\mathbb{Z}^4 \times \mathbb{Z}_2$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{K})$	$\mathbb{Z}^4 \times \mathbb{Z}_2^2$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{Q})$	$\mathbb{Z}^4 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{C})$
$\mathbb{Z}^2 \times \mathbb{Z}_3^2$	$\mathcal{T}(\Gamma^1_{\mathcal{C}}, \Gamma^2_{M_3(\mathbb{F})})$ $(G_2\,,\ M_3(\mathbb{F})^{(+)})$	—		$\mathbb{Z}^2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^1_{\mathcal{C}}, \Gamma^4_{\mathcal{A}})$ $(G_2\,,\ \mathcal{A})$
$\mathbb{Z}^2 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{M_3(\mathbb{F})})$ $(A_2\,,\ \mathcal{C})$	$\mathbb{Z}^3 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{\mathcal{H}_3(\mathcal{Q})})$ $(C_3\,,\ \mathcal{C})$	—	
$\mathbb{Z}^2 \times \mathbb{Z}_2^3$	$Kan(\tilde{\Gamma}_{\mathcal{X}(\mathbb{F})})$ $(BC_2\,,\ \mathcal{X}(\mathbb{F})_{1/2})$	$\mathbb{Z}^2 \times \mathbb{Z}_2^4$	$Kan(\tilde{\Gamma}_{\mathcal{X}(\mathcal{K})})$ $(BC_2\,,\ \mathcal{X}(\mathcal{K})_{1/2})$	$\mathbb{Z}^2 \times \mathbb{Z}_2^5$	$Kan(\tilde{\Gamma}_{\mathcal{X}(\mathcal{Q})})$ $(BC_2\,,\ \mathcal{X}(\mathcal{Q})_{1/2})$
—		$\mathbb{Z} \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^1_{\mathcal{Q}}, \Gamma^4_{\mathcal{A}})$ $(A_1\,,\ \mathcal{A})$	—	
$\mathbb{Z} \times \mathbb{Z}_2^5$	$Kan(\Gamma^1_{\mathcal{X}(\mathbb{F})})$ $(BC_1\,,\ \mathcal{X}(\mathbb{F}))$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$Kan(\Gamma^1_{\mathcal{X}(\mathcal{K})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{K}))$	$\mathbb{Z} \times \mathbb{Z}_2^7$	$Kan(\Gamma^1_{\mathcal{X}(\mathcal{Q})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{Q}))$
$\mathbb{Z} \times \mathbb{Z}_2^4$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{K} \otimes \mathcal{C})$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{Q} \otimes \mathcal{C})$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{C} \otimes \mathcal{C})$
—		$\mathbb{Z} \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$Kan(\Gamma^2_{\mathcal{X}(\mathcal{K})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{K}))$	$\mathbb{Z} \times \mathbb{Z}_4^3$	$Kan(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{Q}))$

Fine gradings for "series" E, finite universal group

E_6		E_7		E_8	
Universal group	Model	Universal group	Model	Universal group	Model
$\mathbb{Z}_3^4$	$\mathfrak{g}(\Gamma_{\bar{\mathcal{K}}}, \Gamma_{\mathcal{O}})$	—		$\mathbb{Z}_3^5$	$\mathfrak{g}(\Gamma_{\mathcal{O}}, \Gamma_{\mathcal{O}})$
$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^2_{M_3(\mathbb{F})})$	—		—	
$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^4_{\mathcal{A}})$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^4_{\mathcal{A}})$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^4_{\mathcal{A}})$
$\mathbb{Z}_2^7$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathbb{F})})$	$\mathbb{Z}_2^8$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathcal{K})})$	$\mathbb{Z}_2^9$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathcal{Q})})$
$\mathbb{Z}_2^6$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^2_{\mathcal{A}})$	$\mathbb{Z}_2^7$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^2_{\mathcal{A}})$	$\mathbb{Z}_2^8$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^2_{\mathcal{A}})$
$\mathbb{Z}_4^3$	$\mathrm{Der}(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2$	$\mathfrak{str}_0(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2^2$	$\mathfrak{stu}_3(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$
$\mathbb{Z}_4 \times \mathbb{Z}_2^4$	$\mathrm{Der}(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4 \times \mathbb{Z}_2^5$	$\mathfrak{str}_0(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4 \times \mathbb{Z}_2^6$	$\mathfrak{stu}_3(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$
—		$\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$	$\mathfrak{stu}_3(\Gamma^2_{\mathcal{X}(\mathcal{K})})$	—	
—		—		$\mathbb{Z}_5^3$	Jordan grading

The list is known to be complete if $\mathrm{char}\,\mathbb{F} = 0$: Draper–Viruel for E_6 (preprint 2012, published 2016); Yu for all E types over $\mathbb{C}$ (preprint 2014, published 2016) $\Rightarrow$ over any a.c. $\mathbb{F}$ of char 0 (Elduque 2016).

Open problem: given G, classify all G-gradings for E_6, E_7, E_8 up to isomorphism.