On rigidity of Nichols algebras

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Outline

Let $\Bbbk$ be an algebraically closed field of characteristic zero; $\otimes = \otimes_{\Bbbk}$.

Let V be a vector space over $\Bbbk$.

An invertible linear map $c : V \otimes V \to V \otimes V$ is a *braiding* if
$c_1 c_2 c_1 = c_2 c_1 c_2$
where $c_1 = c \otimes \mathrm{id}_V$ and $c_2 = \mathrm{id}_V \otimes c$;

c is *symmetric* if $c^2 = \mathrm{id}_{V \otimes V}$.

Definition

Let A be an algebra (not necessarily associative) with multiplication $\mu : A \otimes A \to A$, and let $c : A \otimes A \to A \otimes A$ be a braiding.
(A, μ, c) is a *braided algebra* if these compatibility conditions hold:

$$c \circ (\mu \otimes \mathrm{id}_A) = (\mathrm{id}_A \otimes \mu) \circ c_1 c_2;$$
$$c \circ (\mathrm{id}_A \otimes \mu) = (\mu \otimes \mathrm{id}_A) \circ c_2 c_1.$$

If c is symmetric then the two conditions are equivalent to one another.

Braided coalgebras are defined similarly.

Let $\mathcal{V}$ be a braided $\Bbbk$-linear category. An *algebra in* $\mathcal{V}$ is a pair (A, μ) where A is an object and $\mu : A \otimes A \to A$ is a morphism. If $\mathcal{V}$ consists of vector spaces and linear maps, e.g. $\mathcal{V} = \mathcal{M}^H$ where (H, β) is a CQT bialgebra, then A is a braided algebra with $c = c_{A,A}$.

Conversely, any f.d. braided vector space (V, c) can be regarded as an object in a suitable braided category $\mathcal{V}$ such that $c = c_{V,V}$, e.g. $\mathcal{V} = \mathcal{M}^H$ for a CQT bialgebra (H, β). Under a certain condition on c, we can make H a CQT Hopf algebra.

If (A, μ, c) is a braided algebra then (H, β) can be replaced with a quotient such that μ is a morphism in $\mathcal{M}^H$ (Takeuchi 2000).

Definition

$(B, m, u, \Delta, \varepsilon, c)$ is a *braided bialgebra* if (B, m, u, c) is a unital associative braided algebra, $(B, \Delta, \varepsilon, c)$ is a counital coassociative braided coalgebra, u is a counital coalgebra map, ε is a unital algebra map, and $\Delta m = (m \otimes m)(\mathrm{id}_B \otimes c \otimes \mathrm{id}_B)(\Delta \otimes \Delta)$.

If $(B, m, u, \Delta, \varepsilon, c)$ is a braided bialgebra then, for suitable (H, β), B is a *bialgebra in* $\mathcal{M}^H$, i.e., $c = c_{B,B}$ and m, u, Δ, ε are morphisms.

Definition (Gurevich 1986)

Let $(L, [,], c)$ be a braided algebra, where c is symmetric.
Then $(L, [,], c)$ is a *c-Lie algebra* if

$[,] \circ (\mathrm{id}_{L\otimes L} + c) = 0$ (braided antisymmetry)

$[,] \circ ([,] \otimes \mathrm{id}_L) \circ (\mathrm{id}_{L\otimes L\otimes L} + c_1 c_2 + c_2 c_1) = 0$ (braided Jacobi).

This generalizes Lie algebras (c is the flip), superalgebras (c is the signed flip) and coloralgebras (c is of diagonal type).

If (A, μ, c) is a braided associative algebra (with symmetric c), then $(A, [,], c)$ is a braided Lie algebra, denoted $A^{(-)}$,
where $[,]$ is the *braided commutator*:

$$[,] = \mu \circ (\mathrm{id}_{A\otimes A} - c).$$

If B is a braided bialgebra with symmetric c, then the space of primitive elements $P(B)$ is closed under the braided commutator. This is false for general c.

Let H be a cotriangular bialgebra with R-form $\beta : H \otimes H \to \Bbbk$.

Recall that the braiding on the category $\mathcal{M}^H$ is given by

$$c_{V,W} : v \otimes w \mapsto \sum \beta(v_{(1)}, w_{(1)}) w_{(0)} \otimes v_{(0)}$$

where $V, W \in \mathcal{M}^H$.

Let L be an algebra in $\mathcal{M}^H$. Then L is a Lie algebra in $\mathcal{M}^H$ iff it satisfies the following identities:

$$\begin{aligned} [a,b] &+ \sum \beta(a_{(1)}, b_{(1)})[b_{(0)}, a_{(0)}] = 0, \\ [[a,b],c] &+ \sum \beta(a_{(1)}b_{(1)}, c_{(1)})[[c_{(0)}, a_{(0)}], b_{(0)}] \\ &+ \sum \beta(a_{(1)}, b_{(1)}c_{(1)})[[b_{(0)}, c_{(0)}], a_{(0)}] = 0. \end{aligned}$$

Such objects are also known as (H, β)-*Lie algebras* (Bahturin–Fischman–Montgomery 1996).

Let $\mathcal{V}$ and $\mathcal{V}'$ be symmetric categories.

Proposition (K 2008)

Let $(\Phi, \varphi_2) : \mathcal{V} \to \mathcal{V}'$ be a braided monoidal functor. Let A be an algebra in $\mathcal{V}$. If A satisfies the (multilinear) polynomial identity $F = 0$, then so does the algebra $\Phi(A)$ in $\mathcal{V}'$.

Suppose (H, β) is a cotriangular bialgebra and $\sigma : H \otimes H \to \mathbb{k}$ a right 2-cocycle. Then (H_σ, β_σ) is again a cotriangular bialgebra where $H_\sigma = H$ as a coalgebra, the multiplication of H_σ is given by

$$h \cdot_\sigma k = \sum \sigma^{-1}(h_{(1)}, k_{(1)}) h_{(2)} k_{(2)} \sigma(h_{(3)}, k_{(3)}),$$

and the R-form

$$\beta_\sigma(h, k) = \sum \sigma^{-1}(k_{(1)}, h_{(1)}) \beta(h_{(2)} k_{(2)}) \sigma(h_{(3)}, k_{(3)}).$$

Also $\Phi = \mathrm{id} : \mathcal{M}^H \to \mathcal{M}^{H_\sigma}$ and

$$\varphi_2(V, W) : v \otimes w \mapsto \sum \sigma(v_{(1)}, w_{(1)}) v_{(0)} \otimes w_{(0)}$$

define an equivalence of braided monoidal categories $\mathcal{M}^H$ and $\mathcal{M}^{H_\sigma}$.

If A is an algebra in $\mathcal{M}^H$ with multiplication $\mu : A \otimes A \to A$, then $\Phi(A) = A$ as an H-comodule and the multiplication of $\Phi(A)$ is given by

$$\mu_\sigma(a \otimes b) = \sum \sigma(a_{(1)}, b_{(1)})\mu(a_{(0)} \otimes b_{(0)}).$$

Corollary

Let L be an (H, β)-Lie algebra. Then L_σ is an (H_σ, β_σ)-Lie algebra. Moreover, L and L_σ have the same H-comodule subalgebras and ideals. L is solvable (resp., nilpotent) iff so is L_σ.

Theorem (Etingof–Gelaki 2001)

Let (H, β) be a cotriangular Hopf algebra. Assume that H is pseudoinvolutive (i.e., for any finite-dimensional subcoalgebra $C \subset H$ we have $\mathrm{tr}(S^2|_C) = \dim C$). Then there exists a 2-cocycle $\sigma : H \otimes H \to \Bbbk$ such that H_σ is commutative and
$\beta_\sigma = \frac{1}{2}(\varepsilon \otimes \varepsilon + \varepsilon \otimes \zeta + \zeta \otimes \varepsilon - \zeta \otimes \zeta)$
for some central grouplike $\zeta \in H^*$ with $\zeta^2 = 1$.

Since $\Bbbk$ is an a. c. field of characteristic zero and H_σ is a commutative Hopf algebra, we have $H_\sigma = \mathcal{O}(G)$, the algebra of regular functions the pro-algebraic group $G = G(H_\sigma^*) = \mathrm{Alg}(H_\sigma, \Bbbk)$.

A right H_σ-comodule is a vector space on which G acts linearly and algebraically.

Corollary

Let (H, β), σ, and ζ be as above.
Let $L_0 := \{a \in L | \zeta \cdot a = a\}$, $L_1 := \{a \in L | \zeta \cdot a = -a\}$.
Then $L \mapsto L_\sigma = L_0 \oplus L_1$ is an equivalence of the category of (H, β)-Lie algebras and the category of Lie superalgebras with an algebraic G-action by automorphisms of graded algebras.

Dually, one can work with a triangular Hopf algebra (H, R) and a dual cocycle $J \in H \otimes H$.

Theorem (Etingof–Gelaki 2003)

Let (H, R) be a finite-dimensional triangular Hopf algebra. Then there exists a dual cocycle $J \in H \otimes H$ such that (H^J, R^J) is a modified supergroup algebra.

Let V be a f.d. vector space, G a finite group that acts linearly on V, and $\zeta \in Z(G)$ such that $\zeta^2 = 1$ and $\zeta \cdot v = -v$ for all $v \in V$.

Let $H = \Lambda(V)\#\Bbbk G$. Define comultiplication Δ on H by
$\Delta g = g \otimes g$ for $g \in G$, $\Delta v = v \otimes 1 + \zeta \otimes v$ for $v \in V$
and the R-matrix $R_\zeta = \frac{1}{2}(1 \otimes 1 + 1 \otimes \zeta + \zeta \otimes 1 - \zeta \otimes \zeta)$.

Then (H, R_ζ) is a triangular Hopf algebra, called a *modified supergroup algebra*. (The algebra $\Lambda(V)\#\Bbbk G$ carries the natural structure of a Hopf superalgebra, but we modified it to obtain an ordinary Hopf algebra.)

Let G be a finite group and J a dual cocycle for $\Bbbk G$.

If G acts by automorphisms on a Lie algebra L, then L is a $((\Bbbk G)^*, \varepsilon \otimes \varepsilon)$-Lie algebra, so L_J is an (H, β)-Lie algebra where $H = ((\Bbbk G)^J)^*$ and $\beta = J_{21}^{-1} J$.

Take $G = A \rtimes K$, $A = \langle a \rangle_4$, $K = \langle g \rangle_2 \times \langle h \rangle_2$, and $g \cdot a = a$, $g \cdot a = a^{-1}$. Let $\pi : K \to A$ be a 1-cocycle defined by $\pi(g) = a^2$ and $\pi(h) = a$. Then π is bijective and

$$J = \frac{1}{|A|} \sum_{x \in A, y^* \in \widehat{A}} \langle x, y^* \rangle \pi^{-1}(x) \otimes y^*$$

is a minimal dual cocycle for $\Bbbk G$ (Etingof–Gelaki).

Example (K 2008)

Take $L = sl_2(\mathbb{k}) \times sl_2(\mathbb{k}) = \langle e_k, f_k, h_k \,|\, k = 1, 2\rangle$ and let G act on L:
1) g swaps the two $sl_2(\mathbb{k})$ components;
2) h acts by $Ad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on each component;
3) a acts by $Ad \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$ on each component, where ω is a primitive 8-th root of unity.

Multiplication table of the twisted $sl_2 \times sl_2$

	h_1	e_1	f_1	h_2	e_2	f_2
h_1	0	0	$2f_1$	0	$-2e_2$	0
e_1	$-2e_1$	0	0	0	$-h_2$	0
f_1	$2f_1$	0	h_1	0	0	0
h_2	0	$-2e_1$	0	0	0	$2f_2$
e_2	0	$-h_1$	0	$-2e_2$	0	0
f_2	0	0	0	$2f_2$	0	h_2

Braiding on the twisted $sl_2 \times sl_2$

	h_1	e_1	f_1	h_2	e_2	f_2
h_1	$h_1 \otimes h_1$	$-e_1 \otimes h_2$	$-f_1 \otimes h_1$	$h_2 \otimes h_1$	$-e_2 \otimes h_2$	$-f_2 \otimes h_1$
e_1	$-h_2 \otimes e_1$	$f_1 \otimes f_2$	$e_1 \otimes f_1$	$-h_1 \otimes e_1$	$f_2 \otimes f_2$	$e_2 \otimes f_1$
f_1	$-h_1 \otimes f_1$	$f_2 \otimes e_2$	$e_2 \otimes e_1$	$-h_2 \otimes f_1$	$f_1 \otimes e_2$	$e_1 \otimes e_1$
h_2	$h_1 \otimes h_2$	$-e_1 \otimes h_1$	$-f_1 \otimes h_2$	$h_2 \otimes h_2$	$-e_2 \otimes h_1$	$-f_2 \otimes h_2$
e_2	$h_2 \otimes e_2$	$f_1 \otimes f_1$	$e_1 \otimes f_2$	$-h_1 \otimes e_2$	$f_2 \otimes f_1$	$e_2 \otimes f_2$
f_2	$-h_1 \otimes f_2$	$f_2 \otimes e_1$	$e_2 \otimes e_2$	$-h_2 \otimes f_2$	$f_1 \otimes e_1$	$e_1 \otimes e_2$

This is an example of an (H, β)-Lie algebra which is not a Lie coloralgebra.

Let $V = \langle x \rangle$, $G = \langle g \rangle_2$, and $\zeta = g$. Then the modified supergroup algebra is the Sweedler algebra of dimension 4:

$$H_4 = \langle x, g \mid g^2 = 1,\ x^2 = 0,\ gx = -xg \rangle, \quad \Delta g = g \otimes g,\ \Delta x = x \otimes 1 + g \otimes x.$$

Set $J_\lambda = 1 \otimes 1 - \frac{\lambda}{2} gx \otimes g$, $\lambda \in \Bbbk$. Then $(H_4)^{J_\lambda} = H_4$, but with a different R-matrix: $(R_g)^{J_\lambda} = R_g - \frac{\lambda}{2}(x \otimes x + gx \otimes x - x \otimes gx + gx \otimes gx)$.

Example (K 2008)

Take $L = sl_{2,1}(\Bbbk) = \langle h, e, f, z \rangle \oplus \langle E_{13}, E_{23}, E_{31}, E_{32} \rangle$, where h, e, f is the standard basis of $sl_2(\Bbbk)$ in the upper left corner of $sl_{2,1}(\Bbbk)$ and $z = \mathrm{diag}(1,1,2)$. Let g act by parity and let x act by $\mathrm{ad}\, E_{13}$.
Then L is an H_4-module algebra. One can check that $[\,,\,]_{J_\lambda}$ coincides with $[\,,\,]$ on all basis elements except the following:

$$[f, E_{31}]_{J_\lambda} = -\tfrac{\lambda}{2} E_{23}, \qquad [E_{31}, f]_{J_\lambda} = -\tfrac{\lambda}{2} E_{23},$$
$$[f, E_{32}]_{J_\lambda} = -E_{31} + \tfrac{\lambda}{2} E_{13}, \qquad [E_{32}, f]_{J_\lambda} = E_{31} + \tfrac{\lambda}{2} E_{13},$$
$$[E_{31}, E_{32}]_{J_\lambda} = -\tfrac{\lambda}{2} e, \qquad [E_{32}, E_{31}]_{J_\lambda} = \tfrac{\lambda}{2} e.$$

Let L be a c-Lie algebra (with symmetric c). Define the *universal enveloping algebra* $\mathcal{U}_c(L)$ as the quotient of the tensor algebra $T(L)$ by the relations:

$$x \otimes y - c(x \otimes y) - [x, y], \quad \forall x, y \in L.$$

Then there is a canonical map $\eta : L \to \mathcal{U}_c(L)$ that satisfies the usual universal property: for any unital associative braided algebra A and a braided algebra map $f : L \to A^{(-)}$ there exists a unique unital algebra map $F : \mathcal{U}_c(L) \to A$ such that $f = F \circ \eta$.

The usual increasing filtration of $T(L)$ gives rise to the *standard filtration* of $\mathcal{U}_c(L)$.

Theorem (Kharchenko 2007)

The graded algebra $\mathrm{gr}\,\mathcal{U}_c(L)$ *associated to the standard filtration of* $\mathcal{U}_c(L)$ *is naturally isomorphic to* $\mathcal{U}_c(L^\circ)$ *where* L° *denotes the braided Lie algebra with the same underlying braided vector space as* L *but with zero bracket.*

In particular, η is an embedding.

Let B be a connected braided bialgebra (hence a Hopf algebra) with symmetric c.

Definition

B is called *c-cocommutative* if $c \circ \Delta = \Delta$.

If L is a c-Lie algebra, then c extends to a braiding on $\mathcal{U}_c(L)$. There exists a unique structure of a braided bialgebra on $\mathcal{U}_c(L)$ such that

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \forall x \in L.$$

Theorem (Kharchenko 2007)

The functors $L \mapsto \mathcal{U}_c(L)$ and $B \mapsto P(B)$ determine an equivalence between the category of c-Lie algebras and the category of connected c-cocommutative bialgebras.

Masuoka proved a dual version of the above theorem: there is an equivalence between the category of locally nilpotent c-Lie coalgebras and the category of irreducible c-commutative Hopf algebras.

Recall: the *Nichols algebra* of a braided vector space (V, c), denoted by $\mathcal{B}(V, c)$ or just $\mathcal{B}(V)$, is the unique (up to isomorphism) graded braided bialgebra $\mathcal{B} = \bigoplus_{n\geq 0} \mathcal{B}_n$ with $\mathcal{B}_0 = \Bbbk$, $\mathcal{B}_1 = V$ such that the restriction of the braiding of $\mathcal{B}$ to V is c, $\mathcal{B}$ is generated by V as an algebra, and $V = P(\mathcal{B})$.

If L is a c-Lie algebra with symmetric c then $\operatorname{gr}\mathcal{U}_c(L) \cong \mathcal{U}_c(L^\circ) \cong \mathcal{B}(L, c)$. Thus, $\mathcal{U}_c(L)$ is a *graded deformation* or *lifting* of the graded braided bialgebra $\mathcal{B}(L, c)$.

Let $\mathcal{B}$ be a bialgebra in a braided tensor category $\mathcal{V}$ (consisting of vector spaces and linear maps) equipped with a grading, as an object in $\mathcal{V}$, $\mathcal{B} = \bigoplus_{n\geq 0} \mathcal{B}_n$, which is at the same time an algebra and a coalgebra grading, i.e., $\mathcal{B}_i\mathcal{B}_j \subseteq \mathcal{B}_{i+j}$ and $\Delta(\mathcal{B}_k) \subseteq \bigoplus_{i+j=k} \mathcal{B}_i \otimes \mathcal{B}_j$, for all $i, j, k \geq 0$.

A *lifting* $(\mathcal{U}, \pi)$ of $\mathcal{B}$ consists of a filtered bialgebra $\mathcal{U}$ and a filtered vector space isomorphism $\pi\colon \mathcal{U} \to \mathcal{B}$ such that $\operatorname{gr}\pi\colon \operatorname{gr}\mathcal{U} \to \operatorname{gr}\mathcal{B} = \mathcal{B}$ is an isomorphism of graded bialgebras. An *equivalence* between liftings $(\mathcal{U}, \pi)$ and $(\mathcal{U}', \pi')$ is a filtered bialgebra isomorphism $f\colon \mathcal{U} \to \mathcal{U}'$ such that $\operatorname{gr}\pi \circ \operatorname{gr} f = \operatorname{gr}\pi'$.

Consider the polynomial algebra $\Bbbk[t]$ equipped with its standard grading. By extending scalars from $\Bbbk$ to $\Bbbk[t]$, the braided tensor category $\mathcal{V}$ gives rise to the braided tensor category $\mathcal{V}_{\Bbbk[t]}$.

Definition (following Du–Chen–Ye 2007)

A (formal) *graded deformation* of a graded bialgebra $(\mathcal{B}, m, \Delta)$ in $\mathcal{V}$ is a $\Bbbk[t]$-linear graded structure (m_t, Δ_t) on $\mathcal{B}[t] = \mathcal{B} \otimes \Bbbk[t]$ such that $(\mathcal{B}[t], m_t, \Delta_t)$ is a graded bialgebra in $\mathcal{V}_{\Bbbk[t]}$ and $(m_t, \Delta_t)|_{t=0} = (m, \Delta)$.

We say that two graded deformations, $(\mathcal{B}[t], m_t, \Delta_t)$ and $(\mathcal{B}[t], m'_t, \Delta'_t)$, are *equivalent* if there exists a $\Bbbk[t]$-linear graded bialgebra isomorphism $f\colon (\mathcal{B}[t], m_t, \Delta_t) \to (\mathcal{B}[t], m'_t, \Delta'_t)$.

Graded deformations of $\mathcal{B}$ are controlled by the *truncated graded bialgebra cohomology* $\widehat{\mathrm{H}}^*_{\mathrm{b}}(\mathcal{B})_\ell$. In particular, if $\widehat{\mathrm{H}}^2_{\mathrm{b}}(\mathcal{B})_\ell = 0$ for all $\ell < 0$, then $\mathcal{B}$ is *rigid*, i.e., has no nontrivial graded deformations.

A graded deformation is given by a sequence of pairs of maps (m_i, Δ_i), $i \geq 0$, of degree $-i$ such that $m_t|_{\mathcal{B}\otimes\mathcal{B}} = m + \sum_{i\geq 1} m_i t^i$ and $\Delta_t|_{\mathcal{B}} = \Delta + \sum_{i\geq 1} \Delta_i t^i$.

The concepts of lifting and graded deformation are equivalent:

A graded deformation $(\mathcal{B}[t], m_t, \Delta_t)$ defines a lifting $(\mathcal{U}, \pi)$, where $\mathcal{U}$ is $\mathcal{B}$ as a filtered vector space, π is identity, and $(m_{\mathcal{U}}, \Delta_{\mathcal{U}}) = (m_t, \Delta_t)|_{t=1}$.

If $(\mathcal{U}, \pi)$ is a lifting, then the linear maps
$\tilde{m}\colon \mathcal{B}\otimes\mathcal{B} \xrightarrow{\pi^{-1}\otimes\pi^{-1}} \mathcal{U}\otimes\mathcal{U} \xrightarrow{m_{\mathcal{U}}} \mathcal{U} \xrightarrow{\pi} \mathcal{B}$, $\tilde{\Delta}\colon \mathcal{B} \xrightarrow{\pi^{-1}} \mathcal{U} \xrightarrow{\Delta_{\mathcal{U}}} \mathcal{U}\otimes\mathcal{U} \xrightarrow{\pi\otimes\pi} \mathcal{B}\otimes\mathcal{B}$
decompose into direct sums of homogeneous components m_i, Δ_i of degrees $-i$ for $i \geq 0$, and the structure maps $(m_t, \Delta_t) = (\sum_i m_i t^i, \sum_i \Delta_i t^i)$ on $\mathcal{B}[t]$ define a formal graded deformation of $\mathcal{B}$.

Up to equivalence, these correspondences are inverses of each other.

Theorem

Let (H, β) be a cotriangular Hopf algebra that is either pseudo-involutive or finite-dimensional. Let V be a finite-dimensional H-comodule with the corresponding braiding c. If the Nichols algebra $\mathcal{B}(V, c)$ is finite-dimensional then it does not admit nontrivial graded deformations as an augmented algebra or bialgebra in $\mathcal{M}^H$.

Idea of proof: a graded deformation of $\mathcal{B}(V, c)$ is given by a c-Lie algebra structure on V, which can be twisted to a superalgebra structure using Etigof–Gelaki, but for a Lie superlagebra L, $\dim \mathcal{U}(L) < \infty$ only if $L_0 = 0$ and hence the bracket is zero.

Corollary

Let (V, c) be a finite-dimensional braided vector space such that c can be obtained from a coaction by a finite-dimensional cotriangular Hopf algebra. If $\mathcal{B}(V, c)$ is finite-dimensional then it does not admit nontrivial graded deformations as a braided augmented algebra or bialgebra.

Theorem

Let V be a an object in $\mathcal{V}$ and $T(V)$ its (braided) tensor bialgebra. Let $R \subset T(V)_{(2)}$ be a graded subspace that is an object in $\mathcal{V}$ and generates a biideal in $T(V)$. Consider the quotient $\mathcal{B} = T(V)/\langle R\rangle$, which is a graded bialgebra in $\mathcal{V}$, and assume that the multiplication map $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \to (\mathcal{B}^+)^2$ splits in $\mathcal{V}$. If for some negative ℓ we have that $\mathrm{Hom}(R, P(\mathcal{B}))_\ell = 0$*, then* $\widehat{\mathrm{H}}^2_{\mathrm{b}}(\mathcal{B})_\ell = 0$.
In particular, if $\mathrm{Hom}(R, P(\mathcal{B}))_\ell = 0$ *for all negative ℓ, then $\mathcal{B}$ is rigid.*

Theorem

If $\mathcal{B}(V, c)$ is a Nichols algebra of diagonal type with finite root system then $\mathcal{B}(V, c)$ does not admit nontrivial graded deformations as a braided bialgebra.

The second theorem applies to finite-dimensional Nichols algebras and positive parts of quantum groups.
The first theorem applies to some non-diagonal situations such as the Fomin–Kirillov algebras.