Gradings on trialitarian algebras and simple Lie algebras of type D_4

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Outline

Definition of a group grading

Let $\mathcal{A}$ be a nonassociative algebra over a field $\mathbb{F}$. Let G be a group.

Definition

- A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$. $\mathcal{A}_g$ is called the *homogeneous component* of degree g.
- The *support* of Γ is the set $S = \operatorname{Supp} \Gamma := \{g \in G \mid \mathcal{A}_g \neq 0\}$.
- The *universal (abelian) group* $U(\Gamma)$ is the (abelian) group with generating set S and defining relations $s_1 s_2 = s_3$ whenever $0 \neq \mathcal{A}_{s_1} \mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$.

Γ can be regarded as a $U(\Gamma)$-grading.
$\exists!$ homomorphism $U(\Gamma) \to G$ that restricts to id_S.

We assume that $\dim \mathcal{A} < \infty$ and G is abelian.

Examples of gradings

Example

The following is a $\mathbb{Z}$-grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where

$$\mathfrak{g}_{-1} = \mathrm{Span}\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\},\ \mathfrak{g}_0 = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\},\ \mathfrak{g}_1 = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}.$$

This can also be regarded as a $\mathbb{Z}_m$-grading for any $m > 2$, but the universal group is $\mathbb{Z}$.

Example (Cartan grading)

Let $\mathfrak{g}$ be a s.s. Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\Big)$$

can be viewed as a grading by the root lattice $G = \langle \Phi \rangle$.
$\mathrm{Supp}\,\Gamma = \{0\} \cup \Phi$; $U(\Gamma) = G \cong \mathbb{Z}^r$ where $r = \dim \mathfrak{h}$.

Examples continued

Example (Pauli grading)

A grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Namely, $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ where $\mathbb{Z}_2^2 = \{e, a, b, c\}$ and

$$\mathfrak{g}_a = \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \mathfrak{g}_b = \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}, \mathfrak{g}_c = \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}.$$

Example (Generalized Pauli grading)

If $\varepsilon \in \mathbb{F}$, there is a grading on $\mathcal{R} = M_n(\mathbb{F})$ ($\Rightarrow$ on $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$) by $G = \mathbb{Z}_n^2$:

$$X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \varepsilon^{n-1} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where ε is a primitive n-th root of 1. Choose generators a and b of G and set $\mathcal{R}_{a^i b^j} = \mathbb{F} X^i Y^j$.

Isomorphism and equivalence of gradings

Definition

- Two G-gradings on $\mathcal{A}$, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}'_g$, are *isomorphic* if there exists an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$.
- A G-grading $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and an H-grading $\mathcal{A} = \bigoplus_{h\in G} \mathcal{A}'_h$, with supports S an S', respectively, are *equivalent* if there exists an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ and a bijection $\alpha\colon S \to S'$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_{\alpha(g)}$ for all $g \in S$.

In the def of equivalent gradings, if G and H are universal grading groups then α extends to a unique isomorphism of groups $G \to H$.

Example

All Pauli gradings on $M_n(\mathbb{F})$ or $\mathfrak{sl}_n(\mathbb{F})$ are equivalent. For $M_n(\mathbb{F})$, there are $\phi(n)$ (Euler function) non-isomorphic $\mathbb{Z}_n^2$-gradings among them. Hence $\frac{1}{2}\phi(n)$ for $\mathfrak{sl}_n(\mathbb{F})$ if $n > 2$.

Fine gradings

Definition

Consider a G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and an H-grading $\Gamma' : \mathcal{A} = \bigoplus_{h\in G} \mathcal{A}'_h$. We say that Γ' is a *coarsening* of Γ (or Γ is a *refinement* of Γ') if for any $g \in G$ there exists $h \in H$ such that $\mathcal{A}_g \subset \mathcal{A}'_h$. If we have $\neq$ for some $g \in \operatorname{Supp}\Gamma$, then Γ a *proper* refinement of Γ'. A grading is *fine* if it does not have proper refinements.

Example

$\mathfrak{sl}_2(\mathbb{C}) = \operatorname{Span}\left\{\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\right\} \oplus \operatorname{Span}\left\{\left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right]\right\}$ is a $\mathbb{Z}_2$-grading that is a proper coarsening of the Cartan grading and also of the Pauli grading. Up to equivalence, there are exactly 2 fine ab. group gradings on $\mathfrak{sl}_2(\mathbb{F})$, $\operatorname{char}\mathbb{F} \neq 2$: the Cartan grading and the Pauli grading.

If $\mathbb{F}$ is a.c., $\operatorname{char}\mathbb{F} = 0$, then (equivalence classes of) fine gradings on $\mathcal{A}$ $\leftrightarrow$ (conjugacy classes of) maximal quasitori in $\operatorname{Aut}(\mathcal{A})$.

Gradings and automorphism group schemes

Let G be an abelian group. Over an arbitrary field $\mathbb{F}$, a G-grading Γ on a f.-d. algebra $\mathcal{U}$ is equivalent to a morphism of affine group schemes $\eta_\Gamma\colon G^D \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{U})$.
Recall that an *affine group scheme* is a representable functor from $\mathrm{Alg}_{\mathbb{F}}$ (unital commutative associative $\mathbb{F}$-algebras) to groups. The representing object is automatically a (commutative) Hopf algebra.
The *Cartier dual* G^D is represented by the group algebra $\mathbb{F}G$.
The automorphism group scheme $\mathbf{Aut}_{\mathbb{F}}(\mathcal{U})$ sends $\mathcal{R} \in \mathrm{Alg}_{\mathbb{F}}$ to the group $\mathrm{Aut}_{\mathcal{R}}(\mathcal{U} \otimes \mathcal{R})$.
The morphism η_Γ is defined as follows: for any $\mathcal{R} \in \mathrm{Alg}_{\mathbb{F}}$, the corresponding homomorphism of groups $(\eta_\Gamma)_{\mathcal{R}} : \mathrm{Alg}_{\mathbb{F}}(\mathbb{F}G, \mathcal{R}) \to \mathrm{Aut}_{\mathcal{R}}(\mathcal{U} \otimes \mathcal{R})$ is defined by

$$(\eta_\Gamma)_{\mathcal{R}}(f)(x \otimes r) = x \otimes f(g)r \text{ for all } x \in \mathcal{U}_g, g \in G, r \in \mathcal{R}, f \in \mathrm{Alg}_{\mathbb{F}}(\mathbb{F}G, \mathcal{R}).$$

Consequently, if we have two algebras, $\mathcal{U}$ and $\mathcal{V}$, and a morphism $\theta\colon \mathbf{Aut}_{\mathbb{F}}(\mathcal{U}) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{V})$ then any G-grading Γ on $\mathcal{U}$ gives rise to a G-grading $\theta(\Gamma)$ on $\mathcal{V}$ by setting $\eta_{\theta(\Gamma)} := \theta \circ \eta_\Gamma$.

Definition and types of composition algebras

A (f.-d.) *composition algebra* is a nonassociative algebra $\mathcal{A}$ with a nonsingular quadratic form n such that $n(xy) = n(x)n(y)\ \forall x, y \in \mathcal{A}$.

- *Hurwitz algebras:* the unital composition algebras. They can be obtained using the Cayley–Dickson doubling process, so $\dim \mathcal{A}$ can be 1, 2, 4 or 8 ($\Rightarrow$ the same for any composition algebra). The *standard conjugation*: $\bar{x} = -x + n(x, 1)1$.
- *Symmetric composition algebras:* the polar form of the norm is associative: $n(xy, z) = n(x, yz)\ \forall x, y, z \in \mathcal{A}$.

If $(\mathcal{C}, n)$ is Hurwitz, we can define the *para-Hurwitz* product: $x \bullet y = \bar{x}\bar{y}$, which makes $(\mathcal{C}, n)$ a symmetric composition algebra.

Hurwitz algebras of dim 4 are called *quaternion algebras*; those of dim 8 are called *octonion* or *Cayley algebras*. If $\mathbb{F}$ is a.c. then, up to isomorphism, there is only one Hurwitz algebra in each dim.

If $\mathbb{F}$ is a.c. then there are two symmetric composition algebras of dim 8: the para-Cayley and the Okubo algebra.

Cayley–Dickson doubling process

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 2$. Let $\mathcal{Q}$ be a Hurwitz algebra with norm n. Fix $0 \neq \alpha \in \mathbb{F}$ and let $\mathfrak{CD}(\mathcal{Q}, \alpha) = \mathcal{Q} \oplus \mathcal{Q}w$ be the direct sum of two copies of $\mathcal{Q}$, where we write the element (x, y) as $x + yw$, with multiplication

$$(a + bw)(c + dw) = (ac + \alpha\bar{d}b) + (da + b\bar{c})w,$$

and norm

$$n(x + yw) = n(x) - \alpha n(y).$$

It is well known that $\mathfrak{CD}(\mathcal{Q}, \alpha)$ is a Hurwitz algebra $\Leftrightarrow$ $\mathcal{Q}$ is associative.

Note that $\mathcal{K} := \mathfrak{CD}(\mathbb{F}, \alpha)$ is $\mathbb{Z}_2$-graded, $\mathcal{Q} := \mathfrak{CD}(\mathcal{K}, \beta)$ is $\mathbb{Z}_2^2$-graded and $\mathcal{C} := \mathfrak{CD}(\mathcal{Q}, \gamma)$ is $\mathbb{Z}_2^3$-graded. Explicitly,

$$\mathcal{C} = \bigoplus_{\alpha \in \mathbb{Z}_2^3} \mathbb{F}e_\alpha \quad \text{where } e_\alpha = (w_1^{\alpha_1} w_2^{\alpha_2}) w_3^{\alpha_3}.$$

Triality group and triality Lie algebra

Let $(\mathcal{S}, \star, n)$ be a symmetric composition algebra of dim 8. Its *triality Lie algebra* $\mathfrak{tri}(\mathcal{S}, \star, n)$ is defined as

$$\{(d_1, d_2, d_3) \in \mathfrak{so}(\mathcal{S}, n)^3 \mid d_1(x \star y) = d_2(x) \star y + x \star d_3(y)\ \forall x, y \in \mathcal{S}\},$$

with componentwise multiplication.
"Local triality principle": this definition is symmetric with respect to cyclic permutations of (d_1, d_2, d_3), and each projection determines an isomorphism $\mathfrak{tri}(\mathcal{S}) \to \mathfrak{so}(\mathcal{S}, n)$, so $\mathfrak{tri}(\mathcal{S})$ is a Lie algebra of type D_4.

The *triality group* $\mathrm{Tri}(\mathcal{S}, \star, n)$ is defined as

$$\{(f_1, f_2, f_3) \in \mathrm{O}(\mathcal{S}, n)^3 \mid f_1(x \star y) = f_2(x) \star f_3(y)\ \forall x, y \in \mathcal{S}\},$$

with componentwise multiplication.
"Global triality principle": this definition is symmetric with respect to cyclic permutations of (f_1, f_2, f_3), and $\mathrm{Tri}(\mathcal{S})$ is isomorphic to $\mathrm{Spin}(\mathcal{S}, n)$. In fact, this isomorphism can be defined at the level of the corresponding group schemes.

Definition and examples of cyclic composition algebras

Let $\mathbb{L}$ be a Galois algebra over $\mathbb{F}$ with respect to the cyclic group of order 3. Fix a generator ρ of this group.

A *cyclic composition algebra* [Springer 63] over $(\mathbb{L}, \rho)$ is a free $\mathbb{L}$-module V with a nonsingular $\mathbb{L}$-valued quadratic form Q and an $\mathbb{F}$-bilinear multiplication $(x, y) \mapsto x * y$ that is ρ-semilinear in x and ρ^2-semilinear in y and satisfies the following identities:

$$Q(x * y) = \rho(Q(x))\rho^2(Q(y)),$$
$$b_Q(x * y, z) = \rho(b_Q(y * z, x)) = \rho^2(b_Q(z * x, y)),$$

where $b_Q(x, y) := Q(x + y) - Q(x) - Q(y)$ is the polar form of Q.

If $(\mathcal{S}, \star, n)$ is a symmetric composition algebra then $\mathcal{S} \otimes \mathbb{L}$ becomes a cyclic composition algebra with $Q(x \otimes \ell) = n(x)\ell^2$ and $(x \otimes \ell) * (y \otimes m) = (x \star y) \otimes \rho(\ell)\rho^2(m)$.

If $\mathbb{F}$ is a.c. then any cyclic composition algebra is isomorphic to $\mathcal{S} \otimes \mathbb{L}$ where $\mathcal{S}$ is para-Hurwitz. Hence, the $\mathbb{L}$-rank can be 1, 2, 4 or 8, and there is only one isomorphism class in each rank.

Let $(\mathcal{S}, \star, n)$ be a symmetric composition algebra, $\mathbb{L} = \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ and $\rho(\ell_1, \ell_2, \ell_3) = (\ell_2, \ell_3, \ell_1)$.
Then $V = \mathcal{S} \otimes \mathbb{L} = \mathcal{S} \times \mathcal{S} \times \mathcal{S}$ with $Q = (n, n, n)$ and

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (x_2 \star y_3, x_3 \star y_1, x_1 \star y_2).$$

Also, $\mathfrak{tri}(\mathcal{S}, \star, n)$ can be interpreted as $\mathrm{Der}_{\mathbb{L}}(V, *, Q)$, $\mathrm{Tri}(\mathcal{S}, \star, n)$ as $\mathrm{Aut}_{\mathbb{L}}(V, *, Q)$, and $\mathrm{Tri}(\mathcal{S}, \star, n) \rtimes A_3$ as $\mathrm{Aut}_{\mathbb{F}}(V, \mathbb{L}, \rho, *, Q)$.
At the level of group schemes:

$$\mathbf{Aut}_{\mathbb{F}}(V, \mathbb{L}, \rho, *, Q) = \mathbf{Tri}(\mathcal{S}, \star, n) \rtimes \mathbf{A}_3 \cong \mathbf{Spin}(\mathcal{S}, n) \rtimes \mathbf{A}_3.$$

Recall that $\mathcal{L} := \mathfrak{tri}(\mathcal{S}, \star, n)$ is a Lie algebra of type D_4. We have a morphism $\mathrm{Ad} : \mathbf{Aut}_{\mathbb{F}}(V, \mathbb{L}, \rho, *, Q) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$, but $\mathbf{Aut}_{\mathbb{F}}(\mathcal{L}) \cong \mathbf{PGO}^+(\mathcal{S}, n) \rtimes \mathbf{S}_3$, so we need to look at $\mathrm{End}_{\mathbb{L}}(V)$.

The trialitarian algebra $\mathrm{End}_{\mathbb{L}}(V)$ [KMRT 98]

Let V be a cyclic composition algebra over $(\mathbb{L}, \rho)$ of rank 8. Then $E := \mathrm{End}_{\mathbb{L}}(V)$ is a central separable associative algebra over $\mathbb{L}$, with involution σ determined by the quadratic form Q.
The even Clifford algebra $\mathfrak{Cl}_0(V, Q)$ can be defined purely in terms of (E, σ) as the quotient $\mathfrak{Cl}(E, \sigma)$ of the tensor algebra of E (regarded as an $\mathbb{L}$-module). We have a canonical $\mathbb{L}$-linear map $\kappa\colon E \to \mathfrak{Cl}(E, \sigma)$ (neither injective nor a homomorphism of algebras).
$\mathfrak{Cl}_0(V, Q) \xrightarrow{\sim} \mathfrak{Cl}(E, \sigma)$ by sending $xy \in \mathfrak{Cl}_0(V, Q)$ to $\kappa\big(xb_Q(y, \cdot)\big)$.
The multiplication $*$ of V allows us to define an additional structure on E, namely, an isomorphism of $\mathbb{L}$-algebras with involution:

$$\alpha\colon \mathfrak{Cl}(E, \sigma) \xrightarrow{\sim} {}^{\rho}E \times {}^{\rho^2}E,$$

where ${}^{\rho}E$ is E as an $\mathbb{F}$-algebra with involution, but with the new $\mathbb{L}$-module structure defined by $\ell \cdot a = \rho(\ell)a$.
This is done using the Clifford algebra $\mathfrak{Cl}(V, Q)$, with the final result:

$$\alpha\colon \kappa\big(xb_Q(y, \cdot)\big) \mapsto (l_x r_y, r_x l_y),\ x, y \in V.$$

The Lie algebra of the trialitarian algebra $\mathrm{End}_{\mathbb{L}}(V)$

It turns out that the restriction $\frac{1}{2}\kappa\colon \mathrm{Skew}(E,\sigma) \to \mathrm{Skew}(\mathfrak{Cl}(E,\sigma),\underline{\sigma})$ is an injective homomorphism of Lie algebras over $\mathbb{L}$, and the $\mathbb{F}$-subspace

$$\mathcal{L}(E,\mathbb{L},\rho,\sigma,\alpha) := \{x \in \mathrm{Skew}(E,\sigma) \mid \alpha(\kappa(x)) = 2(x,x)\}$$

is precisely the Lie subalgebra $\mathcal{L} = \mathrm{Der}_{\mathbb{L}}(V,*,Q)$ of type D_4.

Now, the restriction $\mathbf{Aut}_{\mathbb{F}}(E,\mathbb{L},\sigma,\alpha) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism of group schemes [KMRT 98]. Consequently, the Lie algebra $\mathcal{L}$ and the trialitarian algebra $\mathrm{End}_{\mathbb{L}}(V)$ have the same classification of gradings.

Let $\pi\colon \mathbf{Aut}_{\mathbb{F}}(\mathcal{L}) \to \mathbf{S}_3$ be the quotient map. Given a G-grading Γ on $\mathcal{L}$, the image $\pi\eta_\Gamma(G^D)$ is an abelian subgroupscheme of $\mathbf{S_3}$. Since the subgroupschemes of a constant group correspond to subgroups, here we have three possibilities: the image has order 1, 2 or 3. The grading Γ will be said to have *Type I, II or III*, respectively.

Gradings of Types I and II are "matrix gradings", i.e., they are isomorphic to restrictions of gradings on $M_8(\mathbb{F})$ compatible with an orthogonal involution to $\mathfrak{so}_8(\mathbb{F})$ [Elduque 10, Bahturin–K 10].

Classification of fine gradings up to equivalence

We prove that any Type III G-grading on the Lie algebra $\mathcal{L}$ or, equivalently, on the trialitarian algebra $E = \mathrm{End}_{\mathbb{L}}(V)$, is induced by a G-grading on the cyclic composition algebra V, and we classify the latter up to isomorphism, and fine gradings up to equivalence.

There are 15 equivalence classes of fine gradings on $M_8(\mathbb{F})$ with orthogonal involution: 8 of them restrict to Type I and 7 to Type II gradings on $\mathcal{L}$. It turns out that two of the Type I gradings are equivalent to each other. Also, there are 3 equivalence classes of Type III fine gradings if $\mathrm{char}\,\mathbb{F} \neq 3$ and none if $\mathrm{char}\,\mathbb{F} = 3$.

Theorem (Elduque 10 for $\mathrm{char}\,\mathbb{F} = 0$, Elduque–K 14 in general)

Let $\mathcal{L}$ be the simple Lie algebra of type D_4 over an a.c. field $\mathbb{F}$.

- *If $\mathrm{char}\,\mathbb{F} \neq 2, 3$ then there are, up to equivalence, 17 fine gradings on $\mathcal{L}$. Their universal groups and types are below.*
- *If $\mathrm{char}\,\mathbb{F} = 3$ then there are, up to equivalence, 14 fine gradings on $\mathcal{L}$. They correspond to cases (1)—(14) below.*

Classification of fine gradings continued

1. universal group $\mathbb{Z}^4$ (Cartan grading), type $(24,0,0,1)$;
2. universal group $\mathbb{Z}_2 \times \mathbb{Z}^3$, type $(25,0,1)$;
3. universal group $\mathbb{Z}_2^3 \times \mathbb{Z}^2$, type $(26,1)$;
4. universal group $\mathbb{Z}_2^5 \times \mathbb{Z}$, type (28);
5. universal group $\mathbb{Z}_2^7$, type (28);
6. universal group $\mathbb{Z}_2^2 \times \mathbb{Z}^2$, type $(20,4)$;
7. universal group $\mathbb{Z}_2^3 \times \mathbb{Z}$, type $(25,0,1)$;
8. universal group $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}$, type $(24,2)$;
9. universal group $\mathbb{Z}_2^5$, type $(24,0,0,1)$;
10. universal group $\mathbb{Z}_2^3 \times \mathbb{Z}_4$, type $(25,0,1)$;
11. universal group $\mathbb{Z}_2^3 \times \mathbb{Z}_4$, type $(24,2)$;
12. universal group $\mathbb{Z}_2 \times \mathbb{Z}_4^2$, type $(26,1)$;
13. universal group $\mathbb{Z}_2^4 \times \mathbb{Z}$, type (28);
14. universal group $\mathbb{Z}_2^6$, type (28);
15. universal group $\mathbb{Z}^2 \times \mathbb{Z}_3$, type $(26,1)$;
16. universal group $\mathbb{Z}_2^3 \times \mathbb{Z}_3$, type $(14,7)$;
17. universal group $\mathbb{Z}_3^3$, type $(24,2)$.