Graded modules over simple Lie algebras with a group grading

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Outline

Definition of group grading

Let $\mathcal{A}$ be an algebra (not necessarily associative) over a field $\mathbb{F}$.
Let G be a (semi)group.

Definition

A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ such that $\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$.
The *support* of the G-grading is the set $\operatorname{Supp}\mathcal{A} := \{g \in G \mid \mathcal{A}_g \neq 0\}$.

Definition

Two G-gradings on $\mathcal{A}$, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}'_g$, are *isomorphic* if there exists an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$.

Cartan grading

Example

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, and let $\mathfrak{h}$ be a Cartan subalgebra. Then the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\Big)$$

can be viewed as a grading by the root lattice $\langle\Phi\rangle \cong \mathbb{Z}^r$, $r = \dim \mathfrak{h}$.

Cartan grading also exists for simple Lie algebras of types A-G in characteristic $p > 0$.

Pauli matrices

Example

There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Namely, we set

$$\begin{aligned} \mathfrak{g}_{(0,0)} &= 0, & \mathfrak{g}_{(1,0)} &= \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \\ \mathfrak{g}_{(0,1)} &= \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}, & \mathfrak{g}_{(1,1)} &= \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}. \end{aligned}$$

Any G-grading on $\mathfrak{sl}_2(\mathbb{F})$, $\operatorname{char}\mathbb{F} \neq 2$, is induced by the above grading or by the Cartan grading via a group homomorphism $\mathbb{Z}_2^2 \to G$, resp. $\mathbb{Z} \to G$.

Graded modules

Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a grading on an associative (or Lie) algebra.

Definition

An $\mathcal{A}$-module V is said to be *graded* if it is equipped with a vector space decomposition $V = \bigoplus_{g\in G} V_g$ such that $\mathcal{A}_g \cdot V_h \subseteq V_{gh}$ for all $g, h \in G$.
A *homomorphism of graded modules* $\varphi\colon V \to W$ is a homomorphism of modules such that $\varphi(V_g) \subseteq W_g$ for all $g \in G$.

Lemma

Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a G-graded associative algebra where G is any group. Let $N \subset M$ be graded $\mathcal{A}$-modules. If N admits a complement in M as an $\mathcal{A}$-module then it admits a complement as a graded $\mathcal{A}$-module.

Graded modules (continued)

Take $\mathcal{A} = U(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple f.d. Lie algebra over a field of characteristic 0
$\Rightarrow$ the category of graded f.d. $\mathfrak{g}$-modules is semisimple.

Example

Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$ with the Pauli grading
$\Rightarrow$ the natural module $V = \mathrm{Span}\{v_1, v_2\}$ does not admit a grading that would make it a graded $\mathfrak{g}$-module.
But $W = V^2 = \mathrm{Span}\{v_i^j \mid i, j = 1, 2\}$ admits a grading making it a graded-simple $\mathfrak{g}$-module:

$$\begin{aligned} W_{(0,0)} &= \mathrm{Span}\{v_1^1 + v_2^2\}, & W_{(1,0)} &= \mathrm{Span}\{v_1^1 - v_2^2\}, \\ W_{(0,1)} &= \mathrm{Span}\{v_2^1 + v_1^2\}, & W_{(1,1)} &= \mathrm{Span}\{v_2^1 - v_1^2\}. \end{aligned}$$

The natural $\mathfrak{sl}_2$-module has *graded Schur index* 2 wrt Pauli grading.

A structure theorem

$\mathcal{D}$ graded division algebra: all nonzero homogeneous elements are invertible $\Rightarrow$ graded $\mathcal{D}$-modules are free ("graded v.s. / $\mathcal{D}$")

Theorem

Let $\mathcal{R}$ be a G-graded algebra (or ring). Then $\mathcal{R}$ is graded simple and satisfies d.c.c. on graded one-sided ideals $\Leftrightarrow$ there exists a G-graded division algebra $\mathcal{D}$ and a graded right v. s. V of finite dimension over $\mathcal{D}$ such that $\mathcal{R} \cong \mathrm{End}_{\mathcal{D}}(V)$ as a graded algebra.

The graded division algebra $\mathcal{D}$ and graded v.s. V over $\mathcal{D}$ are determined up to isomorphism and shift of grading.
If $\mathcal{R} = M_n(\mathbb{F})$ with a G-grading then $\mathcal{D} \cong M_\ell(\mathbb{F})$ with a *division grading* and $\mathcal{R} \cong M_k(\mathcal{D})$ with a grading determined by a k-tuple $(g_1, \ldots, g_k)$ of elements of G, $k\ell = n$.

Generalized Pauli matrices

A G-grading on $\mathcal{D} = M_\ell(\mathbb{F})$ is a *division grading* if it makes $\mathcal{D}$ a graded division algebra ($\Rightarrow \operatorname{Supp}\mathcal{D}$ is a subgroup).

Example

$$X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \varepsilon^{n-1} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where $\varepsilon \in \mathbb{F}$ is a primitive ℓ-th root of unity. Then the following is a division grading by $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$: $\mathcal{D}_{(i,j)} = \mathbb{F}X^iY^j$.

Theorem (HPP,BSZ for $\operatorname{char}\mathbb{F} = 0$; BZ for $\operatorname{char}\mathbb{F} > 0$)

Let T be an ab. group and $\mathbb{F}$ an a.c. field. Then for any division grading on $\mathcal{D} = M_\ell(\mathbb{F})$ with support T, there exists a decomposition $T = H_1 \times \cdots \times H_r$ such that $H_i \cong \mathbb{Z}_{\ell_i}^2$ and $\mathcal{D} \cong M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ with $M_{\ell_i}(\mathbb{F})$ graded as above.

Classification of division gradings on $M_\ell(\mathbb{F})$

Suppose $\mathcal{D} = M_\ell(\mathbb{F})$, $\mathbb{F}$ a.c. field, has a division grading with support $T \subset G$. Then, for each $t \in T$, $\mathcal{D}_t = \mathbb{F}X_t$ and hence

$$X_s X_t = \sigma(s,t) X_{st}$$

for some 2-cocycle $\sigma\colon T \times T \to \mathbb{F}^\times$, i.e., $\mathcal{D}$ is isomorphic to a *twisted group algebra* $\mathbb{F}^\sigma T$, with its natural T-grading regarded as a G-grading. Set $\beta(s,t) = \sigma(s,t)/\sigma(t,s)$.

Theorem (BK)

If G is abelian, then the isomorphism classes of division gradings on $M_\ell(\mathbb{F})$ are in bijection with the pairs (T,β) where $T \subset G$ is a subgroup of order ℓ^2 and $\beta\colon T \times T \to \mathbb{F}^\times$ is a nondegenerate alternating bicharacter ($\Rightarrow \operatorname{char}\mathbb{F} \nmid \ell$).

Graded Brauer group

Fix an abelian group G and consider G-graded matrix algebras. If $\mathcal{R} = \operatorname{End}_{\mathcal{D}}(W)$, we denote by $[\mathcal{R}]$ the isomorphism class of $\mathcal{D}$.
$\mathcal{R}_1 = \operatorname{End}_{\mathcal{D}_1}(W_1)$ and $\mathcal{R}_2 = \operatorname{End}_{\mathcal{D}_2}(W_2) \Rightarrow \mathcal{R}_1 \otimes \mathcal{R}_2 = \operatorname{End}_{\mathcal{D}}(W)$ for some $\mathcal{D}$ and W.
What is $\mathcal{D}$?
$\mathcal{D}$ depends only on $\mathcal{D}_1$ and $\mathcal{D}_2 \Rightarrow$ define $[\mathcal{R}_1][\mathcal{R}_2] := [\mathcal{R}_1 \otimes \mathcal{R}_2]$.

Let $\widehat{G}$ be the group of characters. For any $\chi \in \widehat{G}$, there is a unique $t \in T$ such that $\chi(\cdot) = \beta(t, \cdot)$ on T.
Define an alternating bicharacter $\hat{\beta}$ on $\widehat{G}$ (with values in $\mathbb{F}^\times$) by setting $\hat{\beta}(\chi_1, \chi_2) = \beta(t_1, t_2)$.
T and β can be recovered from $\hat{\beta}$ as follows: $T = (\operatorname{rad}\hat{\beta})^\perp$ and $\beta(t_1, t_2) = \hat{\beta}(\chi_1, \chi_2)$ where χ_i is any extension of $\beta(t_i, \cdot) \colon T \to \mathbb{F}^\times$ to a character of G ($i = 1, 2$).

Graded Brauer group (continued)

If $[\mathcal{R}_1]$ corresponds to $\hat{\beta}_1$ and $[\mathcal{R}_2]$ corresponds to $\hat{\beta}_1$ then $[\mathcal{R}_1][\mathcal{R}_2]$ corresponds to $\hat{\beta}_1\hat{\beta}_2$.

The G-graded Brauer group of $\mathbb{F}$ is isomorphic to the group of alternating continuous bicharacters of the pro-finite group $\widehat{G_0}$ where G_0 is the torsion subgroup of G if $\operatorname{char}\mathbb{F} = 0$ and the p'-torsion subgroup of G if $\operatorname{char}\mathbb{F} = p$.

The topology of $\widehat{G_0}$ comes from the identification of $\widehat{G_0}$ with the inverse limit of the finite groups $\widehat{H}$ where H ranges over all finite subgroup of G_0.

$\widehat{G_0}$ is a compact and totally discontinuous topological group.

Lemma

If ε is a homogeneous idempotent of $\mathcal{R}$ then $\varepsilon\mathcal{R}\varepsilon$ is a G-graded matrix algebra and $[\varepsilon\mathcal{R}\varepsilon] = [\mathcal{R}]$ in the G-graded Brauer group.

Action of $\widehat{G}$ and twisting of modules

Let $\mathcal{L}$ be a semisimple f.d. Lie algebra over an a.c. field $\mathbb{F}$ of characteristic 0. Given a grading $\mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ by an abelian group G, we want to classify (f.d.) graded-simple $\mathcal{L}$-modules.

For $\chi \in \widehat{G}$, denote by α_χ the corresponding automorphism of $\mathcal{L}$: $\alpha_\chi(x) = \chi * x$ for all $x \in \mathcal{L}$.
If W is a graded $\mathcal{L}$-module then $\widehat{G}$ also acts on W. For $\chi \in \widehat{G}$, denote by φ_χ the corresponding linear transformation of W: $\varphi_\chi(w) = \chi * w$ for all $w \in W$.
For any $\mathcal{L}$-module V and $\alpha \in \mathrm{Aut}(\mathcal{L})$, denote by V^α the corresponding twisted $\mathcal{L}$-module, i.e., the vector space V with a different $\mathcal{L}$-action: $x \cdot v = \alpha(x)v$ for all $x \in \mathcal{L}$ and $v \in V$.
If α is inner then, for any $\mathcal{L}$-module V, the twisted module V^α is isomorphic to V, so the action of $\mathrm{Aut}(\mathcal{L})$ on the isomorphism classes of $\mathcal{L}$-modules factors through $\mathrm{Out}(\mathcal{L}) := \mathrm{Aut}(\mathcal{L})/\mathrm{Int}(\mathcal{L})$. In particular, the orbits are finite.

From simple modules to gradings on matrix algebras

If W is a graded $\mathcal{L}$-module then $\mathcal{L}_g W_h \subset W_{gh}$ for all $g, h \in G$
$\Leftrightarrow \varphi_\chi(xw) = \alpha_\chi(x)\varphi_\chi(w)$ for all $\chi \in \widehat{G}, x \in \mathcal{L}, w \in W$
$\Leftrightarrow \varphi_\chi \colon W \to W^{\alpha_\chi}$ as $\mathcal{L}$-modules, for all $\chi \in \widehat{G}$

For any $\mathcal{L}$-module V, denote V^{α_χ} by V^χ.

Let V_λ be a simple (f.d.) module of highest weight λ. Define the *inertia group* $K_\lambda = \{\chi \in \widehat{G} \mid V_\lambda^\chi \cong V_\lambda\} = \{\chi \in \widehat{G} \mid \tau_\chi(\lambda) = \lambda\}$, where τ_χ is the diagram automorphism corresponding to the image of α_χ in $\mathrm{Out}(\mathcal{L})$. Let $H_\lambda := K_\lambda^\perp \subset G$.

For $V = V_\lambda$, $\rho\colon U(\mathcal{L}) \to \mathrm{End}\,(V)$ is surjective. For any $\chi \in K_\lambda$, there exists $u_\chi \in \mathrm{GL}(V)$ such that $\rho(\alpha_\chi(a)) = u_\chi \rho(a) u_\chi^{-1}$ for all $a \in U(\mathcal{L})$, and the inner automorphism $\widetilde{\alpha}_\chi(x) := u_\chi x u_\chi^{-1}$ of $\mathrm{End}\,(V)$ is uniquely determined.
$\Rightarrow$ we obtain a representation $K_\lambda \to \mathrm{Aut}(\mathrm{End}\,(V))$, $\chi \mapsto \widetilde{\alpha}_\chi$, and a corresponding $\overline{G}$-grading on $\mathrm{End}\,(V)$ where $\overline{G} = G/H_\lambda$ such that $\rho\colon U(\mathcal{L}) \to \mathrm{End}\,(V)$ is a homom of graded algebras.

Graded Brauer invariant and graded Schur index

Definition

The class $[\mathrm{End}\,(V_\lambda)]$ in the (G/H_λ)-graded Brauer group will be called the *Brauer invariant* of λ (or of V_λ) and denoted by $\mathrm{Br}(\lambda)$. The degree of the graded division algebra $\mathcal{D}$ representing $\mathrm{Br}(\lambda)$ will be called the *(graded) Schur index* of λ (or of V_λ).

Proposition

Let $V = V_\lambda$ and $\overline{G} = G/H_\lambda$. The $\mathcal{L}$-module V^k admits a $\overline{G}$-grading that makes it a graded-simple $\mathcal{L}$-module if and only if k equals the Schur index of V. This $\overline{G}$-grading is unique up to isomorphism and shift.

$\mathcal{L} = \mathfrak{sl}_n(\mathbb{F})$ with the Pauli grading, V natural module, i.e., $\lambda = \omega_1$
$\Rightarrow H_\lambda = \{e\}$, G-graded Schur index of V is n.

Induced graded module

Let H be a finite subgroup of G and let $U = \bigoplus_{\overline{g}\in\overline{G}} U_{\overline{g}}$ be a $\overline{G}$-graded v. s. where $\overline{G} = G/H$. Let $K = H^{\perp} \subset \widehat{G}$ and
$W = \mathrm{Ind}_K^{\widehat{G}} U := \mathbb{F}\widehat{G} \otimes_{\mathbb{F}K} U$.
W is a $\widehat{G}$-module $\Rightarrow$ a G-graded space.
Explicitly, take $\widehat{H} = \{\chi_1, \ldots, \chi_s\}$ and extend each χ_i to G
$\Rightarrow \{\chi_1, \ldots, \chi_s\}$ is a transversal for the subgroup K in $\widehat{G}$, so
$\mathrm{Ind}_K^{\widehat{G}} U = \chi_1 \otimes U \oplus \cdots \oplus \chi_s \otimes U$, and, for any $g \in G$,
$W_g = \{\sum_{j=1}^{s} \chi_j \otimes \chi_j(g)^{-1} u \mid u \in U_{\overline{g}}\}$.
We will denote the G-graded space $W = \bigoplus_{g\in G} W_g$ by $\mathrm{Ind}_{G/H}^{G} U$.
If U is a $\overline{G}$-graded $\mathcal{L}$-module then $\mathcal{L}$ acts on W as follows:
$x \cdot (\chi \otimes u) = \chi \otimes \alpha_{\overline{\chi^{-1}}}(x) u$ for all $x \in \mathcal{L}$, $\chi \in \widehat{G}$ and $u \in U$.
If U is simple as a $\overline{G}$-graded $\mathcal{L}$-module and K is its inertia group then W is simple as a G-graded $\mathcal{L}$-module.

Definition of $W(\mathcal{O})$

Let λ be a dominant integral weight. If $\mu \in \widehat{G}\lambda$ then $H_\lambda = H_\mu$ (since $\widehat{G}$ is abelian) and the (G/H_λ)-graded algebras $\mathrm{End}\,(V_\lambda)$ and $\mathrm{End}\,(V_\mu)$ are isomorphic. Hence λ and μ have the same graded Schur index and Brauer invariant.

Definition

For each $\widehat{G}$-orbit $\mathcal{O}$ in the set of dominant integral weights, we select a representative λ and equip $U = V_\lambda^\ell$ with a (G/H_λ)-grading, where ℓ is the graded Schur index of V_λ. Then $W(\mathcal{O}) := \mathrm{Ind}_{G/H_\lambda}^{G} U$ is a graded-simple $\mathcal{L}$-module.

Classification theorem

Theorem

For any graded-simple f.d. $\mathcal{L}$-module W, there exist a $\widehat{G}$-orbit $\mathcal{O}$ of dominant integral weights and an element $g \in G$ such that W is isomorphic to $W(\mathcal{O})^{[g]}$.
Two such graded modules, $W(\mathcal{O})^{[g]}$ and $W(\mathcal{O}')^{[g']}$, are isomorphic if and only if $\mathcal{O}' = \mathcal{O}$ and $g'G_\lambda = gG_\lambda$ where G_λ is the pre-image of the support of the Brauer invariant $[\mathrm{End}\,(V_\lambda)]$ under the quotient map $G \to G/H_\lambda$, with λ being a representative of $\mathcal{O}$.

Corollary

A f.d. $\mathcal{L}$-module V admits a G-grading that would make it a graded $\mathcal{L}$-module $\Leftrightarrow$ for any λ, the multiplicities of $V_{\lambda_1}, \ldots, V_{\lambda_s}$ in V are equal to each other and divisible by the graded Schur index of λ, where $\{\lambda_1, \ldots, \lambda_s\}$ is the $\widehat{G}$-orbit of λ.

Product formulas

Let $\mu = \lambda_1 + \lambda_2$ where λ_1 and λ_2 are dominant integral weights of a semisimple f.d. Lie algebra $\mathcal{L}$.

Proposition

Suppose $\mathcal{L}$ is equipped with a grading by an abelian group G such that $H_{\lambda_1} \subset H_\mu$ (equivalently, $H_{\lambda_2} \subset H_\mu$). Then $\mathrm{Br}(\mu) = \mathrm{Br}(\lambda_1)\mathrm{Br}(\lambda_2)$ in the (G/H_μ)-graded Brauer group.

Proposition

Suppose that $V_{\lambda_2} \cong V_{\lambda_1}^*$ and $\mathcal{L}$ is equipped with a grading by an abelian group G such that $\{\lambda_1, \lambda_2\}$ is a $\widehat{G}$-orbit. Then $\mathrm{Br}(\mu)$ is trivial.

Orbits in the set of weights

If $\mathcal{L}$ is simple then $\mathrm{Out}(\mathcal{L})$ has order ≤ 2 in all cases except D_4. For D_4, we have $\mathrm{Out}(\mathcal{L}) \cong S_3$ so the image of the abelian group $\widehat{G}$ in $\mathrm{Out}(\mathcal{L})$ has order ≤ 3.
$\Rightarrow$ $\widehat{G}$-orbits in the set of dominant integral weights have length ≤ 3, and $\widehat{G}$ acts cyclically on each orbit.

Consider $\lambda = \sum_{i=1}^{s} m_i \omega_{k_i}$ where $m_i \in \mathbb{Z}_+$ and $\{\omega_{k_1}, \ldots, \omega_{k_s}\}$ is a $\widehat{G}$-orbit. We have one of the following two possibilities: either all m_i are equal or $H_\lambda = H_{\omega_{k_i}}$ (for any i).
$\Rightarrow$ If we know the Brauer invariant for $\sum_{i=1}^{s} \omega_{k_s}$, then we can compute it for any dominant integral weight.
We have a method for $s = 2$ that reduces the problem to single fundamental weights.

Type I gradings

Consider $\mathcal{L} = \mathfrak{sl}_{r+1}(\mathbb{F})$ of type A_r. Then the simple $\mathcal{L}$-module of highest weight ω_i ($i = 1, \ldots, r$) can be realized as $\wedge^i V$ where V is the natural module of $\mathcal{L}$.

Theorem

Suppose $\mathcal{L}$ is graded by an abelian group G such that the image of $\widehat{G}$ in $\mathrm{Aut}(\mathcal{L})$ consists of inner automorphisms (Type I grading). Then, for any dominant integral weight $\lambda = \sum_{i=1}^r m_i\omega_i$, we have $H_\lambda = \{e\}$ and $\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^r im_i}$, where $\hat{\beta}\colon \widehat{G} \times \widehat{G} \to \mathbb{F}^\times$ is associated to the parameters (T, β) of the grading on $\mathcal{L}$.

Corollary

The simple $\mathcal{L}$-module V_λ admits a G-grading making it a graded $\mathcal{L}$-module $\Leftrightarrow \sum_{i=1}^r im_i$ is divisible by the exponent of the group T.

Type II gradings

Theorem

Suppose $\mathcal{L}$ is graded by an abelian group G such that the image of $\widehat{G}$ in $\mathrm{Aut}(\mathcal{L})$ contains outer automorphisms and hence $H_{\omega_1} = \langle h \rangle$ for some $h \in G$ of order 2 (Type II grading with distinguished element h). Let $K = K_{\omega_1} = \langle h \rangle^{\perp}$. Then, for a dominant integral weight $\lambda = \sum_{i=1}^{r} m_i \omega_i$, we have:

1) *If $m_i \neq m_{r+1-i}$ for some i, then $H_\lambda = \langle h \rangle$, $K_\lambda = K$, and $\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{r} i m_i}$, where $\hat{\beta}\colon K \times K \to \mathbb{F}^\times$ is associated to the parameters $(\overline{T}, \beta)$ of the grading on $\mathcal{L}$.*

2a) *If r is even and $m_i = m_{r+1-i}$ for all i, then $H_\lambda = \{e\}$ and $\mathrm{Br}(\lambda) = 1$.*

2b) *If r is odd and $m_i = m_{r+1-i}$ for all i, then $H_\lambda = \{e\}$ and $\mathrm{Br}(\lambda) = \hat{\gamma}^{m_{(r+1)/2}}$, where $\hat{\gamma}\colon \widehat{G} \times \widehat{G} \to \mathbb{F}^\times$ is a certain extension of $\hat{\beta}^{(r+1)/2}\colon K \times K \to \mathbb{F}^\times$.*

Type II gradings (continued)

The extension $\hat{\gamma}$ is determined explicitly in terms of a certain element $h' \in G$ of order ≤ 2.

If $r \equiv 3 \pmod 4$ then $\hat{\beta}^{(r+1)/2} = 1$ and the support T of the graded division algebra representing $\mathrm{Br}(\omega_{(r+1)/2})$ is $\{e\}$ if $h' = e$ and $\langle h, h' \rangle \cong \mathbb{Z}_2^2$ if $h' \neq e$.

If $r \equiv 1 \pmod 4$ then $\hat{\beta}^{(r+1)/2} = \hat{\beta}$ and the support T of the graded division algebra representing $\mathrm{Br}(\omega_{(r+1)/2})$ is $\overline{T}$ if $h' \in \overline{T}$ and $\overline{T} \times \langle h, h' \rangle \cong \overline{T} \times \mathbb{Z}_2^2$ (orthogonal sum relative to γ) if $h' \notin \overline{T}$.

Corollary

The simple $\mathcal{L}$-module V_λ admits a G-grading making it a graded $\mathcal{L}$-module if and only if 1) for all i, $m_i = m_{r+1-i}$ and 2) either r is even or r is odd and one of the following conditions is satisfied: (i) $m_{(r+1)/2}$ is even or (ii) $r \equiv 3 \pmod 4$ and $\bar{h}' = \bar{e}$ in $\overline{G}$ or (iii) $r \equiv 1 \pmod 4$, $\overline{T} = \{\bar{e}\}$ and $\hat{h}' = \bar{e}$ in $\overline{G}$.

Consider $\mathcal{L} = \mathfrak{sp}_{2r}(\mathbb{F})$ of type C_r. Then all automorphisms of $\mathcal{L}$ are inner and the simple $\mathcal{L}$-module of highest weight ω_i ($i = 1, \ldots, r$) is contained with multiplicity 1 in $\wedge^i V$ where V is the natural module of $\mathcal{L}$.

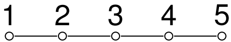
Theorem

Suppose $\mathcal{L}$ is graded by an abelian group G. Then, for any dominant integral weight $\lambda = \sum_{i=1}^{r} m_i\omega_i$, we have $H_\lambda = \{e\}$ and $\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}}$, where $\hat{\beta} \colon \widehat{G} \times \widehat{G} \to \mathbb{F}^\times$ is the commutation factor associated to the parameters (T, β) of the grading on $\mathcal{L}$.

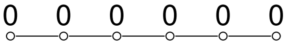
Corollary

The simple $\mathcal{L}$-module V_λ admits a G-grading making it a graded $\mathcal{L}$-module if and only if $T = \{e\}$ or the number $\sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}$ is even.

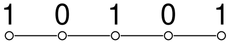
Weights for the marks on Dynkin diagram



inner grading

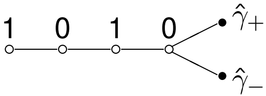


even rank, outer grading, sym λ



outer grading, nonsym λ

odd rank, outer grading, sym λ



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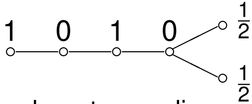
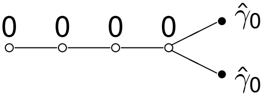
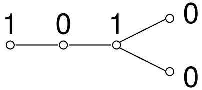


even rank, outer grading, sym λ



outer grading, nonsym λ



odd rank, outer grading, sym λ