Group gradings on structurable algebras and exceptional simple Lie algebras

M. Kotchetov

Department of Mathematics and Statistics
Memorial University of Newfoundland

Universidad de Zaragoza
Zaragoza, 25 June 2014

Outline

Definition of a group grading

Let $\mathcal{A}$ be a nonassociative algebra over a field $\mathbb{F}$. Let G be a group.

Definition

- A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$. $\mathcal{A}_g$ is called the *homogeneous component* of degree g.
- The *support* of Γ is the set $S = \operatorname{Supp} \Gamma := \{g \in G \mid \mathcal{A}_g \neq 0\}$.
- The *universal (abelian) group* $U(\Gamma)$ is the (abelian) group with generating set S and defining relations $s_1 s_2 = s_3$ whenever $0 \neq \mathcal{A}_{s_1} \mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$.

Γ can be regarded as a $U(\Gamma)$-grading.
$\exists!$ homomorphism $U(\Gamma) \to G$ that restricts to id_S.

We assume that $\dim \mathcal{A} < \infty$ and G is abelian.

Examples of gradings

Example

The following is a $\mathbb{Z}$-grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where

$$\mathfrak{g}_{-1} = \operatorname{Span}\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\},\ \mathfrak{g}_0 = \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\},\ \mathfrak{g}_1 = \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}.$$

This can also be regarded as a $\mathbb{Z}_m$-grading for any $m > 2$, but the universal group is $\mathbb{Z}$.

Example (Cartan grading)

Let $\mathfrak{g}$ be a s.s. Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\Big)$$

can be viewed as a grading by the root lattice $G = \langle \Phi \rangle$.
$\operatorname{Supp}\Gamma = \{0\} \cup \Phi$; $U(\Gamma) = G \cong \mathbb{Z}^r$ where $r = \dim \mathfrak{h}$.

Examples continued

Example (Pauli grading)

A grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Namely, $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ where $\mathbb{Z}_2^2 = \{e, a, b, c\}$ and

$$\mathfrak{g}_a = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \ \mathfrak{g}_b = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}, \ \mathfrak{g}_c = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}.$$

Example (Generalized Pauli grading)

If $\varepsilon \in \mathbb{F}$, there is a grading on $\mathcal{R} = M_n(\mathbb{F})$ ($\Rightarrow$ on $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$) by $G = \mathbb{Z}_n^2$:

$$X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \varepsilon^{n-1} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where ε is a primitive n-th root of 1. Choose generators a and b of G and set $\mathcal{R}_{a^i b^j} = \mathbb{F}X^i Y^j$.

Isomorphism and equivalence of gradings

Definition

- Two G-gradings on $\mathcal{A}$, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}'_g$, are *isomorphic* if there exists an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$.
- A G-grading $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and an H-grading $\mathcal{A} = \bigoplus_{h\in G} \mathcal{A}'_h$, with supports S an S', respectively, are *equivalent* if there exists an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ and a bijection $\alpha \colon S \to S'$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_{\alpha(g)}$ for all $g \in S$.

In the def of equivalent gradings, if G and H are universal grading groups then α extends to a unique isomorphism of groups $G \to H$.

Example

All Pauli gradings on $M_n(\mathbb{F})$ or $\mathfrak{sl}_n(\mathbb{F})$ are equivalent. For $M_n(\mathbb{F})$, there are $\phi(n)$ (Euler function) non-isomorphic $\mathbb{Z}_n^2$-gradings among them. Hence $\frac{1}{2}\phi(n)$ for $\mathfrak{sl}_n(\mathbb{F})$ if $n > 2$.

Fine gradings

Definition

Consider a G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and an H-grading $\Gamma' : \mathcal{A} = \bigoplus_{h\in G} \mathcal{A}'_h$. We say that Γ' is a *coarsening* of Γ (or Γ is a *refinement* of Γ') if for any $g \in G$ there exists $h \in H$ such that $\mathcal{A}_g \subset \mathcal{A}'_h$. If we have $\neq$ for some $g \in \operatorname{Supp}\Gamma$, then Γ a *proper* refinement of Γ'.
A grading is *fine* if it does not have proper refinements.

Example

$\mathfrak{sl}_2(\mathbb{C}) = \operatorname{Span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right\} \oplus \operatorname{Span}\left\{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]\right\}$ is a $\mathbb{Z}_2$-grading that is a proper coarsening of the Cartan grading and also of the Pauli grading.
Up to equivalence, there are exacly 2 fine ab. group gradings on $\mathfrak{sl}_2(\mathbb{F})$, $\operatorname{char}\mathbb{F} \neq 2$: the Cartan grading and the Pauli grading.

If $\mathbb{F}$ is a.c., $\operatorname{char}\mathbb{F} = 0$, then (equivalence classes of) fine gradings on $\mathcal{A}$ $\leftrightarrow$ (conjugacy classes of) maximal quasitori in $\operatorname{Aut}(\mathcal{A})$.

Nonreduced root systems

A set Φ of vectors in a Euclidean space E is a *root system* if

(R1) Φ is a finite subset of $E \setminus \{0\}$ that spans E;

(R2) if $\alpha \in \Phi$, then the only scalar multiples of α in Φ are $\pm\alpha$;

(R3) if $\alpha \in \Phi$, then the reflection $\sigma_\alpha \colon u \mapsto u - \frac{2(u,\alpha)}{(\alpha,\alpha)}\alpha$ leaves Φ invariant;

(R4) if $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Irreducible root systems: A_r ($r \geq 1$), B_r ($r \geq 2$), C_r ($r \geq 3$), D_r ($r \geq 4$), E_6, E_7, E_8, F_4, G_2, where the subscript is the dimension of the space.

If Φ satisfies (R1), (R3) and (R4) but not necessarily (R2), it is called a *nonreduced root system*.

Irreducible nonreduced root systems: the above and BC_r ($r \geq 1$).

Lie algebras graded by a reduced root system

Definition (Berman–Moody, 1992)

A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the root system* Φ, or Φ*-graded*, if

1. $\mathcal{L}$ contains as a subalgebra a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$ whose root system is Φ, relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;
2. $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{X \in \mathcal{L} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$; and
3. $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

The subalgebra $\mathfrak{g}$ is said to be a *grading subalgebra* of $\mathcal{L}$.

Example (Tits, 1962)

$\mathcal{L}$ is A_1-graded if it contains an $\mathfrak{sl}_2$-triple (E, F, H) such that $\mathcal{L} = \mathcal{L}(-2) \oplus \mathcal{L}(0) \oplus \mathcal{L}(2)$, where $\mathcal{L}(i) = \{X \in \mathcal{L} \mid [H, X] = iX\}$, and $\mathcal{L}(0) = [\mathcal{L}(-2), \mathcal{L}(2)]$. Hence $\mathcal{L} = (\mathfrak{sl}_2 \otimes \mathcal{A}) \oplus \mathcal{D}$ as an $\mathfrak{sl}_2$-module. $\mathcal{A}$ can be given the structure of a unital algebra ("coordinate algebra").

Definition

Let Φ be the nonreduced root system BC_r ($r \geq 1$). A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by* Φ, or Φ*-graded*, if

1. $\mathcal{L}$ contains as a subalgebra a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi'} \mathfrak{g}_\alpha\right)$ whose root system Φ' relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is the reduced subsystem of type B_r, C_r or D_r contained in Φ;
2. $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{X \in \mathcal{L} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$; and
3. $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

Again, the subalgebra $\mathfrak{g}$ is said to be a *grading subalgebra* of $\mathcal{L}$, and $\mathcal{L}$ is said to be BC_r-graded with grading subalgebra of type X_r, where X_r is the type of $\mathfrak{g}$.

Fine gradings and gradings by root systems

Let $\mathcal{L}$ be a simple Lie algebra over an a. c. field $\mathbb{F}$ of characteristic 0.

Theorem (Elduque, 2013)

Let $\Gamma : \mathcal{L} = \bigoplus_{g\in G}\mathcal{L}_g$ be a fine grading where G is the universal group of Γ. Then the following conditions hold:

1. *$\mathcal{L}_e$ is a toral subalgebra of $\mathcal{L}$.*
2. *$\dim \mathcal{L}_e$ coincides with the free rank of G.*
3. *Let $\overline{G} = G/t(G)$. The induced grading $\overline{\Gamma} : \mathcal{L} = \bigoplus_{\bar{g}\in\overline{G}}\mathcal{L}_{\bar{g}}$ is the weight space decomposition relative to $\mathcal{L}_e$.*
4. *Let $\mathfrak{h} = \mathcal{L}_e$ and $\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathcal{L}(\alpha) \neq 0\}$, so $\overline{G} = \mathbb{Z}\Phi$. Then Φ is an irreducible (nonreduced) root system in $E = \mathbb{R}\otimes_{\mathbb{Q}}\mathbb{Q}\Phi$.*
5. *For each simple root α in Φ, pick its preimage g_α in G. Then $G = \widetilde{G}\oplus t(G)$ where $\widetilde{G}$ is free with basis g_α. Let $\mathfrak{g} = \bigoplus_{g\in\widetilde{G}}\mathcal{L}_g$. Then $\mathcal{L}$ is graded by the root system Φ with grading subalgebra $\mathfrak{g}$. If Φ is not reduced (type BC_r), then $\mathfrak{g}$ is simple of type B_r.*

Definition of a structurable algebra

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 2, 3$. Let $\mathcal{A}$ be a unital algebra over $\mathbb{F}$ and let $x \mapsto \bar{x}$ be an involution of $\mathcal{A}$.
For any $x, y \in \mathcal{A}$, define the operator $V_{x,y} \colon \mathcal{A} \to \mathcal{A}$ as follows:

$$V_{x,y}(z) = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y \quad \text{for all } z \in \mathcal{A},$$

and set $T_x = V_{x,1}$, i.e., $T_x(z) = xz + zx - z\bar{x}$.

Definition (Allison, 1978)

A unital algebra with involution $(\mathcal{A}, ^-)$ is said to be *structurable* if

$$[T_z, V_{x,y}] = V_{T_z(x),y} - V_{x,T_{\bar{z}}(y)} \quad \text{for all } x, y, z \in \mathcal{A}.$$

If $(\mathcal{A}, ^-)$ is structurable then it is *skew-alternative*, i.e.

$$(s, x, y) = -(x, s, y) = (x, y, s) \quad \text{for all } x, y, s \in \mathcal{A} \text{ with } \bar{s} = -s,$$

where $(x, y, z) := (xy)z - x(yz)$.

Examples of structurable algebras

Example

If $(\mathcal{A},^{-})$ is an associative algebra with involution then it is structurable.

Example

If $\mathcal{J}$ is a Jordan algebra then $(\mathcal{J},^{-})$ is structurable where the involution is the identity map.

Recall that a *Hurwitz algebra* is a unital algebra endowed with a nonsingular multiplicative quadratic form (the *norm*). The *standard conjugation* of a Hurwitz algebra $(\mathcal{C}, n)$ is given by $\bar{x} = -x + n(x, 1)1$.

Example

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are Hurwitz algebras then $(\mathcal{C}_1 \otimes \mathcal{C}_2,^{-})$ is structurable where

$$\overline{x_1 \otimes x_2} = \bar{x}_1 \otimes \bar{x}_2 \quad \text{for all } x_1 \in \mathcal{C}_1 \text{ and } x_2 \in \mathcal{C}_2.$$

Lie algebras associated to a structurable algebra $\mathcal{A}$

Let $\mathcal{H} = \{x \in \mathcal{A} \mid \bar{x} = x\}$ and $\mathcal{S} = \{x \in \mathcal{A} \mid \bar{x} = -x\}$.

- The Lie algebra of derivations (commuting with the involution) $\mathrm{Der}(\mathcal{A})$.
- The (inner) *structure Lie algebra* $\mathfrak{str}(\mathcal{A})$, which is the subalgebra of $\mathfrak{gl}(\mathcal{A})$ spanned by the operators $V_{x,y}$, $x, y \in \mathcal{A}$. For simple $\mathcal{A}$, $\mathrm{Der}(\mathcal{A})$ is a subalgebra of $\mathfrak{str}(\mathcal{A})$ and we have a $\mathbb{Z}_2$-grading on $\mathfrak{str}(\mathcal{A})$ with $\mathfrak{str}(\mathcal{A})_{\bar{0}} = \mathrm{Der}(\mathcal{A}) \oplus T_{\mathcal{S}}$ and $\mathfrak{str}(\mathcal{A})_{\bar{1}} = T_{\mathcal{H}}$.
- The *Steinberg unitary Lie algebra* $\mathfrak{stu}_3(\mathcal{A})$ is obtained from three copies of $\mathcal{A}$.
- The *Kantor algebra* $Kan(\mathcal{A})$ is a Lie algebra graded by the nonreduced root system BC_1 with coordinate algebra $\mathcal{A}$.

Any grading on $\mathcal{A}$ by an abelian group G induces a grading on

- $\mathrm{Der}(\mathcal{A})$ by G,
- $\mathfrak{str}(\mathcal{A})$ and its derived algebra $\mathfrak{str}_0(\mathcal{A})$ by $G \times \mathbb{Z}_2$,
- $\mathfrak{stu}_3(\mathcal{A})$ by $G \times \mathbb{Z}_2^2$,
- $Kan(\mathcal{A})$ by $G \times \mathbb{Z}$.

Kantor construction for structurable algebras

Let $\mathcal{A}$ be a structurable algebra. Consider two copies, $\mathcal{A}^+$ and $\mathcal{A}^-$, of $\mathcal{A}$, and two copies, $\mathcal{S}^+$ and $\mathcal{S}^-$, of $\mathcal{S}$. Let $\mathcal{L} = \mathfrak{str}(\mathcal{A})$ and let ε be the grading automorphism of $\mathcal{L}$. Define $Kan(\mathcal{A}) = \mathcal{S}^- \oplus \mathcal{A}^- \oplus \mathcal{L} \oplus \mathcal{A}^+ \oplus \mathcal{S}^+$. This is a $\mathbb{Z}$-grading with support $\{-2, -1, 0, 1, 2\}$ (if $\mathcal{S} \neq 0$). Consider also the coarsening mod 2: $Kan(\mathcal{A}) = Kan(\mathcal{A})_{\bar{0}} \oplus Kan(\mathcal{A})_{\bar{1}}$.
We can identify the element $(t, y, A, x, s) \in Kan(\mathcal{A})$ with

$$\begin{pmatrix} A & L_s \\ L_t & A^\varepsilon \end{pmatrix} \oplus \begin{pmatrix} x \\ y \end{pmatrix} \in Kan(\mathcal{A})_{\bar{0}} \oplus Kan(\mathcal{A})_{\bar{1}},$$

and define a skew-symmetric product by

$$[C, D] = CD - DC, \quad [C, u] = Cu, \quad [u, v] = u * v - v * u,$$

for $C, D \in Kan(\mathcal{A})_{\bar{0}}$, $u, v \in Kan(\mathcal{A})_{\bar{1}}$, where

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} V_{x,w} & U_{x,z} \\ U_{y,w} & V_{y,z} \end{pmatrix},$$

where $U_{x,z}(y) := V_{x,y}(z)$, $A^\varepsilon := A - T_{A(1)+\overline{A(1)}}$.

Steinberg unitary Lie algebras

If $\mathcal{A}$ is a unital associative algebra with involution, define an involution φ on $M_n(\mathcal{A})$ by $X \mapsto \gamma^{-1}\overline{X}^t\gamma$ where $\gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, $\gamma_i \in \mathbb{F}^\times$.
$\Rightarrow$ the unitary Lie algebra $\mathfrak{u}_n(\mathcal{A}, \gamma) = \{X \in M_n(\mathcal{A}) \mid \varphi(X) = -X\}$.

Definition (Allison–Faulkner, 1993)

Let $\mathcal{A}$ be any unital algebra with involution. Define $\mathfrak{stu}_n(\mathcal{A}, \gamma)$ ($n \geq 3$) as the Lie algebra generated by the symbols $u_{ij}(a)$, $i \neq j$, $a \in \mathcal{A}$, subject to the following relations: $u_{ij}(a) = u_{ji}(-\gamma_i\gamma_j^{-1}\overline{a})$; $a \mapsto u_{ij}(a)$ is $\mathbb{F}$-linear;
$[u_{ij}(a), u_{jk}(b)] = u_{ik}(ab)$ for distinct i, j, k;
$[u_{ij}(a), u_{kl}(b)] = 0$ for distinct i, j, k, l.

If $\mathcal{A}$ is associative, then $u_{ij}(a) \mapsto a[ij] := ae_{ij} - \gamma_i\gamma_j^{-1}\overline{a}e_{ji}$ yields an isomorphism $\mathfrak{stu}_n(\mathcal{A}, \gamma)/\mathfrak{z} \to \mathfrak{psu}_n(\mathcal{A}, \gamma)$.

Theorem (Allison–Faulkner, 1993)

The linear maps $a \mapsto u_{ij}(a)$ are injective $\Leftrightarrow$ either $n \geq 4$ and $\mathcal{A}$ is associative or $n = 3$ and $\mathcal{A}$ is structurable.

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 2$. Let $\mathcal{Q}$ be a Hurwitz algebra with norm n. Fix $0 \neq \alpha \in \mathbb{F}$ and let $\mathfrak{CD}(\mathcal{Q},\alpha) = \mathcal{Q} \oplus \mathcal{Q}w$ be the direct sum of two copies of $\mathcal{Q}$, where we write the element (x, y) as $x + yw$, with multiplication

$$(a + bw)(c + dw) = (ac + \alpha\bar{d}b) + (da + b\bar{c})w,$$

and norm

$$n(x + yw) = n(x) - \alpha n(y).$$

It is well known that $\mathfrak{CD}(\mathcal{Q},\alpha)$ is a Hurwitz algebra $\Leftrightarrow$ $\mathcal{Q}$ is associative.

Note that $\mathcal{K} := \mathfrak{CD}(\mathbb{F},\alpha)$ is $\mathbb{Z}_2$-graded, $\mathcal{Q} := \mathfrak{CD}(\mathcal{K},\beta)$ is $\mathbb{Z}_2^2$-graded and $\mathcal{C} := \mathfrak{CD}(\mathcal{Q},\gamma)$ is $\mathbb{Z}_2^3$-graded. Explicitly,

$$\mathcal{C} = \bigoplus_{\alpha\in\mathbb{Z}_2^3} \mathbb{F}e_\alpha \quad \text{where } e_\alpha = (w_1^{\alpha_1} w_2^{\alpha_2})w_3^{\alpha_3}.$$

Thus, any Cayley algebra $\mathcal{C}$ can be realized as a twisted group algebra $\mathbb{F}^\sigma\mathbb{Z}_2^3$. If $\mathbb{F}$ is a.c. then w_i can be normalized (Albuquerque–Majid, 1999) so that $\sigma(\alpha,\beta)=(-1)^{\psi(\alpha,\beta)}$, where $\psi(\alpha,\beta)=\beta_1\alpha_2\alpha_3+\alpha_1\beta_2\alpha_3+\alpha_1\alpha_2\beta_3+\sum_{i\leq j}\alpha_i\beta_j$.

If $\mathbb{F}$ is a.c. then the quaternion algebra $\mathcal{Q}\cong M_2(\mathbb{F})$, and the $\mathbb{Z}_2^2$-grading induced by the Cayley–Dickson process is isomorphic to the Pauli grading.
More generally, if $\mathbb{F}$ contains a primitive n-th root of 1 then $M_n(\mathbb{F})$ can be realized as a twisted group algebra $\mathbb{F}^\sigma\mathbb{Z}_n^2$ (here σ is a 2-cocycle).

These gradings are *division gradings* in the sense that (nonzero) homogeneous elements are invertible.
If $\mathbb{F}$ is a.c. and $M_n(\mathbb{F})$ is endowed with a division grading by G then the support $T\subset G$ is a subgroup and $M_n(\mathbb{F})\cong\mathbb{F}^\sigma T$. Such gradings are classified up to isomorphism (Bahturin–K, 2010) by the pairs (T,β) where $\beta(a,b)=\sigma(a,b)/\sigma(b,a)$ is a nondegenerate alternating bicharacter $T\times T\to\mathbb{F}^\times$, $T\subset G$, $|T|=n^2$.

First Tits construction

Let $\mathbb{F}$ be an a.c. field, $\operatorname{char}\mathbb{F} \neq 2$. The simple exceptional Jordan algebra $\mathcal{A} = \mathcal{H}_3(\mathcal{C})$, with multiplication $x \circ y = \frac{1}{2}(xy + yx)$, can be realized as the sum of three copies of $\mathcal{R} = M_3(\mathbb{F})$.
Any $x \in \mathcal{R}$ satisfies the Cayley–Hamilton equation

$$x^3 - \mathrm{tr}(x)x^2 + s(x)x - \det(x)1 = 0,$$

where $s(x) = \frac{1}{2}\left(\mathrm{tr}(x)^2 - \mathrm{tr}(x^2)\right)$. Define $x^\sharp = x^2 - \mathrm{tr}(x)x + s(x)1$, so $xx^\sharp = x^\sharp x = \det(x)1$ for any $x \in \mathcal{R}$, and its linearization

$$x \times y = \frac{1}{2}\Big(xy + yx - \big(\mathrm{tr}(x)y + \mathrm{tr}(y)x\big) + \big(\mathrm{tr}(x)\mathrm{tr}(y) - \mathrm{tr}(xy)\big)1\Big).$$

Set $\bar{x} = x \times 1 = \frac{1}{2}\big(\mathrm{tr}(x)1 - x\big)$. Then $\mathcal{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2$, where $\mathcal{R}$ is linearly isomorphic to $\mathcal{R}_i$ ($x \mapsto x_i$), with the following multiplication:

	a'_0	b'_1	c'_2
a_0	$(a \circ a')_0$	$(\bar{a}b')_1$	$(c'\bar{a})_2$
b_1	$(\overline{a'}b)_1$	$(b \times b')_2$	$(\overline{bc'})_0$
c_2	$(c\overline{a'})_2$	$(\overline{b'c})_0$	$(c \times c')_1$

Albert algebra as a twisted group algebra

Assume $\operatorname{char}\mathbb{F} \neq 2, 3$ and let ω be a primitive cubic root of 1. Let

$$x = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

which satisfy $xy = \omega yx$.
Then we have a $\mathbb{Z}_3^3$-grading on $\mathcal{A}$ defined by

$$\mathcal{A} = \bigoplus_{\alpha \in \mathbb{Z}_3^3} \mathbb{F}e_\alpha \quad \text{where } e_\alpha = \omega^{\alpha_1\alpha_2}(x^{\alpha_1}y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3}.$$

This is a division grading and identifies (Griess, 1990) $\mathcal{A}$ with $\mathbb{F}^\sigma\mathbb{Z}_3^3$ where

$$\sigma(\alpha, \beta) = \begin{cases} \omega^{\psi(\alpha,\beta)} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\psi(\alpha,\beta)} & \text{otherwise,} \end{cases}$$

and $\psi(\alpha, \beta) = (\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_3 - \beta_3)$.

Cayley–Dickson doubling process for Jordan algebras

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 2$. For a separable (finite-dimensional) Jordan algebra $(\mathcal{J}, \cdot)$ of degree 4 , we can define a structurable algebra by means of the following doubling process.

Let $\mathcal{A} = \mathcal{J} \oplus v\mathcal{J}$ with multiplication determined by the following rules:

$$ab = a \cdot b,\ a(vb) = v(a^\theta \cdot b),\ (va)b = v(a^\theta \cdot b^\theta)^\theta,\ (va)(vb) = (a \cdot b^\theta)^\theta,$$

where $\theta \colon \mathcal{J} \to \mathcal{J}$ is a linear map defined by $1^\theta = 1$ and $a^\theta = -a$ for any element a whose generic trace is zero. The involution of $\mathcal{A}$ is defined by $\overline{a + vb} = a - vb^\theta$.

Note that the only skew-symmetric elements are the scalar multiples of v. Since $v^2 = 1$, it follows that all automorphisms of $\mathcal{A}$ (commuting with involution) send v to $\pm v$ and all derivations (commuting with involution) send v to 0. Any automorphism or derivation of $\mathcal{J}$ extends uniquely to $\mathcal{A}$. A grading on $\mathcal{J}$ by an abelian group G induces a grading on $\mathcal{A}$ by $G \times \mathbb{Z}_2$.

The structurable algebra $\mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$

Let $\mathcal{Q}$ be the split quaternion algebra over $\mathbb{F}$, equipped with its standard involution. Upon the identification $\mathcal{Q} \cong M_2(\mathbb{F})$, the involution switches E_{11} with E_{22} and multiplies both E_{12} and E_{21} by -1. The subalgebra $\mathcal{K} = \mathrm{Span}\{E_{11}, E_{22}\}$ is isomorphic to $\mathbb{F} \times \mathbb{F}$ with exchange involution.

Consider the associative algebra $M_4(\mathcal{Q})$ with involution $(q_{ij})^* = (\overline{q}_{ji})$. Since $M_4(\mathcal{Q}) \cong M_4(\mathbb{F}) \otimes \mathcal{Q}$, we can alternatively write the elements of $M_4(\mathcal{Q})$ as sums of tensor products or as 2×2 matrices over $M_4(\mathbb{F})$.

Consider the Jordan subalgebra of symmetric elements

$$\begin{aligned}\mathcal{H}_4(\mathcal{Q}) &= \{a \in M_4(\mathcal{Q}) \mid a^* = a\} \\ &= \{\left(\begin{smallmatrix} z & x \\ y & z^t \end{smallmatrix}\right) \mid x, y, z \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\}.\end{aligned}$$

Note that the subalgebra $\mathcal{H}_4(\mathcal{K}) \subset \mathcal{H}_4(\mathcal{Q})$ is isomorphic to $M_4(\mathbb{F})^{(+)}$.

The Cayley–Dickson double $\mathcal{A} = \mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$ is a simple structurable algebra of dimension 56. The simple Lie algebras of "series" E can be constructed in terms of $\mathcal{A}$ as follows: $\mathrm{Der}(\mathcal{A})$ has type E_6, $\mathfrak{str}_0(\mathcal{A})$ has type E_7 and $\mathfrak{stu}_3(\mathcal{A})$ has type E_8.

Construction of the $\mathbb{Z}_4^3$-grading

Assume $\mathbb{F}$ contains a 4-th root of 1. The construction will proceed in two steps:

- define a $\mathbb{Z}_4$-grading on $\mathcal{A} = \mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$,
- refine it using two commuting automorphisms of order 4.

The even components of the $\mathbb{Z}_4$-grading are just $\mathcal{A}_{\bar{0}} = \mathcal{H}_4(\mathcal{K})$ and $\mathcal{A}_{\bar{2}} = v\mathcal{H}_4(\mathcal{K})$. The odd components are as follows:

$$\mathcal{A}_{\bar{1}} = \{x \otimes E_{12} + v(y \otimes E_{21}) \mid x, y \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\} \quad \text{and}$$
$$\mathcal{A}_{\bar{3}} = \{x \otimes E_{21} + v(y \otimes E_{12}) \mid x, y \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\} = v\mathcal{A}_{\bar{1}}.$$

The group $\mathrm{GL}_4(\mathbb{F})$ acts on $\mathcal{H}_4(\mathcal{Q})$ via $g \mapsto \mathrm{Ad}\begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix}$. Let φ and ψ be the automorphisms of $\mathcal{H}_4(\mathcal{Q})$ corresponding to the generalized Pauli matrices X and Y in $\mathrm{GL}_4(\mathbb{F})$. We denote their extensions to $\mathcal{A}$ by the same symbols.

Note that φ and ψ have order 4 and preserve the $\mathbb{Z}_4$-grading of $\mathcal{A}$,but they do not commute!

The automorphism π

The automorphisms φ and ψ of $\mathcal{A}$ commute on the even component $\mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{2}} = \mathcal{H}_4(\mathcal{K}) \oplus v\mathcal{H}_4(\mathcal{K})$ and anticommute on the odd component $\mathcal{A}_{\bar{1}} \oplus \mathcal{A}_{\bar{3}}$.

We will construct another automorphism π of order 4 that preserves the $\mathbb{Z}_4$-grading, commutes with each of φ and ψ on $\mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{2}}$ and anticommutes on $\mathcal{A}_{\bar{1}} \oplus \mathcal{A}_{\bar{3}}$.

Let $U = \{x \otimes E_{12} \mid x^t = -x\}$ and $V = \{y \otimes E_{21} \mid y^t = -y\}$, so $\mathcal{A}_{\bar{1}} = U \oplus vV$ and $\mathcal{A}_{\bar{3}} = V \oplus vU$.

U and V are dual $\mathrm{GL}_4(\mathbb{F})$-modules, but isomorphic as $\mathrm{SL}_4(\mathbb{F})$-modules. We construct an $\mathrm{SL}_4(\mathbb{F})$-isomorphism $U \to V$, $x \otimes E_{12} \mapsto \widehat{x} \otimes E_{21}$, using the Pfaffian $\mathrm{pf}(x) = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$ for skew $x = (x_{ij}) \in M_4(\mathbb{F})$.

Finally, we define $\pi \colon \mathcal{A} \to \mathcal{A}$ as identity on $\mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{2}}$ and

$$\begin{aligned} \pi(x \otimes E_{12}) &= -v(\widehat{x} \otimes E_{21}), & \pi(v(x \otimes E_{12})) &= -\widehat{x} \otimes E_{21}, \\ \pi(x \otimes E_{21}) &= v(\widehat{x} \otimes E_{12}), & \pi(v(x \otimes E_{21})) &= \widehat{x} \otimes E_{12}. \end{aligned}$$

Fix $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$, so $X = \mathrm{diag}(1, \mathbf{i}, -1, -\mathbf{i})$. We will keep ψ and replace φ by $\widetilde{\varphi}$, which is the composition of π and the action of $\widetilde{X} = \mathrm{diag}(\omega, \omega^3, \omega^5, \omega^7)$ where $\omega^2 = \mathbf{i}$. (We can temporarily extend $\mathbb{F}$.)

Then $\widetilde{\varphi}$ and ψ are commuting automorphisms of order 4 and hence we get a $\mathbb{Z}_4^3$-grading of $\mathcal{A}$ by setting

$$\mathcal{A}_{(\bar{j},\bar{k},\bar{\ell})} = \{a \in \mathcal{A}_{\bar{j}} \mid \psi(a) = \mathbf{i}^k,\ \widetilde{\varphi}(a) = (-\mathbf{i})^\ell\}.$$

Explicitly, the homogeneous components are given by

$$\begin{aligned}
\mathcal{A}_{(\bar{0},\bar{k},\bar{\ell})} &= \mathbb{F}(X^k Y^\ell \otimes E_{11} + (X^k Y^\ell)^t \otimes E_{22});\\
\mathcal{A}_{(\bar{2},\bar{k},\bar{\ell})} &= \mathbb{F}v(X^k Y^\ell \otimes E_{11} + (X^k Y^\ell)^t \otimes E_{22});\\
\mathcal{A}_{(\bar{1},\bar{0},\bar{\ell})} &= \mathbb{F}(\xi_1 \otimes E_{12} + \mathbf{i}^\ell v(\xi_1 \otimes E_{21})),\ \ell = 1,3;\\
\mathcal{A}_{(\bar{1},\bar{2},\bar{\ell})} &= \mathbb{F}(\xi_2 \otimes E_{12} + \mathbf{i}^\ell v(\xi_2 \otimes E_{21})),\ \ell = 1,3;\\
\mathcal{A}_{(\bar{1},\bar{1},\bar{\ell})} &= \mathbb{F}(\xi_3 \otimes E_{12} - \mathbf{i}^\ell v(\xi_3 \otimes E_{21})),\ \ell = 1,3;\\
\mathcal{A}_{(\bar{1},\bar{3},\bar{\ell})} &= \mathbb{F}(\xi_4 \otimes E_{12} - \mathbf{i}^\ell v(\xi_4 \otimes E_{21})),\ \ell = 1,3;\\
\mathcal{A}_{(\bar{1},\bar{1},\bar{\ell})} &= \mathbb{F}(\xi_5 \otimes E_{12} + \mathbf{i}^\ell v(\xi_5 \otimes E_{21})),\ \ell = 0,2;\\
\mathcal{A}_{(\bar{1},\bar{3},\bar{\ell})} &= \mathbb{F}(\xi_6 \otimes E_{12} + \mathbf{i}^\ell v(\xi_6 \otimes E_{21})),\ \ell = 0,2.
\end{aligned}$$

Construction of the $\mathbb{Z}_4^3$-grading (completed)

On the previous slide, $\{\xi_1,\ldots,\xi_6\}$ is the following basis of $\mathcal{K}_4(\mathbb{F})$:

$$\xi_{1,2}=\begin{bmatrix}0&1&0&\mp 1\\ &0&\pm 1&0\\ &&0&1\\ \text{skew}&&&0\end{bmatrix},\ \xi_{3,4}=\begin{bmatrix}0&1&0&\pm \mathbf{i}\\ &0&\pm \mathbf{i}&0\\ &&0&-1\\ \text{skew}&&&0\end{bmatrix},\ \xi_{5,6}=\begin{bmatrix}0&0&1&0\\ &0&0&\pm \mathbf{i}\\ &&0&0\\ \text{skew}&&&0\end{bmatrix}$$

Since all nonzero homogeneous components have dimension 1, our $\mathbb{Z}_4^3$-grading on $\mathcal{A}=\mathcal{H}_4(\mathcal{Q})\oplus v\mathcal{H}_4(\mathcal{Q})$ is fine.
Up to equivalence, it is a unique $\mathbb{Z}_4^3$-grading with this property;
it is a division grading (Aranda–Elduque–K, 2013).
The support S is a proper subset of $\mathbb{Z}_4^3$ of size $\dim\mathcal{A}=56$, which can be characterized as follows:
$g\notin S \Leftrightarrow g^2=h$ where $h=(\bar{2},\bar{0},\bar{0})$ is the degree of v.

Recall that the even component $\mathcal{A}_{\bar{0}}\oplus\mathcal{A}_{\bar{2}}$ is the double of the Jordan algebra $M_4(\mathbb{F})^{(+)}$, so this double receives a fine grading by $\mathbb{Z}_2\times\mathbb{Z}_4^2$.
Here the support is the entire group; there is a distinguished element $h=(\bar{1},\bar{0},\bar{0})$ of order 2 (the degree of v).

Fine gradings for G_2 and F_4

Assume that the ground field $\mathbb{F}$ is a.c., $\operatorname{char} \mathbb{F} \neq 2, 3$.

Up to equivalence, there are exactly two fine gradings on the Cayley algebra $\mathcal{C}$: the division $\mathbb{Z}_2^3$-grading and the Cartan $\mathbb{Z}^2$-grading (Elduque, 1998). They yield two fine gradings on the simple Lie algebra $\mathrm{Der}(\mathcal{C})$ of type G_2, which is a complete list (Draper–Martin, 2006; Elduque–K, 2012).

Up to equivalence, there are exactly four fine (abelian) gradings on the Albert algebra $\mathcal{J}$, with universal groups $\mathbb{Z}^4$, $\mathbb{Z} \times \mathbb{Z}_2^3$, $\mathbb{Z}_2^5$ and $\mathbb{Z}_3^3$ (Draper–Martin, 2009; Elduque–K, 2012). They yield four fine gradings on the simple Lie algebra $\mathrm{Der}(\mathcal{J})$ of type F_4, which is a complete list. Two of these can be obtained from the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$, regarded as a structurable algebra, using the models *Kan*$(\mathcal{C})$ and $\mathfrak{stu}_3(\mathcal{C})$ for F_4.

Fine gradings for “series” E, infinite universal group

The ground field $\mathbb{F}$ is assumed a.c., $\operatorname{char}\mathbb{F} \neq 2, 3$.

E_6		E_7		E_8	
Universal group	Model	Universal group	Model	Universal group	Model
$\mathbb{Z}^6$	Cartan	$\mathbb{Z}^7$	Cartan	$\mathbb{Z}^8$	Cartan
$\mathbb{Z}^4 \times \mathbb{Z}_2$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{K})$	$\mathbb{Z}^4 \times \mathbb{Z}_2^2$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{Q})$	$\mathbb{Z}^4 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{C})$
$\mathbb{Z}^2 \times \mathbb{Z}_3^2$	$\mathcal{T}(\Gamma^1_{\mathcal{C}}, \Gamma^2_{M_3(\mathbb{F})})$ $(G_2\,,\ M_3(\mathbb{F})^{(+)})$	——		$\mathbb{Z}^2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^1_{\mathcal{C}}, \Gamma^4_{\mathcal{A}})$ $(G_2\,,\ \mathcal{A})$
$\mathbb{Z}^2 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{M_3(\mathbb{F})})$ $(A_2\,,\ \mathcal{C})$	$\mathbb{Z}^3 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{\mathcal{H}_3(\mathcal{Q})})$ $(C_3\,,\ \mathcal{C})$	——	
$\mathbb{Z}^2 \times \mathbb{Z}_2^3$	$Kan(\tilde{\Gamma}_{\mathcal{X}(\mathbb{F})})$ $(BC_2\,,\ \mathcal{X}(\mathbb{F})_{1/2})$	$\mathbb{Z}^2 \times \mathbb{Z}_2^4$	$Kan(\tilde{\Gamma}_{\mathcal{X}(\mathcal{K})})$ $(BC_2\,,\ \mathcal{X}(\mathcal{K})_{1/2})$	$\mathbb{Z}^2 \times \mathbb{Z}_2^5$	$Kan(\tilde{\Gamma}_{\mathcal{X}(\mathcal{Q})})$ $(BC_2\,,\ \mathcal{X}(\mathcal{Q})_{1/2})$
——		$\mathbb{Z} \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^1_{\mathcal{Q}}, \Gamma^4_{\mathcal{A}})$ $(A_1\,,\ \mathcal{A})$	——	
$\mathbb{Z} \times \mathbb{Z}_2^5$	$Kan(\Gamma^1_{\mathcal{X}(\mathbb{F})})$ $(BC_1\,,\ \mathcal{X}(\mathbb{F}))$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$Kan(\Gamma^1_{\mathcal{X}(\mathcal{K})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{K}))$	$\mathbb{Z} \times \mathbb{Z}_2^7$	$Kan(\Gamma^1_{\mathcal{X}(\mathcal{Q})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{Q}))$
$\mathbb{Z} \times \mathbb{Z}_2^4$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{K} \otimes \mathcal{C})$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{Q} \otimes \mathcal{C})$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{C} \otimes \mathcal{C})$
——		$\mathbb{Z} \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$Kan(\Gamma^2_{\mathcal{X}(\mathcal{K})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{K}))$	$\mathbb{Z} \times \mathbb{Z}_4^3$	$Kan(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$ $(BC_1\,,\ \mathcal{X}(\mathcal{Q}))$

Fine gradings for “series” E, finite universal group

E_6		E_7		E_8	
Universal group	Model	Universal group	Model	Universal group	Model
$\mathbb{Z}_3^4$	$\mathfrak{g}(\Gamma_{\bar{\mathcal{K}}}, \Gamma_{\mathcal{O}})$	—		$\mathbb{Z}_3^5$	$\mathfrak{g}(\Gamma_{\mathcal{O}}, \Gamma_{\mathcal{O}})$
$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$	$\mathcal{T}(\Gamma_{\mathcal{C}}^2, \Gamma_{M_3(\mathbb{F})}^2)$	—		—	
$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma_{\mathcal{A}}^4)$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma_{\mathcal{Q}}^2, \Gamma_{\mathcal{A}}^4)$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma_{\mathcal{C}}^2, \Gamma_{\mathcal{A}}^4)$
$\mathbb{Z}_2^7$	$\mathfrak{stu}_3(\Gamma_{\mathcal{X}(\mathbb{F})}^1)$	$\mathbb{Z}_2^8$	$\mathfrak{stu}_3(\Gamma_{\mathcal{X}(\mathcal{K})}^1)$	$\mathbb{Z}_2^9$	$\mathfrak{stu}_3(\Gamma_{\mathcal{X}(\mathcal{Q})}^1)$
$\mathbb{Z}_2^6$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma_{\mathcal{A}}^2)$	$\mathbb{Z}_2^7$	$\mathcal{T}(\Gamma_{\mathcal{Q}}^2, \Gamma_{\mathcal{A}}^2)$	$\mathbb{Z}_2^8$	$\mathcal{T}(\Gamma_{\mathcal{C}}^2, \Gamma_{\mathcal{A}}^2)$
$\mathbb{Z}_4^3$	$\mathrm{Der}(\Gamma_{\mathcal{X}(\mathcal{Q})}^2)$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2$	$\mathfrak{str}_0(\Gamma_{\mathcal{X}(\mathcal{Q})}^2)$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2^2$	$\mathfrak{stu}_3(\Gamma_{\mathcal{X}(\mathcal{Q})}^2)$
$\mathbb{Z}_4 \times \mathbb{Z}_2^4$	$\mathrm{Der}(\Gamma_{\mathcal{X}(\mathcal{Q})}^3)$	$\mathbb{Z}_4 \times \mathbb{Z}_2^5$	$\mathfrak{str}_0(\Gamma_{\mathcal{X}(\mathcal{Q})}^3)$	$\mathbb{Z}_4 \times \mathbb{Z}_2^6$	$\mathfrak{stu}_3(\Gamma_{\mathcal{X}(\mathcal{Q})}^3)$
—		$\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$	$\mathfrak{stu}_3(\Gamma_{\mathcal{X}(\mathcal{K})}^2)$	—	
—		—		$\mathbb{Z}_5^3$	Jordan grading

The list is known to be complete (up to equivalence) for E_6 if $\mathrm{char}\,\mathbb{F} = 0$ (Draper–Viruel, preprint 2012).
If $\mathbb{F} = \mathbb{C}$, the completeness of the list for E_r, $r = 6, 7, 8$, follows from the results of Yu on compact real Lie algebras of these types, announced in 2014. By the result of Elduque on the independence of the ground field, also announced in 2014, this holds in general if $\mathrm{char}\,\mathbb{F} = 0$.