Group gradings on the algebras of block-triangular matrices

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Definition of group grading

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ and let G be a (semi)group.

Definition

A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$.

Definition

Two G-gradings on $\mathcal{A}$, $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}'_g$, are *isomorphic* if there exists an algebra automorphism $\psi \colon \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$.

Problem: given an algebra $\mathcal{A}$ and an abelian group G, classify the G-gradings on $\mathcal{A}$ up to isomorphism.
Solved for f.d. s.s. associative ($\mathbb{F}$ is alg. closed or real closed) and Jordan ($\mathbb{F}$ is a.c., $\operatorname{char} \mathbb{F} \neq 2$) algebras, also for simple Lie A-D, F_4, G_2 ($\mathbb{F}$ is a.c., $\operatorname{char} \mathbb{F} \neq 2$) and real forms A-D, G_2.

Cartan grading of a semisimple Lie algebra

Historically the first grading to be studied (and still the most important):

Example (Cartan grading)

Let $\mathfrak{g}$ be a f.-d. semisimple Lie algebra over an a.c. field of char 0, and let $\mathfrak{h}$ be a Cartan subalgebra. Then the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\Big)$$

can be viewed as a grading by the root lattice $\langle \Phi \rangle \cong \mathbb{Z}^r$, $r = \dim \mathfrak{h}$. The support is $\{0\} \cup \Phi$.

Cartan grading also exists for simple Lie algebras of types A-G in characteristic $p > 0$.

Pauli matrices

Example (Pauli grading)

There is a grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Namely, we set

$$\begin{aligned} \mathfrak{g}_{(0,0)} &= 0, & \mathfrak{g}_{(1,0)} &= \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \\ \mathfrak{g}_{(0,1)} &= \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}, & \mathfrak{g}_{(1,1)} &= \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}. \end{aligned}$$

The Pauli grading can be defined for $\mathfrak{sl}_2(\mathbb{F})$, $\operatorname{char}\mathbb{F} \neq 2$.
We can induce G-gradings on $\mathfrak{sl}_2(\mathbb{F})$ from the Pauli and Cartan grading via a group homomorphism $\mathbb{Z}_2^2 \to G$, resp. $\mathbb{Z} \to G$.

Gradings induced by group homomorphisms

Given $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, a homomorphism $\alpha\colon G \to H$ induces ${}^{\alpha}\Gamma : \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}_h$ where $\mathcal{A}_h = \bigoplus_{g\in\alpha^{-1}(h)} \mathcal{A}_g$.

Example ($\mathbb{Z}_2$-gradings on $\mathfrak{sl}_2(\mathbb{F})$)

Let $\Gamma : \mathfrak{sl}_2(\mathbb{F}) = \mathrm{Span}\left\{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]\right\} \oplus \mathrm{Span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right\} \oplus \mathrm{Span}\left\{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right\}$ be the Cartan grading and $\alpha\colon \mathbb{Z} \to \mathbb{Z}_2$ be the quotient map. Then ${}^{\alpha}\Gamma : \mathfrak{sl}_2(\mathbb{F}) = \mathrm{Span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right\} \oplus \mathrm{Span}\left\{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right\}$.
If $\mathbb{F}$ is a.c. then any nontrivial homomorphisms $\mathbb{Z}_2^2 \to \mathbb{Z}_2$ induces from the Pauli grading on $\mathfrak{sl}_2(\mathbb{F})$ a $\mathbb{Z}_2$-grading isomorphic to the above.
If $\mathbb{F} = \mathbb{R}$ then one of the homomorphisms $\mathbb{Z}_2^2 \to \mathbb{Z}_2$ induces the $\mathbb{Z}_2$-grading $\mathfrak{sl}_2(\mathbb{F}) = \mathrm{Span}\left\{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\right\} \oplus \mathrm{Span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right\}$, which is not isomorphic to the above.

In general, it is hard to determine which of the induced gradings are isomorphic to each other.

A structure theorem

$\mathcal{D}$ graded division algebra: all nonzero homogeneous elements are invertible ($\Rightarrow$ graded $\mathcal{D}$-modules have a graded basis).

Theorem ("Graded Wedderburn Theorem")

Let $\mathcal{R}$ be a G-graded algebra (or ring). Then $\mathcal{R}$ is graded simple and satisfies d.c.c. on graded one-sided ideals $\Leftrightarrow$ there exists a G-graded division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$ of finite rank such that $\mathcal{R} \cong \mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as a graded algebra.

$\mathcal{D}$ and $\mathcal{V}$ are determined up to isomorphism and shift of grading.

If $\mathcal{R} = M_n(\mathbb{F})$ with a G-grading then $\mathcal{D} \cong M_\ell(\mathbb{F})$ with a *division grading* and $\mathcal{R} \cong M_k(\mathcal{D}) \cong M_k(\mathbb{F}) \otimes \mathcal{D}$ with a grading determined by a k-tuple $\gamma = (g_1, \ldots, g_k)$ of elements of G, $k\ell = n$, as follows: $\deg(E_{ij} \otimes d) = g_i(\deg d)g_j^{-1}$.

Let $T = \mathrm{Supp}\,\mathcal{D}$ (a subgroup of G) and define the *multiplicity function* $\kappa : G/T \to \mathbb{Z}_{\geq 0}$ by $\kappa(x) = |\{i \mid g_i T = x\}|$.

Generalized Pauli matrices

A G-grading on $\mathcal{D} = M_\ell(\mathbb{F})$ is a *division grading* if it makes $\mathcal{D}$ a graded division algebra ($\Rightarrow \operatorname{Supp}\mathcal{D}$ is a subgroup).

Example

$$X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \varepsilon^{\ell-1} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where $\varepsilon \in \mathbb{F}$ is a primitive ℓ-th root of unity. Then the following is a division grading by $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$: $\mathcal{D}_{(i,j)} = \mathbb{F}X^iY^j$.

Theorem (HPP98, BSZ01 for $\operatorname{char}\mathbb{F} = 0$; BZ03 for $\operatorname{char}\mathbb{F} > 0$)

Let T be an ab. group and $\mathbb{F}$ an a.c. field. Then for any division grading on $\mathcal{D} = M_\ell(\mathbb{F})$ with support T, there exists a decomposition $T = H_1 \times \cdots \times H_r$ such that $H_i \cong \mathbb{Z}_{\ell_i}^2$ and $\mathcal{D} \cong M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ with $M_{\ell_i}(\mathbb{F})$ graded as above.

Classification of abelian group gradings on $M_n(\mathbb{F})$

Suppose $\mathcal{D} = M_\ell(\mathbb{F})$, $\mathbb{F}$ a.c. field, has a division grading with support $T \subset G$. Then, for each $t \in T$, $\mathcal{D}_t = \mathbb{F}X_t$ and hence

$$X_s X_t = \sigma(s,t) X_{st}$$

for some 2-cocycle $\sigma\colon T \times T \to \mathbb{F}^\times$, i.e., $\mathcal{D}$ is isomorphic to a *twisted group algebra* $\mathbb{F}^\sigma T$, with its natural T-grading regarded as a G-grading. Set $\beta(s,t) = \sigma(s,t)/\sigma(t,s)$.

If G is abelian, then the isomorphism classes of division G-gradings on $M_\ell(\mathbb{F})$ are in bijection with the pairs (T,β) where $T \subset G$ is a subgroup of order ℓ^2 and $\beta\colon T \times T \to \mathbb{F}^\times$ is a nondegenerate alternating bicharacter ($\Rightarrow \operatorname{char}\mathbb{F} \nmid \ell$).

Corollary (Bahturin–K, 2010)

The isomorphism classes of G-gradings on $M_n(\mathbb{F})$ are parametrized by (T,β,κ) where $\kappa : G/T \to \mathbb{Z}_{\geq 0}$, $|\kappa|\sqrt{|T|} = n$.

The algebra of upper block-triangular matrices

Let V be a f.-d. vector space over $\mathbb{F}$ and let $\mathscr{F}$ be a flag:

$$0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_s = V.$$

Let $n = \dim V$ and $n_i = \dim V_i/V_{i-1}$, for $i = 1, 2, \ldots, s$.

Denote by $U(\mathscr{F})$ the subalgebra of $\mathrm{End}(V)$ consisting of all endomorphisms preserving $\mathscr{F}$. Fixing a basis of V adapted to the flag, we can identify $U(\mathscr{F}) = UT(n_1, \ldots, n_s) \subset M_n(\mathbb{F})$.

Theorem (Valenti–Zaicev, 2012)

Let G be a finite ab. group and $\mathbb{F}$ be an a.c. field, $\mathrm{char}\,\mathbb{F} = 0$. For any G-grading on $UT(n_1, \ldots, n_s)$, there exists an integer ℓ dividing each n_i such that $UT(n_1, \ldots, n_s) \cong UT(k_1, \ldots, k_s) \otimes \mathcal{D}$, as graded algebras, where $k_i\ell = n_i$, $\mathcal{D} \cong M_\ell(\mathbb{F})$ has a division grading, and the grading on $UT(k_1, \ldots, k_s)$ is determined by a k-tuple $(g_1, \ldots, g_k) \in G^k$, $k\ell = n$, as follows: $\deg E_{ij} = g_i g_j^{-1}$.

Admissible $G^{\#}$-gradings on $M_n(\mathbb{F})$

Consider the $\mathbb{Z}$-grading $M_n(\mathbb{F}) = \bigoplus_{m\in\mathbb{Z}} J_m$ defined by n-tuple

$$(\underbrace{-1,\ldots,-1}_{n_1 \text{ times}},\underbrace{-2,\ldots,-2}_{n_2 \text{ times}},\ldots,\underbrace{-s,\ldots,-s}_{n_s \text{ times}}).$$

We have $UT(n_1,\ldots,n_s) = \bigoplus_{m\geq 0} J_m$, and we call this the *natural* $\mathbb{Z}$*-grading* of $UT(n_1,\ldots,n_s)$. The associated filtration consists of the powers of the Jacobson radical J of $UT(n_1,\ldots,n_s)$, i.e., $\bigoplus_{i\geq m} J_i = J^m$ for all $m \geq 0$.

For an abelian group G, denote $G^{\#} = \mathbb{Z}\times G$. We identify G with the subgroup $\{0\}\times G \subset G^{\#}$ and $\mathbb{Z}$ with $\mathbb{Z}\times\{1_G\}\subset G^{\#}$.

Definition

A $G^{\#}$-grading on $M_n(\mathbb{F})$ is *admissible* if $UT(n_1,\ldots,n_s)$ with its natural $\mathbb{Z}$-grading is a graded subalgebra of $M_n(\mathbb{F})$, where $M_n(\mathbb{F})$ is $\mathbb{Z}$-graded via the projection $G^{\#}\to\mathbb{Z}$. An isomorphism class is *admissible* if it contains an admissible grading.

Classification of G-gradings on $UT(n_1, \ldots, n_s)$

For any admissible $G^\#$-grading on $M_n(\mathbb{F})$, the $\mathbb{Z}$-grading induced by the projection $G^\# \to \mathbb{Z}$ has J_m as its homogeneous component of degree m.

Hence, every admissible $G^\#$-grading $M_n(\mathbb{F}) = \bigoplus_{(m,g)\in G^\#} A_{(m,g)}$ restricts to a $G^\#$-grading on $UT(n_1, \ldots, n_s)$, hence the projection $G^\# \to G$ induces a G-grading on $UT(n_1, \ldots, n_s)$, namely, $UT(n_1, \ldots, n_s) = \bigoplus_{g\in G} B_g$ where $B_g = \bigoplus_{m\geq 0} A_{(m,g)}$.

Theorem (K–Yasumura, 2018)

Let G be an abelian group and $\mathbb{F}$ be an a.c. field. Then the mapping of an admissible $G^\#$-grading on $M_n(\mathbb{F})$ to a G-grading on $UT(n_1, \ldots, n_s)$, given by restriction and coarsening, yields a bijection between the admissible isomorphism classes of $G^\#$-gradings on $M_n(\mathbb{F})$ and the isomorphism classes of G-gradings on $UT(n_1, \ldots, n_s)$.

Classification of G-gradings on $UT(n_1, \ldots, n_s)$ cont'd

The isom. class of $G^{\#}$-gradings on $M_n(\mathbb{F})$ with parameters (T, β, κ), $T \subset G$, $\beta : T \times T \to \mathbb{F}^{\times}$, $\kappa : G^{\#}/T = \mathbb{Z} \times G/T \to \mathbb{Z}_{\geq 0}$, is admissible if and only if there exist $a \in \mathbb{Z}$ and $\kappa_1, \ldots, \kappa_s : G/T \to \mathbb{Z}_{\geq 0}$ with $|\kappa_i|\sqrt{|T|} = n_i$ such that $\kappa(a - i, x) = \kappa_i(x)$ for all $i \in \{1, 2, \ldots, s\}$, $x \in G/T$, and $\kappa(a - i, x) = 0$ if $i \notin \{1, 2, \ldots, s\}$.

Corollary

The isomorphism classes of G-gradings on $UT(n_1, \ldots, n_s)$ are parametrized by the triples $(T, \beta, (\kappa_1, \ldots, \kappa_s))$ as above, with $(T, \beta, (\kappa_1, \ldots, \kappa_s))$ and $(T', \beta', (\kappa'_1, \ldots, \kappa'_s))$ corresponding to the same class iff $T' = T$, $\beta' = \beta$ and $\exists g \in G$ $\kappa'_i = g\kappa_i$ for all i.

Example (Valenti-Zaicev'07; Di Vincenzo-Koshlukov-Valenti'04)

The isom. classes of G-gradings on $UT_n(\mathbb{F})$ are parametrized by n-tuples $(g_1, \ldots, g_n) \in G^n$, determined up to shift.

Reduction modulo the center

We want to classify G-gradings on the Lie algebra $UT(n_1, \ldots, n_s)^{(-)}$, which has a nontrivial center, $\mathbb{F}1$.

Theorem (K–Yasumura, 2018)

Let G be a group and let Γ_1 and Γ_2 be two G-gradings on a Lie algebra L over an arbitrary field. If Γ_1 and Γ_2 restrict to the same grading on $\mathfrak{z}(L)$ and induce the same grading on $L/\mathfrak{z}(L)$, then Γ_1 and Γ_2 are isomorphic.

Corollary

Assume $\operatorname{char}\mathbb{F} = 0$. Two G-gradings on $UT(n_1, \ldots, n_s)^{(-)}$ are isomorphic iff they assign the same degree to the identity matrix and induce isomorphic gradings on $UT(n_1, \ldots, n_s)^{(-)}/\mathbb{F}1$.

Thus, the classification problem reduces to the Lie algebra $UT(n_1, \ldots, n_s)_0$ of zero-trace upper block-triangular matrices.

Classification of G-gradings on $UT(n_1, \ldots, n_s)_0$

Theorem (K–Yasumura, 2018)

Assume $\operatorname{char}\mathbb{F} = 0$. *The support of any group grading on* $UT(n_1, \ldots, n_s)_0$ *generates an abelian subgroup.*

Thus, without loss of generality, we may assume that G is abelian. Recall $G^\# = \mathbb{Z} \times G$.

Theorem (K–Yasumura, 2018)

Let G be an abelian group and $\mathbb{F}$ be an a.c. field, $\operatorname{char}\mathbb{F} = 0$. *Then the mapping of an admissible $G^\#$-grading on $\mathfrak{sl}_n(\mathbb{F})$ to a G-grading on $UT(n_1, \ldots, n_s)_0$, given by restriction and coarsening, yields a bijection between the admissible isomorphism classes of $G^\#$-gradings on $\mathfrak{sl}_n(\mathbb{F})$ and the isomorphism classes of G-gradings on $UT(n_1, \ldots, n_s)_0$.*

Group gradings on $\mathfrak{sl}_n(\mathbb{F})$ were classified up to isomorphism in [BK10]; a better parametrization is given in [BKR18].

Classification of G-gradings on $UT(n_1, \dots, n_s)_0$ cont'd

Type I gradings are obtained by restriction from the associative algebra $UT(n_1, \dots, n_s)$. Other gradings, called Type II, occur only if $n > 2$ and $n_i = n_{s-i+1}$ for all i.

Corollary

The isomorphism classes of Type I gradings are parametrized by $(T, \beta, (\kappa_1, \dots, \kappa_s))$, where $\beta : T \times T \to \mathbb{F}^\times$ is nondegenerate and $\kappa_i : G/T \to \mathbb{Z}_{\geq 0}$ satisfy $|\kappa_i| = n_i\sqrt{|T|}$ for all i.
The isomorphism classes of Type II gradings are parametrized by $(T, \beta, g_0, (\kappa_1, \dots, \kappa_s))$, where $g_0 \in G$, T is 2-elementary, $\beta : T \times T \to \mathbb{F}^\times$ has a radical of size 2, and $\kappa_i : G/T \to \mathbb{Z}_{\geq 0}$ satisfy $|\kappa_i|\sqrt{|T|/2} = n_i$ for all i and two more conditions.

Example (Koshlukov–Yasumura, 2017)

Parameters of Type II gradings on $UT_n(\mathbb{F})_0$ are $f \in G$ of order 2 and $(g_1, \dots, g_n) \in G^n$ such that $g_i g_{n-i+1}$ does not depend on i.

Classification of G-gradings on $UT(n_1,\dots,n_s)^{(+)}$

If φ is an anti-automorphism of an associative algebra R, then φ is an automorphism of the Jordan algebra $R^{(+)}$ and $-\varphi$ is an automorphism of the Lie algebra $R^{(-)}$.

Every automorphism of $UT(n_1,\dots,n_s)_0$ is the restriction of an automorphism or minus anti-automorphism of $UT(n_1,\dots,n_s)$ [Marcoux–Sourour, 1999; Cecil, 2016], and every automorphism of $UT(n_1,\dots,n_s)^{(+)}$ is either an automorphism or anti-automorphism of $UT(n_1,\dots,n_s)$ [Boboc–Dăscălescu–van Wyk, 2016]. Hence:

Lemma

If $n>2$, $\mathrm{Aut}(UT(n_1,\dots,n_s)^{(+)})\cong\mathrm{Aut}(UT(n_1,\dots,n_s)_0)$.

For a.c. $\mathbb{F}$, $\operatorname{char}\mathbb{F}=0$, and any ab. group G, the G-gradings on $UT(n_1,\dots,n_s)^{(+)}$ are in bijection with those on $UT(n_1,\dots,n_s)_0$ (Type I: reduction mod $\mathbb{F}1$, Type II: reduction and shift by f).