Applications of affine group schemes to the study of gradings by abelian groups

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Definition of group grading

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ and let G be a (semi)group.

Definition

A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ such that $\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$.

Definition

Two G-gradings on $\mathcal{A}$, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}'_g$, are *isomorphic* if there exists an algebra automorphism $\psi\colon \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$.

Problem: given an algebra $\mathcal{A}$ and an abelian group G, classify the G-gradings on $\mathcal{A}$ up to isomorphism.
Solved for f.d. s.s. associative ($\mathbb{F}$ is alg. closed or real closed) and Jordan ($\mathbb{F}$ is a.c., $\operatorname{char}\mathbb{F} \neq 2$) algebras, also for simple Lie A-G except E ($\mathbb{F}$ is a.c., $\operatorname{char}\mathbb{F} \neq 2$) and their real forms.

Cartan grading of a semisimple Lie algebra

Historically the first grading to be studied (and still the most important):

Example (Cartan grading)

Let $\mathfrak{g}$ be a f.-d. semisimple Lie algebra over an a.c. field of char 0, and let $\mathfrak{h}$ be a Cartan subalgebra. Then the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$$

can be viewed as a grading by the root lattice $\langle\Phi\rangle \cong \mathbb{Z}^r$, $r = \dim \mathfrak{h}$. The support is $\{0\} \cup \Phi$.

Cartan grading also exists for simple Lie algebras of types A-G in characteristic $p > 0$.

Pauli matrices

Example (Pauli grading)

There is a grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by the group a $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Namely, we set

$$\begin{aligned} \mathfrak{g}_{(0,0)} &= 0, & \mathfrak{g}_{(1,0)} &= \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \\ \mathfrak{g}_{(0,1)} &= \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}, & \mathfrak{g}_{(1,1)} &= \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}. \end{aligned}$$

The Pauli grading can be defined for $\mathfrak{sl}_2(\mathbb{F})$, $\operatorname{char}\mathbb{F} \neq 2$.
Any G-grading on $\mathfrak{sl}_2(\mathbb{F})$ is induced by the Pauli or Cartan grading via a group homomorphism $\mathbb{Z}_2^2 \to G$, resp. $\mathbb{Z} \to G$.
Given $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, a homomorphism $\alpha\colon G \to H$ induces ${}^{\alpha}\Gamma : \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}_h$ where $\mathcal{A}_h = \bigoplus_{g\in\alpha^{-1}(h)} \mathcal{A}_g$.

Affine schemes

Let $\mathrm{Alg}_{\mathbb{F}}$ be the category of associative commutative unital algebras over a field $\mathbb{F}$.

Definition

An *affine scheme* over $\mathbb{F}$ is a representable functor $\mathbf{S}\colon \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Sets}$.

By Yoneda's Lemma, the representing object is unique up to isomorphism. It is denoted $\mathbb{F}[\mathbf{S}]$. Thus, $\mathbf{S}(\mathcal{R}) = \mathrm{Alg}_{\mathbb{F}}(\mathbb{F}[\mathbf{S}], \mathcal{R})$, for any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$ (the set of $\mathcal{R}$-points of $\mathbf{S}$).
The set of morphisms (=natural transformations) $\mathbf{S}_1 \to \mathbf{S}_2$ is in bijection with $\mathrm{Alg}_{\mathbb{F}}(\mathbb{F}[\mathbf{S}_2], \mathbb{F}[\mathbf{S}_1])$. A morphism $\theta\colon \mathbf{S}_1 \to \mathbf{S}_2$ corresponds to the *comorphism* $\theta^* : \mathbb{F}[\mathbf{S}_2] \to \mathbb{F}[\mathbf{S}_1]$: for any $\mathcal{R} \in \mathrm{Alg}_{\mathbb{F}}$, $\theta_{\mathcal{R}}$ is the pre-composition with θ^*.
$\mathbf{S}$ is *finite* if $\mathbb{F}[\mathbf{S}]$ is finite-dimensional;
$\mathbf{S}$ is *algebraic* if $\mathbb{F}[\mathbf{S}]$ is finitely generated.

Affine group schemes and commutative Hopf algebras

Definition

An *affine group scheme* over $\mathbb{F}$ is a representable functor $\mathbf{G}\colon \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Groups}$.

It follows from Yoneda's Lemma that $\mathbb{F}[\mathbf{G}]$ is a (comm) Hopf algebra; for any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$, $\mathbf{G}(\mathcal{R}) = \mathrm{Alg}_{\mathbb{F}}(\mathbb{F}[\mathbf{G}], \mathcal{R})$ is a group under the convolution product (the group of $\mathcal{R}$-points of $\mathbf{G}$).

$\theta\colon \mathbf{G}_1 \to \mathbf{G}_2$ is a homomorphism
$\Leftrightarrow \theta^*\colon \mathbb{F}[\mathbf{G}_2] \to \mathbb{F}[\mathbf{G}_1]$ is a Hopf algebra map.

$\boxed{\text{subgroupschemes of } \mathbf{G}} \leftrightarrow \boxed{\text{Hopf ideals of } \mathbb{F}[\mathbf{G}]}$

$\boxed{\text{quotients of } \mathbf{G}} \leftrightarrow \boxed{\text{Hopf subalgebras of } \mathbb{F}[\mathbf{G}]}$

If $\mathbf{G}$ is algebraic then $\mathbf{G}(\overline{\mathbb{F}})$ is an (affine) algebraic group.
$\mathbf{G}$ is *smooth* if $\overline{\mathbb{F}}[\mathbf{G}] := \mathbb{F}[\mathbf{G}] \otimes \overline{\mathbb{F}}$ is reduced (=no nilpotents).
In characteristic 0, all comm Hopf algebras are reduced.

Examples of affine group schemes

Example (Automorphism group schemes)

Let $\mathcal{A}$ be an $\mathbb{F}$-algebra, not necessarily associative. Define $\mathbf{Aut}_{\mathbb{F}}(\mathcal{A})(\mathcal{R}) := \mathrm{Aut}_{\mathcal{R}}(\mathcal{A}_{\mathcal{R}})$ where $\mathcal{A}_{\mathcal{R}} := \mathcal{A} \otimes \mathcal{R}$, for any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$. If $\mathcal{A}$ is f.d. then $\mathbf{Aut}_{\mathbb{F}}(\mathcal{A})$ is an algebraic group scheme.

For instance, $\mathbf{Aut}_{\mathbb{F}}(M_n(\mathbb{F})) = \mathbf{PGL}_n$.

Example ("Constant" group schemes)

Let M be a finite group. Then the comm Hopf algebra $(\mathbb{F}M)^* = \mathrm{Fun}(M, \mathbb{F})$ represents a finite group scheme, which we denote $\mathbf{M}$. If $\mathcal{R}$ has no nontrivial idempotents, then $\mathbf{M}(\mathcal{R}) = M$.

For instance, $\mathbf{Aut}_{\mathbb{F}}(\mathbb{F} \times \mathbb{F} \times \mathbb{F}) = \mathbf{S}_3$.

Example (Diagonalizable group schemes)

Let G be an abelian group. Then the comm Hopf algebra $\mathbb{F}G$ represents a group scheme (algebraic iff G is f.g.), denoted G^D.

Examples of affine group schemes continued

For instance, $\mathbb{Z}^D = \boldsymbol{\mu}$ where $\boldsymbol{\mu}(\mathcal{R}) := \mathcal{R}^\times$ (multiplicative group); $(\mathbb{Z}/n\mathbb{Z})^D = \boldsymbol{\mu}_n$ where $\boldsymbol{\mu}_n(\mathcal{R}) := \{x \in \mathcal{R} \mid x^n = 1\}$ (roots of 1).

Definition (Étale group schemes)

A finite group scheme is *étale* if its representing (f.d.) Hopf algebra is étale (i.e., separable as an algebra).

$\mathbf{G}$ is étale $\Leftrightarrow \mathbb{F}_{\mathrm{sep}}[\mathbf{G}] := \mathbb{F}[\mathbf{G}] \otimes \mathbb{F}_{\mathrm{sep}}$ is spanned by idempotents. Thus, étale group schemes are precisely the twisted forms of "constant" group schemes. Let $\mathcal{G} = \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$.

Definition

A $\mathcal{G}$-group is a group M with a continuous $\mathcal{G}$-action by automorphisms.

$\boxed{\text{étale group schemes}} \leftrightarrow \boxed{\text{finite } \mathcal{G}\text{-groups}}$

Action by the character group $\widehat{G}$

Let G be an abelian group. Then the character group $\widehat{G} := \mathrm{Hom}(G, \mathbb{F}^\times)$ acts on any G-graded vector space $V = \bigoplus_{g \in G} V_g$ as follows:

$$\chi \cdot v = \chi(g)v \text{ for all } v \in V_g \text{ and } g \in G \text{ (extended by linearity).}$$

In other words, $\chi \cdot v = (\mathrm{id} \otimes \chi)\rho(v)$ for all $v \in V$, where $\rho \colon V \to V \otimes \mathbb{F}G$ is the coaction corresponding to the grading.

If $\mathbb{F}$ is a.c. and $\mathrm{char}\,\mathbb{F} = 0$ (or $\mathrm{char}\,\mathbb{F} = p$ and G has no p-torsion) then $\widehat{G}$ separates the points of G and hence the grading can be recovered from the $\widehat{G}$-action:

$$V_g = \{v \in V \mid \chi \cdot v = \chi(g)v \ \forall \chi \in \widehat{G}\}.$$

But: not every linear $\widehat{G}$-action on V corresponds to a G-grading: only *algebraic* actions.

Action by the diagonalizable group scheme G^D

Note that $\widehat{G} = \mathrm{Alg}_{\mathbb{F}}(\mathbb{F}G, \mathbb{F})$, so $\widehat{G}$ is the group of $\mathbb{F}$-points of G^D.

Definition (Linear representation of an affine group scheme)

A *representation* of $\mathbf{G}$ on V is a homomorphism (=natural transformation) $\mathbf{G} \to \mathbf{GL}(V)$, where $\mathbf{GL}(V)(\mathcal{R}) := \mathrm{End}_{\mathcal{R}}(V_{\mathcal{R}})^{\times}$ and $V_{\mathcal{R}} := V \otimes \mathcal{R}$, for any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$.

If $\dim V = n < \infty$ then $\mathbf{GL}(V) \cong \mathbf{GL}_n$ is representable.

$\boxed{G\text{-gradings on } V} \leftrightarrow \boxed{\text{representations of } G^D \text{ on } V}$

$\Gamma : V = \bigoplus_{g \in G} V_g$ corresponds to $\eta = \eta_\Gamma : G^D \to \mathbf{GL}(V)$, where $\eta_{\mathcal{R}} : G^D(\mathcal{R}) \to \mathrm{End}_{\mathcal{R}}(V_{\mathcal{R}})^{\times}$, for any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$, is given by

$\eta_{\mathcal{R}}(\chi)(v \otimes r) = v \otimes \chi(g)r$ for all $\chi \in G^D(\mathcal{R}) = \mathrm{Hom}(G, \mathcal{R}^{\times})$, $r \in \mathcal{R}$, $v \in V_g$ and $g \in G$ (extended by linearity).

In other words, $\eta_{\mathcal{R}}(\chi)(v \otimes r) = ((\mathrm{id} \otimes \chi)\rho(v))r$ for all $v \in V$ and $r \in \mathcal{R}$, where $\rho : V \to V \otimes \mathbb{F}G$ is the coaction.

A transfer theorem

Let $\mathbb{F}$ be an arbitrary field. Let $\mathcal{A}$ and $\mathcal{B}$ be f.d. (nonassociative) algebras over $\mathbb{F}$, possibly equipped with some additional structure (for example, an $\mathbb{F}$-linear involution).

Theorem

Suppose we have a homomorphism $\theta\colon \mathbf{Aut}_{\mathbb{F}}(\mathcal{A}) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{B})$. Then, for any abelian group G, we have a mapping, $\Gamma \mapsto \theta(\Gamma)$, from G-gradings on $\mathcal{A}$ to G-gradings on $\mathcal{B}$. If Γ and Γ' are isomorphic then $\theta(\Gamma)$ and $\theta(\Gamma')$ are isomorphic.

The grading $\theta(\Gamma)$ is given by the homomorphism $\theta \circ \eta_\Gamma\colon G^D \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{B})$, where $\eta_\Gamma\colon G^D \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{A})$ determines Γ. Note that, for any homom $\alpha\colon G \to H$, we have $\theta({}^\alpha\Gamma) = {}^\alpha(\theta(\Gamma))$.

Corollary

If θ is an isomorphism then $\mathcal{A}$ and $\mathcal{B}$ have the same classification of G-gradings.

Tangent Lie algebra and smoothness

Let $\mathbf{G}$ be an affine group scheme over $\mathbb{F}$.

Definition (Tangent Lie algebra)

$\mathrm{Lie}(\mathbf{G})$ is the kernel of the homomorphism $\mathbf{G}(\pi)\colon \mathbf{G}(\mathbb{F}[\tau]) \to \mathbf{G}(\mathbb{F})$ where $\mathbb{F}[\tau] = \mathbb{F} \oplus \mathbb{F}\tau$ with $\tau^2 = 0$, and $\pi\colon \mathbb{F}[\tau] \to \mathbb{F}$ sends $\tau \mapsto 0$.

$\mathrm{Lie}(\mathbf{G})$ can be identified with $\mathrm{Prim}(\mathbb{F}[\mathbf{G}]^\circ)$, so it is actually a Lie algebra (restricted if $\mathrm{char}\,\mathbb{F} = p$).
Lie is a functor from the category of group schemes to the category of Lie algebras: a homomorphism $\theta\colon \mathbf{G}_1 \to \mathbf{G}_2$ yields a Lie algebra map $\mathrm{d}\theta\colon \mathrm{Lie}(\mathbf{G}_1) \to \mathrm{Lie}(\mathbf{G}_2)$.

Theorem (Differential criterion of smoothness)

Suppose $\mathbf{G}$ is algebraic. Then $\mathbf{G}$ is smooth iff $\dim \mathrm{Lie}(\mathbf{G}) = \dim \mathbf{G}$.

An isomorphism criterion

$\dim \mathbf{G} := \mathrm{Krull.dim}\, \mathbb{F}[\mathbf{G}]$(=dimension of the alg. group $\mathbf{G}(\overline{\mathbb{F}})$).
In general, $\dim \mathrm{Lie}(\mathbf{G}) \geq \dim \mathbf{G}$.

Example (Automorphism group scheme)

$\mathrm{Lie}(\mathbf{Aut}_{\mathbb{F}}(\mathcal{A})) = \mathrm{Der}_{\mathbb{F}}(\mathcal{A})$, so $\mathbf{Aut}_{\mathbb{F}}(\mathcal{A})$ is smooth iff $\dim \mathrm{Der}_{\mathbb{F}}(\mathcal{A}) = \dim \mathrm{Aut}_{\overline{\mathbb{F}}}(\mathcal{A}_{\overline{\mathbb{F}}})$.

All automorphisms and derivations of $M_n(\mathbb{F})$ are inner, hence $\mathbf{Aut}_{\mathbb{F}}(M_n(\mathbb{F})) = \mathbf{PGL}_n$ is smooth.

Theorem

Let $\theta\colon \mathbf{G}_1 \to \mathbf{G}_2$ be a homomorphism of alg. group schemes. Assume that $\mathbf{G}_1$ or $\mathbf{G}_2$ is smooth. Then θ is an isomorphism iff $\theta_{\overline{\mathbb{F}}}\colon \mathbf{G}_1(\overline{\mathbb{F}}) \to \mathbf{G}_2(\overline{\mathbb{F}})$ and $\mathrm{d}\theta\colon \mathrm{Lie}(\mathbf{G}_1) \to \mathrm{Lie}(\mathbf{G}_2)$ are bijective.

If $\mathrm{char}\, \mathbb{F} = 0$ then the bijectivity of $\theta_{\overline{\mathbb{F}}}$ is sufficient.

Types G_2 and F_4

Let $\mathcal{C}$ be a Cayley algebra over $\mathbb{F}$. Then $\mathbf{Aut}_{\mathbb{F}}(\mathcal{C})$ is smooth.
Assume $\operatorname{char}\mathbb{F} \neq 2, 3$. Then $\operatorname{Aut}_{\overline{\mathbb{F}}}(\mathcal{C}_{\overline{\mathbb{F}}})$ is a simple alg. group of type G_2 and $\mathcal{L} := \operatorname{Der}_{\mathbb{F}}(\mathcal{C})$ is a simple Lie algebra of type G_2.
$\operatorname{Ad} : \mathbf{Aut}_{\mathbb{F}}(\mathcal{C}) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism.

Assume $\operatorname{char}\mathbb{F} \neq 2$ and let $\mathcal{A} = \mathcal{H}_3(\mathcal{C})$. $\mathcal{A}$ is an exceptional simple Jordan algebra (also called Albert algebra).
Then $\operatorname{Aut}_{\overline{\mathbb{F}}}(\mathcal{A}_{\overline{\mathbb{F}}})$ is a simple alg. group of type F_4 and $\mathcal{L} := \operatorname{Der}_{\mathbb{F}}(\mathcal{C})$ is a simple Lie algebra of type F_4.
$\operatorname{Ad} : \mathbf{Aut}_{\mathbb{F}}(\mathcal{A}) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism.

Therefore, Ad gives a bijection between (iso classes) of G-gradings on $\mathcal{C}$, resp. $\mathcal{A}$, and G-gradings on $\mathcal{L}$.
Ad maps a grading $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ to the following grading on $\mathcal{L}$:
$\operatorname{End}_{\mathbb{F}}(\mathcal{C})$ is graded by $\operatorname{End}_{\mathbb{F}}(\mathcal{C})_g := \{T\colon \mathcal{C} \to \mathcal{C} \mid T(\mathcal{C}_h) \subseteq \mathcal{C}_{gh}\ \forall h \in G\}$, and $\operatorname{Der}_{\mathbb{F}}(\mathcal{C})$ is a graded subspace of $\operatorname{End}_{\mathbb{F}}(\mathcal{C})$.

Types B_r, C_r and D_r except D_4

Assume $\operatorname{char}\mathbb{F} \neq 2$. Let R be a f.d. central simple associative algebra over $\mathbb{F}$, $\dim R = n^2$, and φ an involution on R such that

B_r: $n = 2r + 1$ ($\Rightarrow R \cong M_n(\mathbb{F})$ and φ is orthogonal), $r \geq 2$;

C_r: $n = 2r$ and φ is symplectic, $r \geq 2$;

D_r: $n = 2r$ and φ is orthogonal, $r \geq 3$.

Let $\mathcal{L} = \operatorname{Skew}(R, \varphi)$. Then $\mathcal{L}$ is a simple Lie algebra of the indicated type, and the restriction map $\mathbf{Aut}_{\mathbb{F}}(R, \varphi) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism, except in the case D_4.

If we give R any G-grading Γ, then (R, Γ) is graded simple, hence isomorphic to $\operatorname{End}_{\mathcal{D}}(\mathcal{V})$ where $\mathcal{D}$ is a graded division algebra (i.e., all nonzero homogeneous elements are invertible) and $\mathcal{V}$ is a graded right $\mathcal{D}$-module (hence free).

φ is given by $B(rv, w) = B(v, \varphi(r)w)$ for all $v, w \in \mathcal{V}$, where $B\colon \mathcal{V} \times \mathcal{V} \to \mathcal{D}$ is a homogeneous nondegenerate sesquilinear form with respect to an involution φ_0 of $\mathcal{D}$.

Type A_r

Assume $\operatorname{char}\mathbb{F} \neq 2$. If R is central simple of dimension n^2 over $\mathbb{F}$, then $\mathcal{L} = [R, R]/Z(R) \cap [R, R]$ is a simple Lie algebra of type A_{n-1}, but the "restriction" map $\mathbf{Aut}_{\mathbb{F}}(R) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is not an isomorphism unless $n = 2$.

Instead, take $(\widetilde{R}, \varphi)$ to be a f.d. s.s. associative algebra with involution such that $Z(\widetilde{R}) = \mathbb{K}$, where $\mathbb{K}$ is a quadratic étale algebra over $\mathbb{F}$ (either $\mathbb{F} \times \mathbb{F}$ or a quadratic field extension of $\mathbb{F}$), and φ is of the second kind (i.e., $\varphi|_{\mathbb{K}} \neq \mathrm{id}$). Hence $\dim \widetilde{R} = 2n^2$. Let $\mathcal{L}$ be the quotient of the derived algebra of $\mathrm{Skew}(\widetilde{R}, \varphi)$ modulo its center. If $n > 2$, then the "restriction" map $\mathbf{Aut}_{\mathbb{F}}(\widetilde{R}, \varphi) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism except in the case $n = 3 = \operatorname{char}\mathbb{F}$.

If we give $\widetilde{R}$ a grading Γ, then $(\widetilde{R}, \Gamma)$ is graded simple unless $\mathbb{K} = \mathbb{F} \times \mathbb{F}$ and $\mathbb{K}$ is trivially graded. But, in this case, the corresponding grading on $\mathcal{L}$ comes from R.

Type D_4

Assume $\operatorname{char}\mathbb{F} \neq 2$. If R is central simple of degree 8 (i.e., dimension 64) over $\mathbb{F}$ and φ is an orthogonal involution, then $\mathcal{L} = \mathrm{Skew}(R, \varphi)$ is a simple Lie algebra of type D_4, but the restriction map $\mathbf{Aut}_{\mathbb{F}}(R, \varphi) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is a closed embedding whose image has index 3.

Instead, take a *trialitarian algebra* $(E, \mathbb{L}, \rho, \sigma, \alpha)$.
Here $\mathbb{L}$ is a cubic étale algebra over $\mathbb{F}$ (e.g., $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$),
E is a "central simple algebra of degree 8" over $\mathbb{L}$,
σ is an orthogonal $\mathbb{L}$-linear involution on E, ρ is a 3-cycle, and $\alpha\colon \mathfrak{Cl}(E, \sigma) \to (E \otimes \Delta)^{\rho}$ is an isomorphism of $\mathbb{L}$-algebras with involution, where Δ is the discriminant of $\mathbb{L}$.

Then there is a canonical Lie $\mathbb{F}$-subalgebra $\mathcal{L}$ of $\mathrm{Skew}(E, \sigma)$, which is simple of type D_4, and the restriction map $\mathbf{Aut}_{\mathbb{F}}(E, \mathbb{L}, \sigma, \alpha) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism.

Type D_4 continued

$$\mathbf{1} \to \mathbf{Aut}_{\mathbb{F}}(E, \mathbb{L}, \sigma, \alpha)^0 \to \mathbf{Aut}_{\mathbb{F}}(E, \mathbb{L}, \sigma, \alpha) \xrightarrow{\pi} \mathbf{Aut}_{\mathbb{F}}(\mathbb{L}) \to \mathbf{1}$$

where π is the restriction.
$\mathbf{Aut}_{\mathbb{F}}(\mathbb{L})$ is a twisted form of $\mathbf{S}_3$, so its representing object is $\mathrm{Fun}(S_3, \mathbb{F}_{\mathrm{sep}})^{\mathcal{G}}$, where $\mathcal{G} = \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$.

If we give the trialitarian algebra E a G-grading Γ, then the image of the homomorphism $\pi \circ \eta_\Gamma \colon G^D \to \mathbf{Aut}_{\mathbb{F}}(\mathbb{L})$ is a diagonalizable subgroupscheme, so it corresponds to a $\mathcal{G}$-invariant abelian subgroup of S_3, which can have order 1, 2 or 3. We say that the grading Γ has Type I, II or III, resp.

If $\mathbb{L}$ is not a field then $\mathcal{L} \cong \mathrm{Skew}(R, \varphi)$ where R is a central simple algebra of degree 8 over $\mathbb{F}$. If Γ has Type I or II then R can be chosen G-graded.

If $\mathbb{F}$ is a.c. ($\Rightarrow \mathbb{L} = \mathbb{F} \times \mathbb{F} \times \mathbb{F}$), then Type III gradings exist iff $\mathrm{char}\, \mathbb{F} \neq 3$. If $\mathbb{F} = \mathbb{R}$, then Type III gradings exist iff $\mathbb{L} = \mathbb{R} \times \mathbb{C}$.