Graded-simple algebras and modules via the loop construction

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Definition of group grading

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ and let G be a (semi)group.

Definition

A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$.
The *support* of the G-grading is the set $\{g \in G \mid \mathcal{A}_g \neq 0\}$.

Definition

Two G-gradings on $\mathcal{A}$, $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}'_g$, are *isomorphic* if there exists an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$.

Problem: given an algebra $\mathcal{A}$ and an abelian group G, classify the G-gradings on $\mathcal{A}$ up to isomorphism.
Solved over an a.c. field of char 0 for simple f.-d. associative and Jordan algebras, also for Lie except E_6, E_7, E_8.

Cartan grading of a semisimple Lie algebra

Historically the first grading to be studied (and still the most important):

Example (Cartan grading)

Let $\mathfrak{g}$ be a f.-d. semisimple Lie algebra over an a.c. field of char 0, and let $\mathfrak{h}$ be a Cartan subalgebra. Then the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\Big)$$

can be viewed as a grading by the root lattice $\langle\Phi\rangle \cong \mathbb{Z}^r$, $r = \dim \mathfrak{h}$. The support is $\{0\} \cup \Phi$.

Cartan grading also exists for simple Lie algebras of types A-G in characteristic $p > 0$.

Pauli matrices

Example (Pauli grading)

There is a grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by the group a $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Namely, we set

$$\begin{aligned} \mathfrak{g}_{(0,0)} &= 0, & \mathfrak{g}_{(1,0)} &= \operatorname{Span}\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}, \\ \mathfrak{g}_{(0,1)} &= \operatorname{Span}\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}, & \mathfrak{g}_{(1,1)} &= \operatorname{Span}\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}. \end{aligned}$$

The Pauli grading can be defined for $\mathfrak{sl}_2(\mathbb{F})$, $\operatorname{char} \mathbb{F} \neq 2$.
Any G-grading on $\mathfrak{sl}_2(\mathbb{F})$ is induced by the Pauli or Cartan grading via a group homomorphism $\mathbb{Z}_2^2 \to G$, resp. $\mathbb{Z} \to G$.
Given $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$, a homomorphism $\alpha \colon G \to H$ induces ${}^{\alpha}\Gamma : \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$ where $\mathcal{A}_h = \bigoplus_{g \in \alpha^{-1}(h)} \mathcal{A}_g$.

Definition of graded module

Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a grading on an associative (or Lie) algebra.

Definition

An $\mathcal{A}$-module V is said to be *graded* if it is equipped with a vector space decomposition $V = \bigoplus_{g\in G} V_g$ such that $\mathcal{A}_g \cdot V_h \subseteq V_{gh}$ for all $g, h \in G$.
A *homomorphism of graded modules* $\varphi\colon V \to W$ is a homomorphism of modules such that $\varphi(V_g) \subseteq W_g$ for all $g \in G$.

Lemma

Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a G-graded associative algebra where G is any group. Let $N \subset M$ be graded $\mathcal{A}$-modules. If N admits a complement in M as an $\mathcal{A}$-module then it admits a complement as a graded $\mathcal{A}$-module.

Example: $\mathfrak{sl}_2(\mathbb{F})$ with Pauli grading

In the above lemma, take $\mathcal{A} = U(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple f.d. Lie algebra over a field of char 0, graded by an abelian group
$\Rightarrow$ the category of graded f.d. $\mathfrak{g}$-modules is semisimple.

Example (simple vs. graded-simple)

Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$ with the Pauli grading
$\Rightarrow$ the natural module $V = \mathrm{Span}\{v_1, v_2\}$ does not admit a grading that would make it a graded $\mathfrak{g}$-module.
But $W = V^2$ admits a grading making it a graded-simple $\mathfrak{g}$-module (isomorphic to $M_2(\mathbb{F})$, with the Pauli grading, under left multiplication): $W = \mathrm{Span}\{v_i^j \mid i, j = 1, 2\}$ where

$$W_{(0,0)} = \mathrm{Span}\{v_1^1 + v_2^2\}, \quad W_{(1,0)} = \mathrm{Span}\{v_1^1 - v_2^2\},$$
$$W_{(0,1)} = \mathrm{Span}\{v_2^1 + v_1^2\}, \quad W_{(1,1)} = \mathrm{Span}\{v_2^1 - v_1^2\}.$$

The natural $\mathfrak{sl}_2$-module has *graded Schur index* 2 with respect to the Pauli grading.

Gradings on semisimple algebras

Recall: for an algebra $\mathcal{A}$, the *multiplication algebra* $M(\mathcal{A})$ is the subalgebra of $\mathrm{End}(\mathcal{A})$ generated by the operators of left and right multiplication by elements of $\mathcal{A}$.

$\mathcal{A}$ is simple $\Leftrightarrow$ $M(\mathcal{A}) \neq 0$ and $\mathcal{A}$ is a simple $M(\mathcal{A})$-module.

If $\mathcal{A}$ is graded by a group G then so is $M(\mathcal{A})$, and $\mathcal{A}$ is a graded $M(\mathcal{A})$-module.

Let $\mathcal{A}$ be an algebra such that $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_k$ where each $\mathcal{A}_i$ is an ideal of $\mathcal{A}$ and a simple algebra. In other words, $\mathcal{A}$ is a semisimple $M(\mathcal{A})$-module.

By lemma, if $\mathcal{A}$ is G-graded then $\mathcal{A} = \mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_s$ where each $\mathcal{B}_i$ is a graded ideal of $\mathcal{A}$ and a graded-simple algebra.

Problem: assuming we know simple algebras of a certain class, obtain all graded-simple algebras of this class.

Loop construction for algebras

Let G be an abelian group (written multiplicatively), $H \leq G$ and $\overline{G} = G/H$. Denote $\pi : G \to \overline{G}$ the natural homomorphism.

Definition (Generalized loop algebras)

Let $\mathcal{A}$ be a $\overline{G}$-graded algebra. Then $\mathcal{A} \otimes \mathbb{F}G$ is an algebra with multiplication $(a \otimes g)(a' \otimes g') = aa' \otimes gg'$. The subalgebra
$\mathrm{L}_\pi(\mathcal{A}) = \mathrm{Span}\,\{a \otimes g \mid a \in \mathcal{A}_{\bar{g}},\, g \in G\}$
can be given a G-grading, namely, $\bigoplus_{g \in G}(\mathcal{A}_{\bar{g}} \otimes g)$.

Example (Classical loop algebras)

$G = \mathbb{Z}$, $H = m\mathbb{Z}$, $\mathbb{F}G \cong \mathbb{F}[t, t^{-1}]$ (Laurent polynomials), $\mathcal{A} = \mathfrak{g}$ semisimple Lie algebra with a $\mathbb{Z}_m$-grading: $\mathfrak{g} = \bigoplus_{\bar{k} \in \mathbb{Z}_m} \mathfrak{g}_{\bar{k}}$.
Then $\mathrm{L}_\pi(\mathfrak{g}) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\bar{k}} t^k \subset \mathfrak{g}[t, t^{-1}]$.

V. Kac: if $\mathcal{L}$ is an affine Kac–Moody Lie algebra then $[\mathcal{L}, \mathcal{L}]/Z(\mathcal{L}) \cong \mathrm{L}_\pi(\mathfrak{g})$ as in the above example.

Graded-central-simple algebras with split centroid

Recall: the *centroid* of an algebra $\mathcal{A}$ is $C(\mathcal{A}) = \mathrm{End}_{M(\mathcal{A})}(\mathcal{A})$. In other words, $C(\mathcal{A})$ is the centralizer of the multiplication algebra in $\mathrm{End}(\mathcal{A})$. If $\mathcal{A}$ is simple then $C(\mathcal{A})$ is a field extension of $\mathbb{F}$. $\mathcal{A}$ is said to be *central* if $C(\mathcal{A}) = \mathbb{F}$.

If $\mathcal{A}$ is G-graded then homogeneous operators on $\mathcal{A}$ span a subalgebra of $\mathrm{End}(\mathcal{A})$ denoted by $\mathrm{End}^{\mathrm{gr}}(\mathcal{A})$. Unlike $\mathrm{End}(\mathcal{A})$ in general, this subalgebra is G-graded. Define the *graded centroid* $C^{\mathrm{gr}}(\mathcal{A})$ as the centralizer of $M(\mathcal{A})$ in $\mathrm{End}^{\mathrm{gr}}(\mathcal{A})$.

If $\mathcal{A}$ is graded-simple then $C^{\mathrm{gr}}(\mathcal{A}) = C(\mathcal{A})$ is a graded-field. $\mathcal{A}$ is *graded-central* if the identity component of $C^{\mathrm{gr}}(\mathcal{A})$ is $\mathbb{F}$.

Theorem (Allison–Berman–Faulkner–Pianzola, 2008)

Let $\mathcal{A}$ be a graded-simple algebra such that $C(\mathcal{A}) \cong \mathbb{F}H$ as a graded algebra. Then $\mathcal{A} \cong \mathrm{L}_\pi(\mathcal{B})$ for some central simple algebra $\mathcal{B}$ with a $\overline{G}$-grading.

Twisting by characters

If $\mathbb{F}$ is a.c. and $\mathcal{A}$ is graded-central-simple then $C(\mathcal{A}) \cong \mathbb{F}H$ for some $H \leq G$ (the support of the centroid), so $\mathcal{A} \cong \mathrm{L}_\pi(\mathcal{B})$ where π is associated to this H.

If $\mathbb{F}$ is a.c. then $\mathcal{B}$ is unique up to isomorphism of $\overline{G}$-graded algebras.

In general, $\mathcal{B}$ is unique up to $\overline{G}$-graded isomorphism and the following procedure. Fix a (set) section $\xi : \overline{G} \to G$.

For a character $\chi : H \to \mathbb{F}^\times$, define $\mathcal{B}^\chi = \mathcal{B}$ as a $\overline{G}$-graded vector space, but with modified multiplication:
$b \star b' = \chi(\xi(\overline{g})\xi(\overline{g}')\xi(\overline{gg}')^{-1})bb'$, for all $b \in \mathcal{B}_{\overline{g}}$ and $b' \in \mathcal{B}_{\overline{g}'}$.

The graded isomorphism class of $\mathcal{B}^\chi$ does not depend on the choice of ξ (equivalently, a transversal for H in G).

If χ extends to a character of G (always the case if $\mathbb{F}$ is a.c.) then $\mathcal{B}^\chi \cong \mathcal{B}$.

Loop construction for modules (Mazorchuk–Zhao)

Problem: for a given G-grading on an associative algebra $\mathcal{R}$ and assuming that we know simple $\mathcal{R}$-modules, obtain all graded-simple $\mathcal{R}$-modules.

For example, $\mathcal{R} = U(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple f.d. Lie algebra over an a.c. field of char 0, consider f.d. modules.

Given $\pi : G \to \overline{G} = G/H$, any G-graded vector space (algebra, module) can be regarded as $\overline{G}$-graded, with the grading induced by π ('coarsening'). Thus, we have a 'forgetful functor' F_π from G-graded v.s. to $\overline{G}$-graded v.s.

The loop functor is the right adjoint of F_π: given a $\overline{G}$-graded v.s. V, define $L_\pi(V)$ to be the subspace $\bigoplus_{g\in G} V_{\overline{g}} \otimes g$ of $V \otimes \mathbb{F}G$.

If V is a $\overline{G}$-graded $\mathcal{R}$-module (where $\mathcal{R}$ is considered $\overline{G}$-graded) then $L_\pi(V)$ is a G-graded $\mathcal{R}$-module:
$r(v \otimes g') = rv \otimes gg'$, for all $r \in \mathcal{R}_g$ and $v \in V_{\overline{g}'}$.

Graded-central-simple modules

Recall: the *centralizer* of an $\mathcal{R}$-module W is $C(W) = \mathrm{End}_{\mathcal{R}}(W)$. In other words, $C(W)$ is the centralizer of (the image of) $\mathcal{R}$ in $\mathrm{End}(W)$. If W is simple then $C(W)$ is a division algebra. W is said to be *central* (or *Schurian*) if $C(W) = \mathbb{F}$.

If W is a G-graded $\mathcal{R}$-module then the *graded centralizer* $C^{\mathrm{gr}}(W)$ is the centralizer of $\mathcal{R}$ in $\mathrm{End}^{\mathrm{gr}}(W)$.

If W is graded-simple then $C^{\mathrm{gr}}(W) = C(W)$ is a graded-division-algebra. W is *graded-central* if the identity component of $C^{\mathrm{gr}}(W)$ is $\mathbb{F}$. If $\mathbb{F}$ is a.c. and W is graded-simple with $\dim W < |\mathbb{F}|$ then W is graded-central (Mazorchuk–Zhao).

Proposition

A G-graded $\mathcal{R}$-module W is isomorphic to $L_\pi(V)$ for a $\overline{G}$-graded $\mathcal{R}$-module V if and only if $C^{\mathrm{gr}}(W)$ contains a graded-subfield isomorphic to $\mathbb{F}H$.

Thin pregradings

Let G be an abelian group, $\mathcal{R}$ an associative G-graded algebra, and $\mathcal{V}$ an $\mathcal{R}$-module.

Definition (Billig–Lau)

- A family of subspaces $\Sigma = \{\mathcal{V}_g : g \in G\}$ is called a *G-pregrading* if $\mathcal{V} = \sum_{g\in G} \mathcal{V}_g$ and $\mathcal{R}_g\mathcal{V}_h \subseteq \mathcal{V}_{gh}\ \forall g, h \in G$.
- Given two pregradings $\Sigma^i = \{\mathcal{V}^i_g : g \in G\}$, $i = 1, 2$, Σ^1 is a *refinement* of Σ^2 (or Σ^2 a *coarsening* of Σ^1) if $\mathcal{V}^1_g \subseteq \mathcal{V}^2_g$ for all $g \in G$. If at least one of these containments is strict, the refinement is said to be *proper*.
- Σ is *thin* if it admits no proper refinement.

Let $H \leq G$ and $\overline{G} = G/H$. If $\mathcal{V} = \bigoplus_{\bar{g}\in\overline{G}} \mathcal{V}_{\bar{g}}$ is a $\overline{G}$-grading on $\mathcal{V}$ making it a $\overline{G}$-graded $\mathcal{R}$-module (where $\mathcal{R}$ is considered $\overline{G}$-graded) then the family $\Sigma := \{\mathcal{V}'_g : g \in G\}$, where $\mathcal{V}'_g = \mathcal{V}_{\bar{g}}$ for all $g \in G$, is a G-pregrading of $\mathcal{V}$.

The groupoids $\mathfrak{M}(\pi)$ and $\mathfrak{N}(\pi)$

Fix $H \leq G$ and let $\pi : G \to \overline{G} = G/H$ be the natural homomorphism.

Definition

- $\mathfrak{M}(\pi)$ is the groupoid whose objects are the simple, central, and $\overline{G}$-graded $\mathcal{R}$-modules $V = \bigoplus_{\bar{g}\in\overline{G}} V_{\bar{g}}$ such that the G-pregrading associated to the $\overline{G}$-grading is thin, and whose morphisms are the $\overline{G}$-graded isomorphisms.
- $\mathfrak{N}(\pi)$ is the groupoid whose objects are the pairs $(W, \mathcal{F})$, where W is a G-graded-simple $\mathcal{R}$-module and $\mathcal{F}$ is a maximal graded-subfield of $C(W)$ isomorphic to $\mathbb{F}H$, and the morphisms $(W, \mathcal{F}) \to (W', \mathcal{F}')$ are the pairs (ϕ, ψ), where $\phi : W \to W'$ is an isomorphism of G-graded modules, $\psi : \mathcal{F} \to \mathcal{F}'$ is an isomorphism of G-graded algebras, and $\phi(wc) = \phi(w)\psi(c)$ for all $w \in W$ and $c \in \mathcal{F}$.

The loop functor

If V is a $\overline{G}$-graded $\mathcal{R}$-module then $C^{\mathrm{gr}}(V)$ is a $\overline{G}$-graded algebra, so its loop algebra $L_\pi(C^{\mathrm{gr}}(V))$ is a G-graded algebra, which acts naturally on $L_\pi(V)$:
$(v \otimes g)(d \otimes g') = vd \otimes gg' \quad \forall g, g' \in G,\ v \in V_{\bar{g}},\ d \in C(V)_{\bar{g}'}$.

This action centralizes the action of $\mathcal{R}$, so we can identify $L_\pi(C^{\mathrm{gr}}(V))$ with a G-graded subalgebra of $C^{\mathrm{gr}}(L_\pi(V))$.

Also, $L_\pi(\mathbb{F}1)$ is a G-graded subalgebra of $L_\pi(C^{\mathrm{gr}}(V))$.

Theorem

If V is an object in $\mathfrak{M}(\pi)$ then $(L_\pi(V), L_\pi(\mathbb{F}1))$ is an object in $\mathfrak{N}(\pi)$. Every object in $\mathfrak{N}(\pi)$ is isomorphic $(L_\pi(V), L_\pi(\mathbb{F}1))$ for some V in $\mathfrak{M}(\pi)$.

If $\mathbb{F}$ is a.c. then for any graded-central-simple $\mathcal{R}$-module W, $C^{\mathrm{gr}}(W) = C(W)$ contains a maximal graded-subfield $\mathcal{F}$ isomorphic to $\mathbb{F}H$ for some $H \leq G$.

Twisting by characters

Let V be a $\overline{G}$-graded $\mathcal{R}$-module. Fix a (set) section $\xi : \overline{G} \to G$.

For a character $\chi : H \to \mathbb{F}^\times$, define $V^\chi = V$ as a $\overline{G}$-graded vector space, but with modified action:
$r \cdot v = \chi(g\xi(\overline{g}')\xi(\overline{gg}')^{-1})rv$, for all $g, g' \in G$, $r \in \mathcal{R}_g$, $v \in V_{\overline{g}'}$.

The graded isomorphism class of V^χ does not depend on the choice of ξ (equivalently, the transversal $\Theta = \xi(\overline{G})$ for H in G).

If χ extends to a character of G (always the case if $\mathbb{F}$ is a.c.) then $V^\chi \cong V^{\alpha_\chi}$, where α_χ is the automorphism of $\mathcal{R}$ induced by the extended character $\chi : G \to \mathbb{F}^\times$:
$\alpha_\chi(r) = \chi(g)r$, for all $g \in G$ and $r \in \mathcal{R}_g$.

Here V^α denotes the twist of V by an automorphism α of $\mathcal{R}$:
$r \cdot v = \alpha(r)v$, for all $r \in \mathcal{R}$, $v \in V$.

The extended loop functor

Define a groupoid $\widetilde{\mathfrak{M}}(\pi)$ by extending $\mathfrak{M}(\pi)$: keep the same objects, but for the morphisms $V \to V'$ take all pairs (φ, χ) where $\chi \in \widehat{H}$ and $\varphi : V^{\chi} \to V'$ is a morphism in $\mathfrak{M}(\pi)$.

Fix a transversal Θ for H in G and extend the loop functor to $\widetilde{L}_{\pi} : \widetilde{\mathfrak{M}}(\pi) \to \mathfrak{N}(\pi)$ as follows:

- define $\widetilde{L}_{\pi}(V) = \big(L_{\pi}(V), L_{\pi}(\mathbb{F}1)\big)$ for objects (the same as before);
- send a morphism (φ, χ) as above to the pair (ϕ, ψ) where $\phi(v \otimes gh) = \chi(h)\varphi(v) \otimes gh$, for all $v \in V_{\bar{g}}$, $g \in \Theta$, $h \in H$, and $\psi(1 \otimes h) = \chi(h)1 \otimes h$.

Theorem

$\widetilde{L}_{\pi} : \widetilde{\mathfrak{M}}(\pi) \to \mathfrak{N}(\pi)$ is an equivalence of categories.

Semisimple case

From now on, assume $\mathbb{F}$ is a.c. For every graded-central-simple W, there exists $\mathcal{F}$ such that $(W, \mathcal{F}) \in \mathfrak{N}(\pi)$ for some $H \leq G$.

Proposition

Let W be a G-graded-simple $\mathcal{R}$-module. TFAE:

- *$C(W)$ contains a maximal graded-subfield $\mathcal{F} \cong \mathbb{F}H$ for some $H \leq G$ where $|H|$ is finite and not divisible by $\mathrm{char}\,\mathbb{F}$;*
- $\dim C(W)$ *is finite and not divisible by $\mathrm{char}\,\mathbb{F}$.*

Under these conditions, W is semisimple as an ungraded module, and $\mathcal{D} := C(W)$ is isomorphic to the twisted group algebra $\mathbb{F}^\sigma T$ where $T \leq G$ is the support of $\mathcal{D}$ and $\sigma : T \times T \to \mathbb{F}^\times$ is a 2-cocycle. The graded isom. class of $\mathcal{D}$ is determined by T and the alternating bicharacter $\beta : T \times T \to \mathbb{F}^\times$ defined by $\beta(s,t) = \sigma(s,t)/\sigma(t,s), \quad \forall s,t \in T$.

$\mathcal{F} \Leftrightarrow$ a maximal (totally) isotropic subgroup in (T, β).

Definition of inertia group and graded Brauer invariant

Let W be a G-graded-simple $\mathcal{R}$-module such that $\dim C(W)$ is finite and not divisible by $\operatorname{char}\mathbb{F}$. Denote $\mathcal{D} = C(W)$ and let $\mathcal{Z}$ be the center of $\mathcal{D}$ and Z the support of $\mathcal{Z}$ (the radical of β).

Definition

- The *inertia group* of W is $K_W := \{\chi \in \widehat{G} : \chi(z) = 1 \ \forall z \in Z\}$ (that is, $K_W = Z^{\perp}$).
- The *graded Brauer invariant* of W is the isomorphism class of the G/Z-graded division algebra $\mathcal{D}\varepsilon$, where ε is any primitive idempotent of $\mathcal{Z}$.
- The *graded Schur index* ℓ of W is the degree of the matrix algebra $\mathcal{D}\varepsilon$.

Let V be a simple (ungraded) submodule of W. Then the simple components of W are precisely the twists V^{α_χ}, $\chi \in \widehat{G}$, and each occurs with multiplicity ℓ. $K_W = \{\chi \in \widehat{G} : V^{\alpha_\chi} \cong V\}$.

Graded Brauer invariants of f.d. modules

Theorem

Let W be a f.d. G-graded-simple $\mathcal{R}$-module such that $\operatorname{char}\mathbb{F}$ does not divide $\dim C(W)$. Let V be a simple (ungraded) submodule of W and let $Z = K_W^\perp$.

- *There is a unique G/Z-grading on $\operatorname{End}(V)$ that makes $\varrho_V : \mathcal{R} \to \operatorname{End}(V)$ a G/Z-graded homomorphism.*
- *The graded Brauer invariant of W is precisely $[\operatorname{End}(V)]$.*
- *For any maximal isotropic subgroup H of the support of $C(W)$, V is endowed with a structure of G/H-graded $\mathcal{R}$-module such that $W \cong L_\pi(V)$.*

Theorem

Assume $\operatorname{char}\mathbb{F} = 0$. A f.d. simple $\mathcal{R}$-module V is isomorphic to a simple submodule of a f.d. G-graded-simple $\mathcal{R}$-module if and only if the subgroup $K_V := \{\chi \in \widehat{G} : V^{\alpha_\chi} \cong V\}$ has finite index.