

ON RIGIDITY OF NICHOLS ALGEBRAS

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ABSTRACT. We study deformations of graded braided bialgebras using cohomological methods. In particular, we show that many examples of Nichols algebras, including the finite-dimensional ones arising in the Andruskiewitsch-Schneider program of classification of pointed Hopf algebras, are rigid. This result can be regarded as nonexistence of “braided Lie algebras” with nontrivial bracket.

1. INTRODUCTION

Let \mathbb{k} be a field of characteristic 0 and V a \mathbb{k} -vector space. The symmetric algebra $S(V) = \bigoplus_{n \geq 0} S^n(V)$ is a graded bialgebra by declaring the elements of V *primitive*, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in V$, and extending to a morphism of (unital) algebras $\Delta: S(V) \rightarrow S(V) \otimes S(V)$. Then Lie brackets on V are in one-to-one correspondence with graded deformations of $S(V)$ as a bialgebra (or just as an augmented algebra).

We are interested in graded deformations of bialgebras generalizing $S(V)$, namely, the Nichols algebras of braided vector spaces, which have become prominent in the theory of Hopf algebras (see the survey [1] and references therein). Recall that a *braided vector space* is a vector space V equipped with a linear isomorphism $c: V \otimes V \rightarrow V \otimes V$ that satisfies the *braid equation*

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c),$$

where $\text{id} = \text{id}_V$. The *Nichols algebra* of (V, c) , denoted by $\mathcal{B}(V, c)$ or just $\mathcal{B}(V)$ if the braiding is clear from the context, is the unique (up to isomorphism) graded braided bialgebra $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}_n$ with $\mathcal{B}_0 = \mathbb{k}$, $\mathcal{B}_1 = V$ such that the restriction of the braiding of \mathcal{B} to V is c , \mathcal{B} is generated by V as an algebra, and V coincides with the space $P(\mathcal{B})$ of primitive elements of \mathcal{B} .

In the case of *symmetric* braiding, i.e., $c^2 = \text{id}$, the concept of braided Lie algebra is well understood [18, 8, 20, 23, 21]. This includes the usual Lie algebras (when c is the flip $v \otimes w \mapsto w \otimes v$), Lie superalgebras (when V is graded by \mathbb{Z}_2 and c is the signed flip $v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v$ where p denotes parity) and color Lie superalgebras. It follows from Kharchenko’s version of PBW Theorem [20, Theorem 7.1] that such Lie structures on (V, c) are in one-to-one correspondence with graded deformations of $\mathcal{B}(V, c)$ as a braided bialgebra with a fixed braiding (see Section 3).

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It is an important and difficult question for what finite-dimensional braided vector spaces the Nichols algebra is also finite-dimensional. This condition puts severe restrictions on c . For example, in the case of signed flip, this happens if and only if the even part of V is zero, in which case the Nichols algebra is the exterior algebra $\Lambda(V)$ and there are no nontrivial graded deformations.

We believe that such rigidity is typical for finite-dimensional Nichols algebras. We establish it for a wide class of symmetric braidings (Theorem 3.3) using the description of finite-dimensional triangular Hopf algebras by Etingof and Gelaki [11, 15, 12]. We also establish a sufficient condition of rigidity (Theorem 5.3) using cohomological techniques, and verify that it is satisfied for finite-dimensional Nichols algebras in the Yetter-Drinfeld category ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}\mathcal{YD}$ over an abelian group Γ (Theorem 6.3) using a description of these Nichols algebras in terms of generators and relations [4]. It follows that any finite-dimensional Nichols algebra arising from a diagonal braiding, i.e., a braiding of the form $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ where $\{x_1, \dots, x_\theta\}$ is a basis of V and $q_{ij} \in \mathbb{k}^\times$, does not admit nontrivial graded deformations (Theorem 6.4).

It should be mentioned that the so-called *bosonizations* of these Nichols algebras often admit nontrivial graded deformations (or “liftings”), as has been shown by Andruskiewitsch and Schneider in the course of their program of classification of pointed Hopf algebras [3].

Our sufficient condition also applies to some interesting infinite-dimensional Nichols algebras (see Section 7) and other braided bialgebras close to Nichols algebras (Theorem 7.1). This may explain the difficulty of constructing new examples in [7], where an attempt is made to define and study braided Lie algebras for non-symmetric braiding.

2. PRELIMINARIES

2.1. Braided tensor categories. It is often more convenient to work in a category rather than with a stand-alone braided vector space. By a *tensor category* we always mean a strict monoidal \mathbb{k} -linear category, see e.g. [24] for details. We are mostly interested in categories of \mathbb{k} -vector spaces endowed with some additional structure. To simplify notation, we omit associativity isomorphisms and parentheses in tensor products. In particular, we denote the tensor powers of an object V by $V^{\otimes n}$ for all $n \geq 0$, where $V^{\otimes 0}$ is the unit object.

A *braided tensor category* is a tensor category \mathcal{V} with a *braiding*, i.e. a natural family of isomorphisms $c_{V,W}: V \otimes W \rightarrow W \otimes V$ in \mathcal{V} satisfying the so-called hexagon axioms:

$$c_{U,V \otimes W} = (\text{id}_V \otimes c_{U,W})(c_{U,V} \otimes \text{id}_W) \quad \text{and} \quad c_{U \otimes V, W} = (c_{U,W} \otimes \text{id}_V)(\text{id}_U \otimes c_{V,W}),$$

for all U, V, W in \mathcal{V} . The braid equation follows:

$$(c_{V,W} \otimes \text{id}_U)(\text{id}_V \otimes c_{U,W})(c_{U,V} \otimes \text{id}_W) = (\text{id}_W \otimes c_{U,V})(c_{U,W} \otimes \text{id}_V)(\text{id}_U \otimes c_{V,W}).$$

The category is said to be *symmetric* if $c_{W,V}c_{V,W} = \text{id}_{V \otimes W}$ for all V, W in \mathcal{V} .

The most well known braided tensor categories are the category of (co)modules over a (co)quasitriangular bialgebra and the category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode. We will now briefly recall the relevant definitions and fix notation; details can be found in textbooks such as [27, 22]. We use the standard Sweedler notation for coalgebras and comodules.

A *coquasitriangular (CQT) bialgebra* is a pair (H, β) where H is a bialgebra and β is a bilinear form $H \times H \rightarrow \mathbb{k}$ that is invertible with respect to convolution and satisfies

$$\begin{aligned}\beta(h_{(1)}, k_{(1)})h_{(2)}k_{(2)} &= \beta(h_{(2)}, k_{(2)})k_{(1)}h_{(1)}, \\ \beta(hk, \ell) &= \beta(h, \ell_{(1)})\beta(k, \ell_{(2)}), \\ \beta(\ell, hk) &= \beta(\ell_{(2)}, h)\beta(\ell_{(1)}, k),\end{aligned}$$

for all $h, k, \ell \in H$. The category of right comodules \mathcal{M}^H is braided as follows:

$$(1) \quad c_{V,W}(v \otimes w) = \beta(v_{(1)}, w_{(1)})w_{(0)} \otimes v_{(0)}, \quad \text{for all } v \in V, w \in W.$$

Similarly, the category of left comodules ${}^H\mathcal{M}$ is braided by

$$c_{V,W}(v \otimes w) = \beta(w_{(-1)}, v_{(-1)})w_{(0)} \otimes v_{(0)}, \quad \text{for all } v \in V, w \in W.$$

If G is a group then the Hopf algebra $H = \mathbb{k}G$ admits a CQT structure β if and only if G is abelian. In this case the possible maps β are just linear extensions of bicharacters $G \times G \rightarrow \mathbb{k}^\times$. Right H -comodules are just G -graded vector spaces, $V = \bigoplus_{g \in G} V_g$, and the braiding is given by $v \otimes w \mapsto \beta(g, h)w \otimes v$ for all $v \in V_g, w \in W_h, g, h \in G$.

An object V of the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ is simultaneously a left module and a left comodule such that the following compatibility condition holds:

$$h_{(1)}v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} = (h_{(1)} \cdot v)_{(-1)}h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)} \quad \text{for all } v \in V, h \in H.$$

A morphism is a linear map preserving both action and coaction. The braiding is given by

$$c_{V,W}: v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)}.$$

The category of right Yetter-Drinfeld modules \mathcal{YD}_H^H is defined in a similar manner. If Γ is a group and $H = \mathbb{k}\Gamma$ then an object in ${}^H_H\mathcal{YD}$ is just a Γ -graded vector space with a left action of Γ such that $g \cdot V_h = V_{ghg^{-1}}$, for all $g, h \in \Gamma$. The braiding is given by $v \otimes w \mapsto g \cdot w \otimes v$, for all $v \in V_g, w \in W$. In particular, if Γ is abelian then the semisimple objects in \mathcal{YD}_H^H are vector spaces graded by the direct product $\Gamma \times \widehat{\Gamma}$ where $\widehat{\Gamma}$ is the character group of Γ . For a vector space V with such a grading, we will denote the homogeneous component of degree (g, χ) by V_g^χ . The braiding becomes $v \otimes w \mapsto \psi(g)w \otimes v$, for all $v \in V_g^\chi$ and $w \in W_h^\psi$.

If a CQT bialgebra (H, β) is a Hopf algebra then its antipode is bijective. Moreover \mathcal{M}^H can be regarded as a full subcategory of the Yetter-Drinfeld category \mathcal{YD}_H^H if we define the right action of H on a right comodule V by means of the usual left action of H^* and the homomorphism of algebras $H^{\text{op}} \rightarrow H^*: h \mapsto \beta(\cdot, h)$, i.e., $v \cdot h = \sum \beta(v_{(1)}, h)v_{(0)}$, for all $v \in V, h \in H$. Similarly, ${}^H\mathcal{M}$ can be regarded as a full subcategory of ${}^H_H\mathcal{YD}$.

If (U, c) is a finite-dimensional braided vector space then the FRT construction [22, 29] yields a CQT bialgebra (H, β) such that $U \in \mathcal{M}^H$ and $c = c_{U,U}$ where $c_{U,U}$ is given by (1). Moreover, for any $V, W \in \mathcal{M}^H$ and a linear map $f: V \rightarrow W$ that *commutes with the braiding with U* in the sense that $(f \otimes \text{id})c_{U,V} = c_{U,W}(\text{id} \otimes f)$ and $(\text{id} \otimes f)c_{V,U} = c_{W,U}(f \otimes \text{id})$, there exists a biideal I of H contained in the left and right kernels of the bilinear form β such that f is a morphism in $\mathcal{M}^{H/I}$ [29, Corollary 1.9]. Hence, replacing (H, β) by $(\bar{H}, \bar{\beta})$, where \bar{H} is the quotient of H by the largest biideal contained in the left and right kernels of β and where $\bar{\beta}$ is induced by β , we obtain a braided category, $\mathcal{M}^{\bar{H}}$, that contains (U, c) and all linear maps that commute with the braiding with U .

There is a Hopf algebra version of the above construction — see e.g. [29] and references therein — for braided vector spaces satisfying a certain condition, called *rigidity* in [29], which allows us to define the braiding operators c_{U,U^*} , $c_{U^*,U}$ and c_{U^*,U^*} , where U^* is the dual space. Namely, there exists a CQT Hopf algebra (H, β) such that $U \in \mathcal{M}^H$ and $c = c_{U,U}$. Again, any linear map that commutes with the braiding with U can be included in the category $\mathcal{M}^{H/I}$ where I is a Hopf ideal contained in the left and right kernels of β , see the proof of [29, Proposition 5.4]. Since the largest biideal contained in the kernels of β is automatically a Hopf ideal, we obtain a CQT Hopf algebra \bar{H} such that $\mathcal{M}^{\bar{H}}$ includes (U, c) and all linear maps that commute with the braiding with U .

We are especially interested in the case of diagonal braiding: $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ where $\{x_1, \dots, x_\theta\}$ is a basis of U and $q_{ij} \in \mathbb{k}^\times$. Here we can take $H = \mathbb{k}G$, where G is the free abelian group \mathbb{Z}^θ , and define the bicharacter β by setting $\beta(e_i, e_j) = q_{ij}$, where $\{e_1, \dots, e_\theta\}$ is the standard basis of \mathbb{Z}^θ . If we make U a G -graded vector space by declaring $x_i \in U_{e_i}$ then we get $c = c_{U,U}$ in \mathcal{M}^H . Alternatively, we can make U an object of ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma} \mathcal{YD}$ for each abelian group Γ containing elements g_1, \dots, g_θ such that there exist characters $\chi_1, \dots, \chi_\theta \in \hat{\Gamma}$ satisfying $\chi_j(g_i) = q_{ij}$; then we declare $x_i \in U_{g_i}^{\chi_i}$ and get $c = c_{U,U}$ in ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma} \mathcal{YD}$. We can choose the group Γ so that it is generated by g_1, \dots, g_θ and the characters $\chi_1, \dots, \chi_\theta$ separate points of Γ . It is easy to see that in this case a linear map $f: V \rightarrow W$ commutes with the braiding with U if and only if f is a morphism in ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma} \mathcal{YD}$.

2.2. Braided bialgebras. A *bialgebra* in a braided tensor category \mathcal{V} with unit object $\mathbb{1}$ is an object \mathcal{B} with four morphisms: multiplication $m: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$, unit $u: \mathbb{1} \rightarrow \mathcal{B}$, comultiplication $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ and counit $\varepsilon: \mathcal{B} \rightarrow \mathbb{1}$ such that (\mathcal{B}, m, u) is a unital algebra, $(\mathcal{B}, \Delta, \varepsilon)$ is a counital coalgebra, and the following compatibility conditions hold:

$$\Delta m = (m \otimes m)(\text{id}_{\mathcal{B}} \otimes c_{\mathcal{B}, \mathcal{B}} \otimes \text{id}_{\mathcal{B}})(\Delta \otimes \Delta), \quad \varepsilon u = \text{id}_{\mathbb{1}}, \quad \varepsilon m = \varepsilon \otimes \varepsilon, \quad \Delta u = u \otimes u.$$

Note that the braiding appears only in the compatibility condition involving m and Δ .

One can define a *braided bialgebra* without reference to any categories [29]: it is a braided vector space (\mathcal{B}, c) with four linear maps, $m: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$, $u: \mathbb{k} \rightarrow \mathcal{B}$, $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ and $\varepsilon: \mathcal{B} \rightarrow \mathbb{k}$, that commute with the braiding induced by c among the tensor powers of \mathcal{B} and satisfy the following conditions: (\mathcal{B}, m, u) is a unital algebra, $(\mathcal{B}, \Delta, \varepsilon)$ is a counital coalgebra, u is a counital coalgebra map, ε is a unital algebra map, and finally $\Delta m = (m \otimes m)(\text{id}_{\mathcal{B}} \otimes c \otimes \text{id}_{\mathcal{B}})(\Delta \otimes \Delta)$.

Obviously, a bialgebra \mathcal{B} in a braided tensor category consisting of vector spaces and linear maps (such as \mathcal{M}^H or ${}^H_H \mathcal{YD}$) satisfies the definition of braided bialgebra with $c = c_{\mathcal{B}, \mathcal{B}}$. Conversely, it is shown in [29] that any finite-dimensional braided bialgebra $(\mathcal{B}, m, u, \Delta, \varepsilon, c)$ can be included in the category \mathcal{M}^H over a suitable CQT bialgebra (Hopf algebra if c is rigid) H such that $m, u, \Delta, \varepsilon$ are morphisms in \mathcal{M}^H and $c = c_{\mathcal{B}, \mathcal{B}}$ in \mathcal{M}^H .

We are mainly interested in the case of the Nichols algebra $\mathcal{B}(V)$ of a finite-dimensional vector space V with a rigid braiding c , which is a braided Hopf algebra, not necessarily finite-dimensional but equipped with a grading over non-negative integers whose components are finite-dimensional. It can be constructed as the quotient of the tensor algebra $T(V)$ by a graded biideal $\mathcal{I}(V)$ [1, Proposition 2.2], which is determined by the braiding c ; indeed the homogeneous components of $\mathcal{I}(V)$ are the kernels of the so-called *quantum symmetrizers* on the tensor powers of V [1, Proposition 2.11]. This construction can be carried out

either with the stand-alone braided vector space (V, c) or in a suitable braided category of comodules or Yetter-Drinfeld modules.

2.3. Graded deformations and liftings. We review the theory of formal graded deformations and liftings from [25], but in a slightly more general setting. The theory of formal bialgebra deformations was introduced by Gerstenhaber and Schack [16], while the graded version and its connection to liftings was considered by Du, Chen and Ye [10]. In this context, a *graded bialgebra* will mean a bialgebra \mathcal{B} in a braided tensor category \mathcal{V} (consisting of vector spaces and linear maps) equipped with a grading, as an object in \mathcal{V} , over non-negative integers, $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}_n$, which is at the same time an algebra and a coalgebra grading, i.e., $\mathcal{B}_i \mathcal{B}_j \subseteq \mathcal{B}_{i+j}$ and $\Delta(\mathcal{B}_k) \subseteq \bigoplus_{i+j=k} \mathcal{B}_i \otimes \mathcal{B}_j$, for all $i, j, k \geq 0$.

Let t be an indeterminate and consider the polynomial algebra $\mathbb{k}[t]$ equipped with its standard grading, i.e., t has degree 1. By extending scalars from \mathbb{k} to $\mathbb{k}[t]$, the braided tensor category \mathcal{V} gives rise to the braided tensor category $\mathcal{V}_{\mathbb{k}[t]}$. A (formal) *graded deformation* of a graded bialgebra (\mathcal{B}, m, Δ) in \mathcal{V} is a $\mathbb{k}[t]$ -linear graded structure (m_t, Δ_t) on $\mathcal{B}[t] = \mathcal{B} \otimes \mathbb{k}[t]$ such that $(\mathcal{B}[t], m_t, \Delta_t)$ is a graded bialgebra in $\mathcal{V}_{\mathbb{k}[t]}$.

We say that two graded deformations, $(\mathcal{B}[t], m_t, \Delta_t)$ and $(\mathcal{B}[t], m'_t, \Delta'_t)$, are *equivalent* if there exists a $\mathbb{k}[t]$ -linear graded bialgebra isomorphism $f: (\mathcal{B}[t], m_t, \Delta_t) \rightarrow (\mathcal{B}[t], m'_t, \Delta'_t)$.

A *lifting* (\mathcal{U}, π) of \mathcal{B} consists of a filtered bialgebra \mathcal{U} and a filtered vector space isomorphism $\pi: \mathcal{U} \rightarrow \mathcal{B}$ such that $\text{gr } \pi: \text{gr } \mathcal{U} \rightarrow \text{gr } \mathcal{B} = \mathcal{B}$ is an isomorphism of graded bialgebras. An *equivalence* between liftings (\mathcal{U}, π) and (\mathcal{U}', π') is a filtered bialgebra isomorphism $f: \mathcal{U} \rightarrow \mathcal{U}'$ such that $\text{gr } \pi \circ \text{gr } f = \text{gr } \pi'$.

A graded deformation is given by a sequence of pairs of maps (m_i, Δ_i) , $i \geq 0$, of degree $-i$ such that $m_t|_{\mathcal{B} \otimes \mathcal{B}} = m + \sum_{i \geq 1} m_i t^i$ and $\Delta_t|_{\mathcal{B}} = \Delta + \sum_{i \geq 1} \Delta_i t^i$. We also denote $(m_0, \Delta_0) = (m, \Delta)$. A graded deformation $(\mathcal{B}[t], m_t, \Delta_t)$ defines a lifting (\mathcal{U}, π) , where \mathcal{U} is \mathcal{B} as a filtered vector space, π is identity, and $(m_{\mathcal{U}}, \Delta_{\mathcal{U}}) = (m_t, \Delta_t)|_{t=1}$.

If (\mathcal{U}, π) is a lifting, then the linear maps $\tilde{m}: \mathcal{B} \otimes \mathcal{B} \xrightarrow{\pi^{-1} \otimes \pi^{-1}} \mathcal{U} \otimes \mathcal{U} \xrightarrow{m_{\mathcal{U}}} \mathcal{U} \xrightarrow{\pi} \mathcal{B}$ and $\tilde{\Delta}: \mathcal{B} \xrightarrow{\pi^{-1}} \mathcal{U} \xrightarrow{\Delta_{\mathcal{U}}} \mathcal{U} \otimes \mathcal{U} \xrightarrow{\pi \otimes \pi} \mathcal{B} \otimes \mathcal{B}$ decompose into direct sums of homogeneous components m_i, Δ_i of degrees $-i$ for $i \geq 0$, and the structure maps $(m_t, \Delta_t) = (\sum_i m_i t^i, \sum_i \Delta_i t^i)$ on $\mathcal{B}[t]$ define a formal graded deformation of \mathcal{B} .

Up to equivalence, these correspondences are inverses of each other.

2.4. Graded bialgebra cohomology. Let \mathcal{B} be a bialgebra in \mathcal{V} . Consider the bisimplicial complex $\mathbf{B} = (\mathbf{B}^{p,q})_{p,q \geq 0}$,

$$\mathbf{B}^{p,q} = \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q}).$$

The left and right diagonal actions and coactions of \mathcal{B} on $\mathcal{B}^{\otimes n}$ will be denoted by $\lambda_l, \lambda_r, \rho_l, \rho_r$, respectively. Note that they involve the braiding. The horizontal faces

$$\partial_i^h: \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q}) \rightarrow \text{Hom}(\mathcal{B}^{\otimes(p+1)}, \mathcal{B}^{\otimes q})$$

and degeneracies

$$\sigma_i^h: \text{Hom}(\mathcal{B}^{\otimes(p+1)}, \mathcal{B}^{\otimes q}) \rightarrow \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q})$$

are those for computing Hochschild cohomology:

$$\begin{aligned}\partial_0^h f &= \lambda_l(\text{id} \otimes f), \\ \partial_i^h f &= f(\text{id} \otimes \dots \otimes m \otimes \dots \otimes \text{id}), \quad 1 \leq i \leq p, \\ \partial_{p+1}^h f &= \lambda_r(f \otimes \text{id}), \\ \sigma_i^h f &= f(\text{id} \otimes \dots \otimes u \otimes \dots \otimes \text{id});\end{aligned}$$

the vertical faces

$$\partial_j^c: \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q}) \rightarrow \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes(q+1)})$$

and degeneracies

$$\sigma_j^c: \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes(q+1)}) \rightarrow \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^q)$$

are those for computing coalgebra (Cartier) cohomology:

$$\begin{aligned}\partial_0^c f &= (\text{id} \otimes f)\rho_l, \\ \partial_j^c f &= (\text{id} \otimes \dots \otimes \Delta \otimes \dots \otimes \text{id})f, \quad 1 \leq j \leq q, \\ \partial_{q+1}^c f &= (f \otimes \text{id})\rho_r, \\ \sigma_i^c f &= (\text{id} \otimes \dots \otimes \varepsilon \otimes \dots \otimes \text{id})f.\end{aligned}$$

The vertical and horizontal differentials are given by the usual alternating sums

$$\partial^h = \sum (-1)^i \partial_i^h, \quad \partial^c = \sum (-1)^j \partial_j^c.$$

By abuse of notation we identify a cosimplicial bicomplex with its associated cochain bicomplex. The *bialgebra cohomology* of \mathcal{B} is then defined as

$$H_b^*(\mathcal{B}) = H^*(\text{Tot } \mathbf{B}).$$

where

$$\text{Tot } \mathbf{B} = \mathbf{B}^{0,0} \rightarrow \mathbf{B}^{1,0} \oplus \mathbf{B}^{0,1} \rightarrow \dots \rightarrow \bigoplus_{p+q=n} \mathbf{B}^{p,q} \xrightarrow{\partial^b} \dots$$

and ∂^b is given by the sign trick (i.e., $\partial^b|_{\mathbf{B}^{p,q}} = \partial^h \oplus (-1)^p \partial^c: \mathbf{B}^{p,q} \rightarrow \mathbf{B}^{p+1,q} \oplus \mathbf{B}^{p,q+1}$).

Let \mathbf{B}_0 denote the bicomplex obtained from \mathbf{B} by replacing the edges by zeroes, i.e., $\mathbf{B}_0^{p,0} = 0 = \mathbf{B}_0^{0,q}$ for all p, q . The *truncated bialgebra cohomology* is

$$\hat{H}_b^*(\mathcal{B}) = H^{*+1}(\text{Tot } \mathbf{B}_0).$$

For computations, it is convenient to use the normalized bicomplex \mathbf{B}^+ , which is obtained from the cochain bicomplex \mathbf{B} by replacing $\mathbf{B}^{p,q} = \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q})$ with the intersection of degeneracies

$$(\mathbf{B}^+)^{p,q} = (\cap \text{Ker } \sigma_i^h) \cap (\cap \text{Ker } \sigma_j^c) \simeq \text{Hom}((\mathcal{B}^+)^{\otimes p}, (\mathcal{B}^+)^{\otimes q}),$$

where $\mathcal{B}^+ = \ker(\varepsilon)$. This change does not affect the cohomology.

We can describe the first two cohomology groups as follows:

$$\hat{H}_b^1(\mathcal{B}) = \{f: \mathcal{B}^+ \rightarrow \mathcal{B}^+ \mid f(ab) = af(b) + f(a)b, \Delta f(a) = a_{(1)} \otimes f(a_{(2)}) + f(a_{(1)}) \otimes a_{(2)}\}$$

and

$$\hat{H}_b^2(\mathcal{B}) = \hat{Z}_b^2(\mathcal{B}) / \hat{B}_b^2(\mathcal{B}),$$

where

$$\begin{aligned}
(2) \quad & \widehat{Z}_b^2(\mathcal{B}) = \{(f, g) \mid f: \mathcal{B}^+ \otimes \mathcal{B}^+ \rightarrow \mathcal{B}^+, g: \mathcal{B}^+ \rightarrow \mathcal{B}^+ \otimes \mathcal{B}^+, \\
& af(b, c) + f(a, bc) = f(ab, c) + f(a, b)c, \\
(3) \quad & c_{(1)} \otimes g(c_{(2)}) + (\text{id} \otimes \Delta)g(c) = (\Delta \otimes \text{id})g(c) + g(c_{(1)}) \otimes c_{(2)}, \\
(4) \quad & (f \otimes m)\Delta(a \otimes b) - \Delta f(a, b) + (m \otimes f)\Delta(a \otimes b) = \\
& -(\Delta a)g(b) + g(ab) - g(a)(\Delta b)\}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{B}_b^2(\mathcal{B}) = \{(f, g) \mid \exists h: \mathcal{B}^+ \rightarrow \mathcal{B}^+, f(a, b) = ah(b) - h(ab) + h(a)b, \\
g(c) = -c_{(1)} \otimes h(c_{(2)}) + \Delta h(c) - h(c_{(1)}) \otimes c_{(2)}\},
\end{aligned}$$

where the elements a, b, c range over \mathcal{B}^+ . All maps above are assumed to be morphisms in \mathcal{V} . By $\Delta(a \otimes b)$ we mean the braided coproduct in $\mathcal{B} \otimes \mathcal{B}$, namely, $(\text{id} \otimes c_{\mathcal{B}, \mathcal{B}} \otimes \text{id})(a_{(1)} \otimes a_{(2)} \otimes b_{(1)} \otimes b_{(2)})$, and we write $f(-, -)$ instead of $f(- \otimes -)$. In the resulting deformation (see the next subsection), Equation (2) will correspond to associativity, Equation (3) to coassociativity and Equation (4) to compatibility.

Now assume that \mathcal{B} is \mathbb{Z} -graded and let \mathbf{B}_ℓ denote the subcomplex of \mathbf{B} consisting of homogeneous maps of degree ℓ , i.e.,

$$\mathbf{B}_\ell^{p,q} = \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q})_\ell = \{f: \mathcal{B}^{\otimes p} \rightarrow \mathcal{B}^{\otimes q} \mid f \text{ is homogeneous of degree } \ell\}.$$

Complexes $(\mathbf{B}_0)_\ell$, \mathbf{B}_ℓ^+ and $(\mathbf{B}_0^+)_\ell$ are defined analogously. The graded bialgebra and truncated graded bialgebra cohomologies are then defined by:

$$\begin{aligned}
H_b^*(\mathcal{B})_\ell &= H^*(\text{Tot } \mathbf{B}_\ell) = H^*(\text{Tot } \mathbf{B}_\ell^+), \\
\widehat{H}_b^*(\mathcal{B})_\ell &= H^{*+1}(\text{Tot}(\mathbf{B}_0)_\ell) = H^{*+1}(\text{Tot}(\mathbf{B}_0^+)_\ell).
\end{aligned}$$

Note that if the support of the grading is finite, in particular if \mathcal{B} is finite-dimensional, then

$$H_b^*(\mathcal{B}) = \bigoplus_{\ell \in \mathbb{Z}} H_b^*(\mathcal{B})_\ell \quad \text{and} \quad \widehat{H}_b^*(\mathcal{B}) = \bigoplus_{\ell \in \mathbb{Z}} \widehat{H}_b^*(\mathcal{B})_\ell.$$

2.5. Cohomological aspects of graded deformations. Given a graded deformation of \mathcal{B} , let r be the smallest positive integer for which $(m_r, \Delta_r) \neq (0, 0)$ (if such an r exists). Then (m_r, Δ_r) is a 2-cocycle in $\widehat{Z}_b^2(\mathcal{B})_{-r}$. Every nontrivial deformation is equivalent to one for which the corresponding (m_r, Δ_r) represents a nontrivial cohomology class [16, 10]. Hence, if $\widehat{H}_b^2(\mathcal{B})_{(\ell)} = 0$ for all $\ell < 0$, then \mathcal{B} is *rigid*, i.e., has no nontrivial graded deformations.

Conversely, given a positive integer r and a 2-cocycle (m', Δ') in $\widehat{Z}_b^2(\mathcal{B})_{-r}$, the maps $m + t^r m'$ and $\Delta + t^r \Delta'$ define a bialgebra structure on $\mathcal{B}[t]/(t^{r+1})$ over $\mathbb{k}[t]/(t^{r+1})$. There may or may not exist (m_{r+k}, Δ_{r+k}) , $k \geq 1$, for which $m_t = m + t^r m' + \sum_{k \geq 1} t^{r+k} m_{r+k}$ and $\Delta_t = \Delta + t^r \Delta' + \sum_{k \geq 1} t^{r+k} \Delta_{r+k}$ make $\mathcal{B}[t]$ into a bialgebra over $\mathbb{k}[t]$.

An r -deformation of \mathcal{B} is a graded deformation of \mathcal{B} over $\mathbb{k}[t]/(t^{r+1})$, i.e. a pair (m_t^r, Δ_t^r) defining a bialgebra structure on $\mathcal{B}[t]/(t^{r+1})$ over $\mathbb{k}[t]/(t^{r+1})$ such that $(m_t^r, \Delta_t^r)|_{t=0} = (m, \Delta)$. For any 2-cocycle (m', Δ') in $\widehat{Z}_b^2(\mathcal{B})_{-r}$, there exists an r -deformation, given by $(m + t^r m', \Delta + t^r \Delta')$.

If a given $(r-1)$ -deformation can be extended to an r -deformation, then all ways of doing so are parametrized by $\widehat{H}_b^2(\mathcal{B})_{-r}$. More precisely, suppose that $(\mathcal{B}[t]/(t^r), m_t^{r-1}, \Delta_t^{r-1})$ is an $(r-1)$ -deformation, where

$$m_t^{r-1} = m + tm_1 + \dots + t^{r-1}m_{r-1}, \quad \Delta_t^{r-1} = \Delta + t\Delta_1 + \dots + t^{r-1}\Delta_{r-1}.$$

If

$$D = (\mathcal{B}[t]/(t^{r+1}), m_t^{r-1} + t^r m_r, \Delta_t^{r-1} + t^r \Delta_r)$$

is an r -deformation, then

$$D' = (\mathcal{B}[t]/(t^{r+1}), m_t^{r-1} + t^r m'_r, \Delta_t^{r-1} + t^r \Delta'_r)$$

is an r -deformation if and only if $(m'_r - m_r, \Delta'_r - \Delta_r) \in \widehat{Z}_b^2(\mathcal{B})_{-r}$. Note also that if $(m'_r - m_r, \Delta'_r - \Delta_r) \in \widehat{B}_b^2(\mathcal{B})_{-r}$, then deformations D and D' are equivalent.

The obstruction to extend r -deformations to $(r+1)$ -deformations lies in $\widehat{H}_b^3(\mathcal{B})_{-r}$.

3. THE CASE OF SYMMETRIC BRAIDING

Let (V, c) be a braided vector space with $c^2 = \text{id}$. Then $\mathcal{B}(V)$ is a quadratic algebra: it is the quotient of $T(V)$ by the ideal generated by the elements $x \otimes y - c(x \otimes y)$, for $x, y \in V$. If c is the flip (respectively, signed flip) then $\mathcal{B}(V) = S(V)$ (respectively, $S(V_0) \otimes \Lambda(V_1)$) and the graded deformations of $\mathcal{B}(V)$ are in one-to-one correspondence with brackets $[\cdot, \cdot]: V \otimes V \rightarrow V$ making V a Lie algebra (respectively, superalgebra). For arbitrary c , we need the following generalization of Lie algebra introduced by Gurevich [18] under the name “Lie c -algebra”.

Definition 3.1. *Let L be a vector space, $c: L \otimes L \rightarrow L \otimes L$ a symmetric braiding, and $[\cdot, \cdot]: L \otimes L \rightarrow L$ a linear map. Then $(L, [\cdot, \cdot], c)$ is a braided Lie algebra if*

$$\begin{aligned} c([\cdot, \cdot] \otimes \text{id}_L) &= (\text{id}_L \otimes [\cdot, \cdot])(c \otimes \text{id}_L)(\text{id}_L \otimes c) && (\text{compatibility}), \\ [\cdot, \cdot](\text{id}_{L \otimes L} + c) &= 0 && (\text{anticommutativity}), \\ [\cdot, \cdot]([\cdot, \cdot] \otimes \text{id}_L) &\left(\text{id}_{L \otimes L \otimes L} + (c \otimes \text{id}_L)(\text{id}_L \otimes c) + (c \otimes \text{id}_L)(\text{id}_L \otimes c) \right) = 0 && (\text{Jacobi identity}). \end{aligned}$$

Note that the compatibility condition (together with $c^2 = \text{id}$) simply means that the bracket commutes with c , and the above Jacobi identity implies a similar identity for $[\cdot, \cdot](\text{id}_L \otimes [\cdot, \cdot])$ instead of $[\cdot, \cdot]([\cdot, \cdot] \otimes \text{id}_L)$. It is straightforward to check that if a vector space A is equipped with a symmetric braiding c and an associative product $m: A \otimes A \rightarrow A$ that commutes with c then $(A, [\cdot, \cdot]_c, c)$ is a braided Lie algebra, where $[\cdot, \cdot]_c$ is the *braided commutator* $m(\text{id}_{A \otimes A} - c)$.

Braided Lie algebras naturally arise as Lie algebras in a symmetric tensor category \mathcal{V} . A Lie algebra in \mathcal{V} is an object L endowed with a morphism $[\cdot, \cdot]: L \otimes L \rightarrow L$ such that the anticommutativity and Jacobi identity hold for $c = c_{L,L}$. If (H, β) is a *cotriangular* bialgebra (i.e., a CQT bialgebra satisfying $\beta^{-1}(h, k) = \beta(k, h)$ for all $h, k \in H$) then the category \mathcal{M}^H is symmetric; Lie algebras in this category were introduced and studied in [8, 9] under the name (H, β) -Lie algebras. By an argument similar to [29] (see Subsection 2.2 above), any finite-dimensional braided Lie algebra can be regarded as an (H, β) -Lie algebra for a suitable cotriangular bialgebra (Hopf algebra if the braiding is rigid).

Given a braided Lie algebra $(L, [\cdot, \cdot], c)$, the *universal enveloping algebra*, which we will denote $\mathcal{U}_c(L)$, is the quotient of the tensor algebra $T(L)$ by the ideal generated by the degree 2 elements $x \otimes y - c(x \otimes y) - [x, y]$ where $x, y \in L$. The usual increasing filtration of $T(L)$ gives rise to the *standard filtration* of $\mathcal{U}_c(L)$. As one would expect, $\mathcal{U}_c(L)$ becomes a braided bialgebra if we declare the elements of L primitive. It is not true in general that, given an ordered basis of L , the corresponding PBW monomials form a basis of $\mathcal{U}_c(L)$. However, the following version of PBW Theorem holds.

Theorem 3.2. [20, Theorem 7.1] *The graded algebra $\text{gr}\mathcal{U}_c(L)$ associated to the standard filtration of $\mathcal{U}_c(L)$ is naturally isomorphic to $\mathcal{U}_c(L^\circ)$ where L° denotes the braided Lie algebra with the same underlying braided vector space as L but with zero bracket.* \square

The standard filtration of $\mathcal{U}_c(L)$ coincides with its coradical filtration. Also $\mathcal{U}_c(L^\circ) = \mathcal{B}(L, c)$.

It follows that graded deformations of $\mathcal{B}(V, c)$ as a braided augmented algebra or as a braided bialgebra (with a fixed braiding) are in one-to-one correspondence with brackets on V making it a braided Lie algebra. Here the “graded deformations” and “braided Lie algebras” can be understood in the sense of a stand-alone object or an object in \mathcal{M}^H for a suitable cotriangular bialgebra (H, β) .

For $H = \mathbb{k}G$, where G is an abelian group, the cotriangular structures on H are linear extensions of skew-symmetric bicharacters $\beta: G \times G \rightarrow \mathbb{k}^\times$. In this case the (H, β) -Lie algebras are known as the *color Lie superalgebras with grading group G and commutation factor β* . Note that the braiding is diagonal and, conversely, any braided Lie algebra with a diagonal braiding can be regarded as a color Lie superalgebra for some G and β .

By a trick going back to Scheunert [28], color Lie superalgebras can be twisted to become ordinary Lie superalgebras. This procedure works in the same way for all color Lie superalgebras with given G and β , and is associated to a suitable cocycle twist of $(\mathbb{k}G, \beta)$ as a CQT bialgebra. Recall that a *right 2-cocycle* on a bialgebra H is a convolution-invertible map $\sigma: H \otimes H \rightarrow \mathbb{k}$ satisfying the following equations for all $h, k, \ell \in H$:

$$\sigma(h, k_{(1)}\ell_{(1)})\sigma(k_{(2)}, \ell_{(2)}) = \sigma(h_{(1)}k_{(1)}, \ell)\sigma(h_{(2)}, k_{(2)}), \quad \sigma(h, 1) = \sigma(1, h) = \varepsilon(h).$$

Also recall that if (H, β) is a cotriangular (more generally, CQT) bialgebra then (H_σ, β_σ) is again a cotriangular (respectively, CQT) bialgebra, see e.g. [22]; here $H_\sigma = H$ as a coalgebra, the multiplication of H_σ is given by

$$h \cdot_\sigma k = \sigma^{-1}(h_{(1)}, k_{(1)})h_{(2)}k_{(2)}\sigma(h_{(3)}, k_{(3)}),$$

and

$$\beta_\sigma(h, k) = \sigma^{-1}(k_{(1)}, h_{(1)})\beta(h_{(2)}k_{(2)})\sigma(h_{(3)}, k_{(3)}).$$

Moreover, σ yields an equivalence of braided tensor categories \mathcal{M}^H and \mathcal{M}^{H_σ} , which is the identity on objects and morphisms and only transforms the tensor product. If A is an algebra (not necessarily associative) in \mathcal{M}^H with multiplication $m: A \otimes A \rightarrow A$, then the corresponding algebra in \mathcal{M}^{H_σ} is A as an H -comodule but with new multiplication:

$$m_\sigma(a \otimes b) = \sigma(a_{(1)}, b_{(1)})m(a_{(0)} \otimes b_{(0)}).$$

We denote this new algebra by A_σ and call it the σ -twist of A . It is shown in [23] that multilinear polynomial identities of A are preserved under σ -twist if we interpret them in

each of the categories \mathcal{M}^H and \mathcal{M}^{H_σ} in terms of the appropriate action of symmetric groups on tensor powers of A . In particular, associative algebras remain associative and (H, β) -Lie algebras become (H_σ, β_σ) -Lie algebras.

If H is cocommutative then $H_\sigma = H$ but β is twisted. If $H = \mathbb{k}G$, with G an abelian group, then there exists a 2-cocycle $\sigma: G \times G \rightarrow \mathbb{k}^\times$ such that β_σ is a “sign bicharacter”:

$$\beta_\sigma(g, h) = \begin{cases} -1 & \text{if } g, h \in G_-, \\ 1 & \text{otherwise;} \end{cases}$$

where $G_- = G \setminus G_+$ and G_+ is a subgroup of index ≤ 2 . It follows that σ twists any color Lie superalgebra L with commutation factor β into a Lie superalgebra with even part L_+ and odd part L_- , where $L_\pm = \bigoplus_{g \in G_\pm} L_g$.

Etingof and Gelaki [11] showed that, under a certain condition on the antipode called *pseudo-involutivity*, a cotriangular Hopf algebra (H, β) can be twisted by a suitable cocycle to become the algebra of regular functions on a pro-algebraic group G such that $\beta_\sigma = \frac{1}{2}(\varepsilon \otimes \varepsilon + \varepsilon \otimes a + a \otimes \varepsilon - a \otimes a)$ for some central element $a \in G$ with $a^2 = 1$. It immediately follows [23, Theorem 4.3] that the same cocycle twists (H, β) -Lie algebras to Lie superalgebras equipped with a G -action. Here the even and odd components are just the eigenspaces with respect to the action of a , with eigenvalues 1 and -1 respectively.

If H is finite-dimensional then pseudo-involutivity of the antipode is equivalent to involutivity and hence to semisimplicity of H . Later, Etingof and Gelaki [12, 15] described all finite-dimensional cotriangular Hopf algebras by showing that (H, β) can be twisted in such a way that its dual triangular Hopf algebra becomes a “modified supergroup algebra”. As a corollary, any (H, β) -Lie algebra is twisted to a Lie superalgebra equipped with a supergroup action [23, Theorem 4.6].

One can use the twisting procedure to transfer known properties of Lie superalgebras to (H, β) -Lie algebras in the above cases. Let $\mathcal{U}_\beta(L)$ be the universal enveloping algebra of an (H, β) -Lie algebra L , i.e., $\mathcal{U}_c(L)$ for $c = c_{L,L}$ determined by β . It is straightforward to verify that $\mathcal{U}_{\beta_\sigma}(L_\sigma)$ is naturally isomorphic to $(\mathcal{U}_\beta(L))_\sigma$. In particular, for V in \mathcal{M}^H and $c = c_{V,V}$ induced by β , the σ -twist of the Nichols algebra $\mathcal{B}(V, c)$ is naturally isomorphic to $\mathcal{B}(V, c')$ where c' is the braiding on V induced by β_σ . This gives an alternative proof of PBW Theorem for (H, β) -Lie algebras [23].

Theorem 3.3. *Let (H, β) be a cotriangular Hopf algebra that is either pseudo-involutive or finite-dimensional. Let V be a finite-dimensional H -comodule with the corresponding braiding c . If the Nichols algebra $\mathcal{B}(V, c)$ is finite-dimensional then it does not admit nontrivial graded deformations as an augmented algebra or bialgebra in \mathcal{M}^H .*

Proof. By our assumption on (H, β) , there exists a cocycle σ such that (H_σ, β_σ) is as described by Etingof and Gelaki. Then the braiding c' induced by β_σ on V is just the signed flip associated to a \mathbb{Z}_2 -grading $V = V_0 \oplus V_1$, so $\mathcal{B}(V, c') = S(V_0) \otimes \Lambda(V_1)$, which is finite-dimensional only if $V_0 = 0$. But in this case V does not admit nontrivial Lie superalgebra structures. It follows that V does not admit nontrivial (H, β) -Lie algebra structures and hence $\mathcal{B}(V, c)$ is rigid in \mathcal{M}^H . \square

Corollary 3.4. *Let (V, c) be a finite-dimensional braided vector space such that c can be obtained from a coaction by a finite-dimensional cotriangular Hopf algebra. If $\mathcal{B}(V, c)$*

is finite-dimensional then it does not admit nontrivial graded deformations as a braided augmented algebra or bialgebra.

Proof. By assumption, V can be regarded as an object in \mathcal{M}^H for some finite-dimensional cotriangular Hopf algebra (H, β) such that $c = c_{V,V}$. Any graded deformation of $\mathcal{B}(V, c)$ can be realized in $\mathcal{M}^{\bar{H}}$ for some quotient $(\bar{H}, \bar{\beta})$ of the cotriangular Hopf algebra (H, β) , so it must be trivial by the above theorem. \square

4. THE VANISHING OF SECOND ALGEBRA COHOMOLOGY FOR A CLASS OF AUGMENTED ALGEBRAS IN A BRAIDED CATEGORY

Let \mathcal{V} be a braided tensor category consisting of vector spaces and linear maps. Let $(\mathcal{B}, \varepsilon)$ be an augmented algebra in \mathcal{V} acting trivially (i.e., via ε) on some U in \mathcal{V} .

◊ A map $f: \mathcal{B} \otimes \mathcal{B} \rightarrow U$ in \mathcal{V} is an ε -cocycle if $f(1, a) = 0 = f(a, 1)$ and $f(xy, z) = f(x, yz)$ for all $a \in \mathcal{B}$ and all $x, y, z \in \mathcal{B}^+$. The space of all ε -cocycles is denoted by $Z_\varepsilon^2(\mathcal{B}, U)$.

◊ An ε -cocycle is an ε -coboundary if there exists a map $t: \mathcal{B} \rightarrow U$ such that $t(1) = 0$ and $f(x, y) = t(xy)$ for all $x, y \in \mathcal{B}^+$. The space of all ε -coboundaries is denoted by $B_\varepsilon^2(\mathcal{B}, U)$.

◊ The quotient of ε -cocycles by ε -coboundaries is denoted by $H_\varepsilon^2(\mathcal{B}, U) = Z_\varepsilon^2(\mathcal{B}, U)/B_\varepsilon^2(\mathcal{B}, U)$.

In what follows $(\mathcal{B}^+)^2$ denotes the range of the multiplication $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{m} \mathcal{B}^+$, i.e., $(\mathcal{B}^+)^2 = \text{span}\{xy \mid x, y \in \mathcal{B}^+\}$.

Lemma 4.1 (cf. [25, Subsection 4.1]). *Let \mathcal{B} be an augmented algebra in \mathcal{V} and let $M = \ker(\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{m} \mathcal{B}^+)$. If the map $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{m} (\mathcal{B}^+)^2$ splits in \mathcal{V} , then for every space $U \in \mathcal{V}$, we have $H_\varepsilon^2(\mathcal{B}, U) = \text{Hom}(M, U)$.*

Proof. Let $\varphi: (\mathcal{B}^+)^2 \rightarrow \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+$ be a splitting of m and let $p: \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \rightarrow \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+$ be the canonical projection. We define a map $\Phi: \text{Hom}(M, U) \rightarrow H_\varepsilon^2(\mathcal{B}, U)$ as follows: if $f: M \rightarrow U$, then the cocycle $\Phi(f): \mathcal{B}^+ \otimes \mathcal{B}^+ \rightarrow U$ is $\Phi(f) = f(p - \varphi m)$. The inverse Ψ of Φ is defined as follows: if $g: \mathcal{B}^+ \otimes \mathcal{B}^+ \rightarrow U$ is a cocycle, then $\Psi(g): M \rightarrow U$ is the unique map such that $\Psi(g)p = g$. Now observe that maps Φ and Ψ are well defined: $\Phi(f)$ is always a cocycle and $\Psi(g) = 0$ whenever g is a coboundary. Note also that $\Psi\Phi = \text{id}$ and that the range of $\Phi\Psi - \text{id}$ consists of coboundaries. \square

Remark 4.2. *A splitting of $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{m} (\mathcal{B}^+)^2$ in \mathcal{V} automatically exists (it is usually not unique) if $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+$ is a semisimple object in \mathcal{V} . This happens whenever \mathcal{V} is either the category of Yetter-Drinfeld modules over a semisimple and cosemisimple Hopf algebra or the category of comodules over a cosemisimple CQT bialgebra. It also happens if \mathcal{V} is the category of Yetter-Drinfeld modules over $\mathbb{k}\Gamma$, where Γ is a possibly infinite abelian group, and \mathcal{B} is a direct sum of its one-dimensional subobjects in \mathcal{V} (e.g., a quotient of the tensor algebra $T(V)$, for some V of finite dimension over \mathbb{k}).*

Let V be an object in \mathcal{V} , $T(V)$ its tensor algebra and I an ideal generated by homogeneous elements of degree at least two. Let $\mathcal{B} = T(V)/I$ and let $\pi: T(V) \rightarrow \mathcal{B}$ be the canonical projection. We also abbreviate $T(V)^+ = \bigoplus_{n \geq 1} V^{\otimes n}$ and $T(V)_{(2)} = \bigoplus_{n \geq 2} V^{\otimes n}$.

Lemma 4.3. *The following is a commutative diagram:*

$$\begin{array}{ccccc}
& & I \otimes T(V)^+ + T(V)^+ \otimes I & \xrightarrow{m} & I \\
& & \downarrow & & \downarrow \\
T(V)^+ \otimes T(V) \otimes T(V)^+ & \xrightarrow{\text{id} \otimes m - m \otimes \text{id}} & T(V)^+ \otimes T(V)^+ & \xrightarrow{m} & T(V)_{(2)} \\
\pi \otimes \pi \otimes \pi \downarrow & & \pi \otimes \pi \downarrow & & \widetilde{\pi \otimes \pi} \downarrow \\
\mathcal{B}^+ \otimes \mathcal{B} \otimes \mathcal{B}^+ & \xrightarrow{\text{id} \otimes m - m \otimes \text{id}} & \mathcal{B}^+ \otimes \mathcal{B}^+ & \xrightarrow{p} & \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \\
& & m \downarrow & & \widetilde{m} \downarrow \\
& & (\mathcal{B}^+)^2 & \xlongequal{\quad} & (\mathcal{B}^+)^2
\end{array}$$

where the maps \widetilde{m} and $\widetilde{\pi \otimes \pi}$ are the universal maps arising from fact (1) below. Moreover, we have the following facts:

- (1) The second and third rows of the diagram are cokernel diagrams.
- (2) The second column of the diagram is exact at $T(V)^+ \otimes T(V)^+$.
- (3) The composition $T(V)_{(2)} \xrightarrow{\widetilde{\pi \otimes \pi}} \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{\widetilde{m}} (\mathcal{B}^+)^2$ is equal to the restriction of π to $T(V)_{(2)}$.
- (4) The map $\widetilde{\pi \otimes \pi}$ is surjective.
- (5) If $\varphi: T(V)_{(2)} \rightarrow T(V)^+ \otimes T(V)^+$ is any splitting of multiplication (e.g., the composition $T(V)_{(2)} \xrightarrow{\sim} V \otimes T(V)^+ \rightarrow T(V)^+ \otimes T(V)^+$ is such a splitting), then $\widetilde{\pi \otimes \pi} = p(\pi \otimes \pi)\varphi$.

Proof. Clearly, each of the squares of the diagram commutes. We prove the remaining claims below:

- (1) The third row is a cokernel diagram by definition. The second row is a cokernel diagram due to the fact that $T(V)^+ = V \otimes T(V)$ as a right $T(V)$ -module (with the obvious action on the second tensor factor), hence $T(V)^+ \otimes_{T(V)} T(V)^+ = V \otimes T(V)^+$, and $V \otimes T(V)^+ \xrightarrow{m} T(V)_{(2)}$ is an isomorphism.
- (2) Clear.
- (3) As π is an algebra map, we have $m(\pi \otimes \pi)m = \pi m$. Hence $\widetilde{m}(\widetilde{\pi \otimes \pi})m = \pi m$. By the universal property of cokernels this means that $\widetilde{m}(\widetilde{\pi \otimes \pi}) = \pi$.
- (4) Follows from the fact that maps p and $\pi \otimes \pi$ are surjective.
- (5) Follows from the universal property of cokernels.

□

Corollary 4.4. *The following sequence is exact:*

$$0 \rightarrow T(V)^+ I + IT(V)^+ \rightarrow I \xrightarrow{\widetilde{\pi \otimes \pi}} \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{\widetilde{m}} (\mathcal{B}^+)^2 \rightarrow 0$$

Therefore, $I/(T(V)^+ I + IT(V)^+) \simeq \ker(\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \rightarrow (\mathcal{B}^+)^2)$.

Proof. To avoid ambiguity, we denote the restriction of $\widetilde{\pi \otimes \pi}$ to I by τ . We first prove that $\ker(\tau) = T(V)^+ I + IT(V)^+$. The inclusion $T(V)^+ I + IT(V)^+ \subseteq \ker(\widetilde{\pi \otimes \pi})$ follows from

$$\widetilde{\pi \otimes \pi}(T(V)^+ I + I T(V)^+) = (\widetilde{\pi \otimes \pi}) m(T(V)^+ \otimes I + I \otimes T(V)^+) = p(\pi \otimes \pi)(T(V)^+ \otimes I + I \otimes T(V)^+) = 0.$$

Let $x \in \ker(\tau)$. Since $m(T(V)^+ \otimes T(V)^+) = T(V)_{(2)}$, there exists $y \in T(V)^+ \otimes T(V)^+$ such that $m(y) = x$. Now $0 = (\widetilde{\pi \otimes \pi}) m(y) = p(\pi \otimes \pi)(y)$, and hence $(\pi \otimes \pi)y = (\text{id} \otimes m - m \otimes \text{id})z$ for some $z \in \mathcal{B}^+ \otimes \mathcal{B} \otimes \mathcal{B}^+$. Let $w \in T(V)^+ \otimes T(V) \otimes T(V)^+$ be such that $(\pi \otimes \pi \otimes \pi)(w) = z$. Define $y' = y - (\text{id} \otimes m - m \otimes \text{id})w$. As $(\pi \otimes \pi)y' = 0$ we have that $y' \in I \otimes T(V)^+ + T(V)^+ \otimes I$ and hence $x = m(y) = m(y') \in I T(V)^+ + T(V)^+ I$.

We now prove that $\widetilde{\pi \otimes \pi}(I) = \ker(\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{\tilde{m}} (\mathcal{B}^+)^2)$. The inclusion \subseteq follows from part (3) of the lemma above: $\tilde{m}(\pi \otimes \pi)(I) = \pi(I) = 0$. The inclusion \supseteq follows from the fact that $\widetilde{\pi \otimes \pi}$ is surjective. \square

Corollary 4.5. *If I is generated by a subobject R , then the induced morphism*

$$R \rightarrow \ker(\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \rightarrow (\mathcal{B}^+)^2)$$

is surjective. \square

We summarize the above results in a theorem which will be needed in the next section to establish rigidity of certain graded bialgebras in \mathcal{V} .

Theorem 4.6. *Let V be an object in \mathcal{V} and $T(V)$ its tensor algebra. Let $R \subset T(V)_{(2)}$ be a graded subspace that is an object in \mathcal{V} . Consider the augmented algebra $\mathcal{B} = T(V)/\langle R \rangle$ and an object U in \mathcal{V} on which \mathcal{B} acts trivially (i.e., via ε). If the multiplication map $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{m} (\mathcal{B}^+)^2$ splits in \mathcal{V} , then there is an injection $H_{\varepsilon}^2(\mathcal{B}, U) \rightarrow \text{Hom}(R, U)$.*

In particular, if f is an ε -cocycle such that for every $u \in \mathcal{B} \otimes \mathcal{B}$ in the range of the composition $R \rightarrow V \otimes T(V)^+ \rightarrow \mathcal{B} \otimes \mathcal{B}$ we have $f(u) = 0$, then f is an ε -coboundary. \square

5. A SUFFICIENT CONDITION FOR RIGIDITY OF GRADED BIALGEBRAS IN A BRAIDED CATEGORY

Let \mathcal{B} be a graded bialgebra in \mathcal{V} . For a homogeneous map $f: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ of degree ℓ and a nonnegative integer r we define $f_r: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ by $f_r|_{(\mathcal{B} \otimes \mathcal{B})_r} = f$ and $f_r|_{(\mathcal{B} \otimes \mathcal{B})_s} = 0$ for $s \neq r$. For $g: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$, we define g_r analogously. We also define $f_{\leq r}$ by $f_{\leq r} = \sum_{i=0}^r f_i$, and $f_{< r}, g_{\leq r}, g_{< r}$ in a similar fashion.

Lemma 5.1 (cf. [25, Lemma 2.3.6]). *Let \mathcal{B} be a graded bialgebra in \mathcal{V} such that $\mathcal{B}_0 = \mathbb{k}$ and \mathcal{B} is generated as an algebra by \mathcal{B}_1 .*

- (1) *If $(f, g) \in Z_{\mathcal{B}}^2(\mathcal{B})_{\ell}$, $r > 1$, $f_{\leq r} = 0$, and $g_{< r} = 0$, then $g_r = 0$.*
- (2) *If $(f, g) \in Z_{\mathcal{B}}^2(\mathcal{B})_{\ell}$, $\ell < 0$, and $f_{\leq r} = 0$, then $g_{\leq r} = 0$.*
- (3) *If $(0, g) \in Z_{\mathcal{B}}^2(\mathcal{B})_{\ell}$, $\ell < 0$, then $g = 0$.*

Proof. The proof in [25] carries over word for word. First note that for every $(f, g) \in Z_{\mathcal{B}}^2(\mathcal{B})$ we have $f_{\leq 1} = 0$ and $g_{\leq 2} = 0$, due to the fact that $(\mathcal{B}^+ \otimes \mathcal{B}^+)_0 = 0 = (\mathcal{B}^+ \otimes \mathcal{B}^+)_1$. Hence (1) easily yields (2) and (3).

For (1) recall that $\partial^c f = -\partial^h g$ by Equation (4). If $r > 1$, $a \in \mathcal{B}_1$ and $b \in \mathcal{B}_{r-1}$, then

$$(\partial^c f)(a, b) = 0 = -(\partial^h g)(a, b) = -(\Delta a)g(b) + g(ab) - g(a)(\Delta b) = g(ab).$$

As \mathcal{B}_r is spanned by such products ab , we have that $g(\mathcal{B}_r) = 0$. \square

Lemma 5.2 (cf. [25, Lemma 2.3.5]). *Let \mathcal{B} be a connected graded bialgebra in \mathcal{V} , let $r \in \mathbb{N}$, and let $f: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ be a homogeneous unital Hochschild cocycle in \mathcal{V} (with respect to left and right regular actions of \mathcal{B} on itself). If $f_{<r} = 0$, then $f_r: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ is an ε -cocycle.*

Proof. This follows directly from $\partial^h f = 0$, see Equation (2). \square

Theorem 5.3 (cf. [25, Lemma 4.2.2]). *Let V be an object in \mathcal{V} and $T(V)$ its (braided) tensor bialgebra. Let $R \subset T(V)_{(2)}$ be a graded subspace that is an object in \mathcal{V} and generates a biideal in $T(V)$. Consider the quotient $\mathcal{B} = T(V)/\langle R \rangle$, which is a graded bialgebra in \mathcal{V} , and assume that the multiplication map $m: \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \rightarrow (\mathcal{B}^+)^2$ splits in \mathcal{V} . If for some negative ℓ we have that $\text{Hom}(R, P(\mathcal{B}))_{\ell} = 0$, then $\widehat{H}_{\mathcal{B}}^2(\mathcal{B})_{\ell} = 0$.*

In particular, if $\text{Hom}(R, P(\mathcal{B}))_{\ell} = 0$ for all negative ℓ , then \mathcal{B} is rigid.

Proof. Let $(f, g) \in Z_{\mathcal{B}}^2(\mathcal{B})_{\ell}$. We will find a map $s = \sum_{r=0}^{\infty} s_r: \mathcal{B} \rightarrow \mathcal{B}$ such that for every nonnegative r , $(f, g)_r = (\partial^h s_r, -\partial^c s_r)$, from where the result trivially follows since $(f, g) = \partial^b \sum_{r=0}^{\infty} s_r$. Here the sum $s = \sum_{r=0}^{\infty} s_r$ is potentially infinite but locally finite. The cases $r = 0, 1$ are clear. Suppose that s_0, \dots, s_{r-1} have been found. Let $(f', g') = (f, g) - \partial^b s_{<r} = (f, g) - \sum_{i=0}^{r-1} (\partial^h s_i, -\partial^c s_i)$. Note that, by assumption, $f'_{<r} = 0$ and hence, by Lemma 5.1, also $g'_{<r} = 0$. Let $u \in (\mathcal{B} \otimes \mathcal{B})$ be in the range of the composition $R \rightarrow V \otimes T(V)^+ \rightarrow \mathcal{B} \otimes \mathcal{B}$. Since $m(u) = 0$ we have from Equation (4) that $f_r(u) \in P(\mathcal{B})$. Therefore, the composition $R \rightarrow V \otimes T(V)^+ \rightarrow \mathcal{B} \otimes \mathcal{B} \xrightarrow{f} \mathcal{B}$ has range in $P(\mathcal{B})$ and must be the zero map. By Theorem 4.6, we get a map $t: \mathcal{B} \rightarrow \mathcal{B}$ such that $f_r = t m$. Now define $s_r = t_r$ and observe that $f'_{\leq r} = f'_r = \partial^h s_r$. Hence, by Lemma 5.1, we also have $g'_r = -\partial^c s_r$. \square

6. NICHOLS ALGEBRAS OF DIAGONAL TYPE

In what follows (V, c) will denote a braided vector space of diagonal type, $\dim V = \theta$, such that the associated Nichols algebra $\mathcal{B}(V)$ has a finite root system Δ_+^V in the sense of [19], i.e., Δ_+^V is the set of \mathbb{N}_0^θ -degrees of generators of a PBW basis. In particular, this is the case if $\mathcal{B}(V)$ is finite-dimensional. Let

$$(5) \quad -c_{ij}^V := \min \{n \in \mathbb{N}_0 \mid (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\}, \quad j \neq i.$$

Now we fix

- a basis $\{x_1, \dots, x_\theta\}$ of V and $q_{ij} \in \mathbb{k}^\times$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$,
- elements $x_\alpha \in \mathcal{B}(V)$ of degree α , $\alpha \in \Delta_+^V$, which generate a PBW basis, see [4].

We use the following notation:

- $\widetilde{q}_{ij} := q_{ij} q_{ji}$ for all $i \neq j$.
- $\chi: \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow \mathbb{k}^\times$ is the bicharacter such that $\chi(\alpha_i, \alpha_j) = q_{ij}$, $1 \leq i, j \leq \theta$, where $\{\alpha_1, \dots, \alpha_\theta\}$ is the canonical basis of \mathbb{Z}^θ .
- N_α is the order of $q_\alpha := \chi(\alpha, \alpha)$, $\alpha \in \Delta_+^V$.
- \mathbb{G}_N is the group of roots of unity of order N and \mathbb{G}'_N is the subset of primitive roots of unity of order N , $N \in \mathbb{N}$.
- $\mathcal{O}(V)$ is the set of Cartan roots of V , i.e., the orbit of Cartan vertices under the action of the Weyl groupoid. Recall that $i \in \{1, \dots, \theta\}$ is a Cartan vertex of V if $\widetilde{q}_{ij} = q_{ii}^{c_{ij}^V}$ for all $j \neq i$ [4, Definition 2.6].

We recall the following result, which gives a presentation by generators and relations for any Nichols algebra of diagonal type with finite root system.

Theorem 6.1. [4] $\mathcal{B}(V)$ is presented by generators x_1, \dots, x_θ and relations:

$$(6) \quad x_\alpha^{N_\alpha}, \quad \alpha \in \mathcal{O}(V);$$

$$(7) \quad (\mathrm{ad}_c x_i)^{1-c_{ij}^V} x_j, \quad q_{ii}^{1-c_{ij}^V} \neq 1;$$

$$(8) \quad x_i^{N_i}, \quad i \text{ is not a Cartan vertex};$$

\diamond if $i, j \in \{1, \dots, \theta\}$ satisfy $q_{ii} = \widetilde{q}_{ij} = q_{jj} = -1$, and there exists $k \neq i, j$ such that $\widetilde{q}_{ik}^2 \neq 1$ or $\widetilde{q}_{jk}^2 \neq 1$,

$$(9) \quad x_{ij}^2;$$

\diamond if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{jj} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{ij}\widetilde{q}_{kj} = 1$, $\widetilde{q}_{ij} \neq -1$,

$$(10) \quad [x_{ijk}, x_j]_c;$$

\diamond if $i, j \in \{1, \dots, \theta\}$ satisfy $q_{jj} = -1$, $q_{ii}\widetilde{q}_{ij} \in \mathbb{G}'_6$, $\widetilde{q}_{ij} \neq -1$, and also $q_{ii} \in \mathbb{G}'_3$ or $-c_{ij}^V \geq 3$,

$$(11) \quad [x_{iij}, x_{ij}]_c;$$

\diamond if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{ii} = \pm \widetilde{q}_{ij} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$, and also $-q_{jj} = \widetilde{q}_{ij}\widetilde{q}_{jk} = 1$ or $q_{jj}^{-1} = \widetilde{q}_{ij} = \widetilde{q}_{jk} \neq -1$,

$$(12) \quad [x_{iijk}, x_{ij}]_c;$$

\diamond if $i, j, k \in \{1, \dots, \theta\}$ satisfy $\widetilde{q}_{ik}, \widetilde{q}_{ij}, \widetilde{q}_{jk} \neq 1$,

$$(13) \quad x_{ijk} - \frac{1 - \widetilde{q}_{jk}}{q_{kj}(1 - \widetilde{q}_{ik})} [x_{ik}, x_j]_c - q_{ij}(1 - \widetilde{q}_{jk}) x_j x_{ik};$$

\diamond if $i, j, k \in \{1, \dots, \theta\}$ satisfy one of the following situations

- (i) $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{ij}^2 = \widetilde{q}_{jk}^{-1}$, $\widetilde{q}_{ik} = 1$, or
- (ii) $\widetilde{q}_{ij} = q_{jj} = -1$, $q_{ii} = -\widetilde{q}_{jk}^2 \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$, or
- (iii) $q_{kk} = \widetilde{q}_{jk} = q_{jj} = -1$, $q_{ii} = -\widetilde{q}_{ij} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$, or
- (iv) $q_{jj} = -1$, $\widetilde{q}_{ij} = q_{ii}^{-2}$, $\widetilde{q}_{jk} = -q_{ii}^3$, $\widetilde{q}_{ik} = 1$, or
- (v) $q_{ii} = q_{jj} = q_{kk} = -1$, $\pm \widetilde{q}_{ij} = \widetilde{q}_{jk} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$,

$$(14) \quad [[x_{ij}, x_{ijk}]_c, x_j]_c;$$

\diamond if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{ij}^3 = \widetilde{q}_{jk}^{-1}$, $\widetilde{q}_{ik} = 1$,

$$(15) \quad [[x_{ij}, [x_{ij}, x_{ijk}]_c]_c, x_j]_c;$$

\diamond if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{jj} = \widetilde{q}_{ij}^2 = \widetilde{q}_{jk} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$,

$$(16) \quad [[x_{ijk}, x_j]_c, x_j]_c;$$

\diamond if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{kk} = q_{jj} = \widetilde{q}_{ij}^{-1} = \widetilde{q}_{jk}^{-1} \in \mathbb{G}'_9$, $\widetilde{q}_{ik} = 1$, $q_{ii} = q_{kk}^6$

$$(17) \quad [x_{iij}, x_{iijk}]_c, x_{ij}]_c;$$

◇ if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{ii} = \widetilde{q_{ij}}^{-1} \in \mathbb{G}'_9$, $q_{jj} = \widetilde{q_{jk}}^{-1} = q_{ii}^5$, $\widetilde{q_{ik}} = 1$, $q_{kk} = q_{ii}^6$

$$(18) \quad [[x_{ijk}, x_j]_c, x_k]_c - (1 + \widetilde{q_{jk}})^{-1} q_{jk} [[x_{ijk}, x_k]_c, x_j]_c;$$

◇ if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{jj} = \widetilde{q_{ij}}^3 = \widetilde{q_{jk}} \in \mathbb{G}'_4$, $\widetilde{q_{ik}} = 1$,

$$(19) \quad [[x_{ijk}, x_j]_c, x_j]_c, x_j]_c;$$

◇ if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{ii} = \widetilde{q_{ij}} = -1$, $q_{jj} = \widetilde{q_{jk}}^{-1} \neq -1$, $\widetilde{q_{ik}} = 1$,

$$(20) \quad [x_{ij}, x_{ijk}]_c;$$

◇ if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{ii} = q_{kk} = -1$, $\widetilde{q_{ik}} = 1$, $\widetilde{q_{ij}} \in \mathbb{G}'_3$, $q_{jj} = -\widetilde{q_{jk}} = \pm \widetilde{q_{ij}}$,

$$(21) \quad [x_i, x_{jjk}]_c - (1 + q_{jj}^2) q_{kj}^{-1} [x_{ijk}, x_j]_c - (1 + q_{jj}^2)(1 + q_{jj}) q_{ij} x_j x_{ijk};$$

◇ if $i, j, k, l \in \{1, \dots, \theta\}$ satisfy $q_{jj} \widetilde{q_{ij}} = q_{jj} \widetilde{q_{jk}} = 1$, $q_{kk} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$, $\widetilde{q_{jk}}^2 = \widetilde{q_{lk}}^{-1} = q_{ll}$,

$$(22) \quad [[x_{ijkl}, x_k]_c, x_j]_c, x_k]_c;$$

◇ if $i, j, k, l \in \{1, \dots, \theta\}$ satisfy $\widetilde{q_{jk}} = \widetilde{q_{ij}} = q_{jj}^{-1} \in \mathbb{G}'_4 \cup \mathbb{G}'_6$, $q_{ii} = q_{kk} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$, $\widetilde{q_{jk}}^3 = \widetilde{q_{lk}}$,

$$(23) \quad [[x_{ijk}, [x_{ijkl}, x_k]_c]_c, x_{jk}]_c;$$

◇ if $i, j, k, l \in \{1, \dots, \theta\}$ satisfy $q_{ll} = \widetilde{q_{ij}}^{-1} = q_{kk} = \widetilde{q_{jk}}^{-1} = q^2$, $\widetilde{q_{ij}} = q_{ii}^{-1} = q^3$ for some $q \in \mathbb{K}^\times$, $q_{jj} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$,

$$(24) \quad [[x_{ijk}, x_j]_c, [x_{ijkl}, x_j]_c]_c, x_{jk}]_c;$$

◇ if $i, j, k, l \in \{1, \dots, \theta\}$ satisfy one of the following situations

(i) $q_{kk} = -1$, $q_{ii} = \widetilde{q_{ij}}^{-1} = q_{jj}^2$, $\widetilde{q_{kl}} = q_{ll}^{-1} = q_{jj}^3$, $\widetilde{q_{jk}} = q_{jj}^{-1}$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$, or

(ii) $q_{ii} = \widetilde{q_{ij}}^{-1} = -q_{ll}^{-1} = -\widetilde{q_{kl}}$, $q_{jj} = \widetilde{q_{jk}} = q_{kk} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$,

$$(25) \quad [[x_{ijkl}, x_j]_c, x_k]_c - q_{jk}(\widetilde{q_{ij}}^{-1} - q_{jj}) [[x_{ijkl}, x_k]_c, x_j]_c;$$

◇ if $i, j, k \in \{1, \dots, \theta\}$ satisfy $\widetilde{q_{jk}} = 1$, $q_{ii} = \widetilde{q_{ij}} = -\widetilde{q_{ik}} \in \mathbb{G}'_3$,

$$(26) \quad [x_i, [x_{ij}, x_{ik}]_c]_c + q_{jk} q_{ik} q_{ji} [x_{iik}, x_{ij}]_c + q_{ij} x_{ij} x_{iik};$$

◇ if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{jj} = q_{kk} = \widetilde{q_{jk}} = -1$, $q_{ii} = -\widetilde{q_{ij}} \in \mathbb{G}'_3$, $\widetilde{q_{ik}} = 1$,

$$(27) \quad [x_{iijk}, x_{ijk}]_c;$$

◇ if $i, j \in \{1, \dots, \theta\}$ satisfy $-q_{ii}, -q_{jj}, q_{ii} \widetilde{q_{ij}}, q_{jj} \widetilde{q_{ij}} \neq 1$,

$$(28) \quad (1 - \widetilde{q_{ij}}) q_{jj} q_{ji} [x_i, [x_{ij}, x_j]_c]_c - (1 + q_{jj})(1 - q_{jj} \widetilde{q_{ij}}) x_{ij}^2;$$

◇ if $i, j \in \{1, \dots, \theta\}$ satisfy that $-c_{ij}^V \in \{4, 5\}$, or $q_{jj} = -1$, $-c_{ij}^V = 3$, $q_{ii} \in \mathbb{G}'_4$,

$$(29) \quad [x_i, x_{3\alpha_i + 2\alpha_j}]_c - \frac{1 - q_{ii} \widetilde{q_{ij}} - q_{ii}^2 \widetilde{q_{ij}}^2 q_{jj}}{(1 - q_{ii} \widetilde{q_{ij}}) q_{ji}} x_{ij}^2;$$

◇ if $i, j \in \{1, \dots, \theta\}$ satisfy $4\alpha_i + 3\alpha_j \notin \Delta_+^V$, $q_{jj} = -1$ or $m_{ji} \geq 2$, and also $-c_{ij}^V \geq 3$, or $-c_{ij}^V = 2$, $q_{ii} \in \mathbb{G}_3'$,

$$(30) \quad x_{4\alpha_i+3\alpha_j} = [x_{3\alpha_i+2\alpha_j}, x_{ij}]_c;$$

◇ if $i, j \in \{1, \dots, \theta\}$ satisfy $3\alpha_i + 2\alpha_j \in \Delta_+^V$, $5\alpha_i + 3\alpha_j \notin \Delta_+^V$, and $q_{ii}^3 \widetilde{q_{ij}}, q_{ii}^4 \widetilde{q_{ij}} \neq 1$,

$$(31) \quad [x_{ii}, x_{3\alpha_i+2\alpha_j}]_c;$$

◇ if $i, j \in \{1, \dots, \theta\}$ satisfy $4\alpha_i + 3\alpha_j \in \Delta_+^V$, $5\alpha_i + 4\alpha_j \notin \Delta_+^V$,

$$(32) \quad x_{5\alpha_i+4\alpha_j} = [x_{4\alpha_i+3\alpha_j}, x_{ij}]_c;$$

◇ if $i, j \in \{1, \dots, \theta\}$ satisfy $5\alpha_i + 2\alpha_j \in \Delta_+^V$, $7\alpha_i + 3\alpha_j \notin \Delta_+^V$,

$$(33) \quad [[x_{iii}, x_{ij}], x_{ij}]_c;$$

◇ if $i, j \in \{1, \dots, \theta\}$ satisfy $q_{jj} = -1$, $5\alpha_i + 4\alpha_j \in \Delta_+^V$,

$$(34) \quad [x_{ii}, x_{4\alpha_i+3\alpha_j}]_c - \frac{b - (1 + q_{ii})(1 - q_{ii}\zeta)(1 + \zeta + q_{ii}\zeta^2)q_{ii}^6\zeta^4}{a q_{ii}^3 q_{ij}^2 q_{ji}^3} x_{3\alpha_i+2\alpha_j}^2,$$

where $\zeta = \widetilde{q_{ij}}$, $a = (1 - \zeta)(1 - q_{ii}^4\zeta^3) - (1 - q_{ii}\zeta)(1 + q_{ii})q_{ii}\zeta$, $b = (1 - \zeta)(1 - q_{ii}^6\zeta^5) - a q_{ii}\zeta$. \square

We fix a realization of (V, c) as a Yetter-Drinfeld module over an abelian group Γ , i.e., there exist $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ such that $\chi_j(g_i) = q_{ij}$ and we make V an object of ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}\mathcal{YD}$ by declaring $x_i \in V_{g_i}^{\chi_i}$. Let \mathcal{R}_V be the set of relations defining $\mathcal{B}(V)$ according to the previous theorem. Note that $\mathbb{k}\mathcal{R}_V$ is a Yetter-Drinfeld submodule of $T(V)$, because each relation is \mathbb{Z}^θ -homogeneous. For each $R \in \mathcal{R}_V$ of degree $(a_1, \dots, a_\theta) \in \mathbb{Z}^\theta$, set

$$(35) \quad g_R := g_1^{a_1} \cdots g_\theta^{a_\theta}, \quad \chi_R := \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \quad \text{so } R \in T(V)_{g_R}^{\chi_R}.$$

The *support* of $R \in \mathcal{R}_V$ is the set $\text{supp } R := \{i \mid a_i \neq 0\}$, i.e., the set of indices of letters x_i appearing in R .

Proposition 6.2. *For every $R \in \mathcal{R}_V$ and $t \in \{1, 2, \dots, \theta\}$, we have $(g_R, \chi_R) \neq (g_t, \chi_t)$.*

Proof. We prove this for each defining relation. For (7), see [6, Proposition 3.1]; the proof does not use that the braiding is of standard type.

We discard easily the cases (6), (8), (9), (14)(**v**), (25)(**ii**), (27), (34) because $\chi_R(g_R) = 1$.

For the remaining cases, note that the propositions in [4, Section 3] show that $(g_R, \chi_R) \neq (g_t, \chi_t)$ for each $t \notin \text{supp } R$. Therefore, we have to consider only the case $t \in \text{supp}(R)$.

For each remaining relation R , we compute $\chi_R(g_R)$ and/or $\{\chi_R(g_t)\chi_t(g_R) \mid t \in \text{supp } R\}$.

(10): we have $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$. Then $\widetilde{q_{ij}} = \chi_R(g_i)\chi_i(g_R) = (q_{ii}\widetilde{q_{ij}})^2$ and $\widetilde{q_{kj}} = \chi_R(g_k)\chi_k(g_R) = (q_{kk}\widetilde{q_{kj}})^2$, so $q_{ii}^2\widetilde{q_{ij}} = q_{kk}^2\widetilde{q_{kj}} = 1$. But such a generalized Dynkin diagram is not in [19], a contradiction.

(11): now $\chi_R(g_R) = q_{ii}^3 \neq q_{ii}$, $\widetilde{q_{ij}} \neq \chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}^2$, so $(g_R, \chi_R) \neq (g_i, \chi_i), (g_j, \chi_j)$.

(12): for both sets of conditions, $\widetilde{q_{ij}}q_{jj}^2\widetilde{q_{jk}} = 1$ so $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$. But $\widetilde{q_{ij}} \neq \chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}^2$, a contradiction.

(13): recall that $\widetilde{q_{ij}}\widetilde{q_{ik}}\widetilde{q_{jk}} = 1$. Suppose that $g_R = g_i$, $\chi_R = \chi_i$. Then $q_{ii} = \chi_R(g_i) = q_{ii}q_{ij}q_{ik}$, so $q_{ij}q_{ik} = 1$. Also $q_{ji}q_{ki} = 1$, so $\widetilde{q_{ij}}\widetilde{q_{ik}} = 1$ and then $\widetilde{q_{jk}} = 1$, a contradiction.

(14)(i) : simply note that $\chi_R(g_R) = -q_{kk} \neq -1, q_{kk}$.

(14)(ii) : as $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$, the remaining case is $t = j$. But also $\widetilde{q_{ij}} = -1 \neq \chi_R(g_i)\chi_i(g_R) = -q_{ii}$.

(14)(iii) : it follows since $\chi_R(g_R) = -q_{ii} \neq -1, q_{ii}$.

(14)(iv) again $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$, so the remaining case is $t = j$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$, so $1 = q_{jj}^2 = \chi_R(g_j)\chi_j(g_R) = \widetilde{q_{ij}}^2 \widetilde{q_{jk}} = -q_{ii}$, a contradiction.

(15): it follows since $\chi_R(g_R) = -q_{kk} \neq -1, q_{kk}$.

(16): again $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$, so

$$q_{jj} = q_{ii}q_{kk}, \quad 1 = \widetilde{q_{ij}}\widetilde{q_{jk}} = \chi_R(g_i)\chi_i(g_R)\chi_R(g_k)\chi_k(g_R) = q_{ii}^2 q_{kk}^2 = q_{jj}^2,$$

which is a contradiction.

(17): it follows from $\chi_R(g_R) = q_{jj}^{-2} \neq q_{ii}, q_{jj}, q_{kk}$.

(18): it follows from $\chi_R(g_R) = q_{jj} \neq q_{ii}, q_{kk}$, and $\chi_R(g_i)\chi_i(g_R) = 1 \neq \widetilde{q_{ij}}$.

(19): the proof is analogous to the one for (16).

(20): As $\chi_R(g_R) = q_{jj}^2 q_{kk}$ and $q_{jj} \neq \pm 1$, we discard the case $t = k$. The case $t = j$ is also discarded because $1 = \chi_R(g_i)\chi_i(g_R) \neq \widetilde{q_{ij}}$. Finally suppose that $\chi_R = \chi_i$, $g_R = g_i$, so $-1 = \widetilde{q_{ij}} = \chi_R(g_j)\chi_j(g_R) = q_{jj}^3$. Then $q_{jj} \in \mathbb{G}'_6$ and $-1 = \chi_R(g_R) = q_{jj}^2 q_{kk}$, so $q_{kk} = q_{jj}$. But this case corresponds to a diagram which is not in [19], a contradiction.

(21): Note that $\chi_R(g_R) = q_{jj}^2 \neq q_{jj}, -1 = q_{ii} = q_{kk}$ because $q_{jj}^2 = \widetilde{q_{ij}}^2 \in \mathbb{G}'_3$.

(22): simply $\chi_R(g_R) = -q_{ii} \neq q_{ii}, q_{jj}, q_{kk}, q_{ll}$ in all the possible cases.

(23): for $t = l$ we have that $\chi_R(g_R) = q_{jj}^3 q_{ll} \neq q_{ll}$, and for $t = i, k$ we have $\chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q_{ij}}, \widetilde{q_{kj}}$. Suppose that $\chi_R = \chi_j$ and $g_R = g_j$. Then $\widetilde{q_{ij}} = \chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}^3$, which is a contradiction because $\widetilde{q_{ij}} \neq \pm 1$.

(24): now, $\chi_R(g_i)\chi_i(g_R) = \chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q_{ij}}, \widetilde{q_{kj}}$, so we discard the cases $t = i, j, k$. Now $\widetilde{q_{kl}} = q_{kk}^{-1} \neq \chi_R(g_k)\chi_k(g_R) = q_{kk}$ so also $(\chi_R, g_R) \neq (\chi_l, g_l)$.

(25)(i) : again $\chi_R(g_i)\chi_i(g_R) = \chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q_{ij}}, \widetilde{q_{kj}}$, and the cases $t = i, j, k$ are solved. As $\widetilde{q_{kl}} = q_{jj}^3 \neq \chi_R(g_k)\chi_k(g_R) = q_{jj}$, we conclude that $(\chi_R, g_R) \neq (\chi_l, g_l)$.

(26): for $t = j, k$ note that $\chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}\widetilde{q_{ik}} \neq \widetilde{q_{ij}}, \widetilde{q_{ik}}$. For $(\chi_R, g_R) = (\chi_i, g_i)$,

$$q_{ii} = \chi_R(g_R) = -q_{jj}q_{kk}, \quad \widetilde{q_{ij}} = \chi_R(g_j)\chi_j(g_R) = \widetilde{q_{ij}}^3 q_{jj}^2, \quad \widetilde{q_{ik}} = \chi_R(g_k)\chi_k(g_R) = \widetilde{q_{ik}}^3 q_{kk}^2,$$

so $q_{jj} = -q_{kk} = \pm q_{ii}^2$, but this diagram is not in [19], a contradiction.

(28): we look for the possible generalized Dynkin diagrams for which we need R .

$$\circ \xrightarrow{\zeta^4} \circ \xrightarrow{\zeta^9} \circ \xrightarrow{\zeta^8} \circ, \quad \zeta \in \mathbb{G}'_{12}: \chi_R(g_R) = 1 \neq q_{ii}, q_{jj}.$$

$$\circ \xrightarrow{\zeta^8} \circ \xrightarrow{\zeta} \circ \xrightarrow{\zeta^8} \circ, \quad \zeta \in \mathbb{G}'_{12}: \chi_R(g_i)\chi_i(g_R) = \chi_R(g_j)\chi_j(g_R) = \zeta^{10} \neq \widetilde{q_{ij}}.$$

$$\circ \xrightarrow{-\zeta} \circ \xrightarrow{\zeta^7} \circ \xrightarrow{\zeta^3} \circ, \quad \zeta \in \mathbb{G}'_9: \chi_R(g_R) = \zeta^8 \neq q_{ii}, q_{jj}.$$

$$\circ \xrightarrow{\zeta^6} \circ \xrightarrow{\zeta^{11}} \circ \xrightarrow{\zeta^8} \circ, \quad \zeta \in \mathbb{G}'_{24}: \chi_R(g_R) = \zeta^4 \neq q_{ii}, q_{jj}.$$

$$\circ \xrightarrow{-\zeta} \circ \xrightarrow{-\zeta^{12}} \circ \xrightarrow{\zeta^5} \circ, \quad \zeta \in \mathbb{G}'_{15}: \chi_R(g_R) = \zeta^{12} \neq q_{ii}, q_{jj}.$$

(29): we consider each possible generalized Dynkin diagram.

$$\begin{aligned}
& \circ \xrightarrow{-\zeta^3} \circ^{-1}, \zeta \in \mathbb{G}'_5: \chi_R(g_R) = 1 \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^3} \circ^{-\zeta^{11}}, \zeta \in \mathbb{G}'_{15}: \chi_R(g_R) = \zeta^{11} \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^8} \circ^{-1}, \zeta \in \mathbb{G}'_{20}: \chi_R(g_R) = \zeta^{12} \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^8} \circ^{-1}, \zeta \in \mathbb{G}'_{20}: \chi_R(g_R) = \zeta^{12} \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{-\zeta^3} \circ^{-1}, \zeta \in \mathbb{G}'_7: \chi_R(g_R) = \zeta^2 \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^2} \circ^{-1}, \zeta \in \mathbb{G}'_8: \chi_R(g_R) = 1 \neq q_{ii}, q_{jj}.
\end{aligned}$$

(30): again consider each possible generalized Dynkin diagram.

$$\begin{aligned}
& \circ \xrightarrow{\zeta^4} \circ^{-1}, \zeta \in \mathbb{G}'_{12}: \chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^8} \circ^{-1}, \zeta \in \mathbb{G}'_{12}: \chi_R(g_R) = \zeta^2 \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^8} \circ^{-1}, \zeta \in \mathbb{G}'_{24}: \chi_R(g_i)\chi_i(g_R) = \zeta, \chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q}_{ij}. \\
& \circ \xrightarrow{\zeta^6} \circ^{-1}, \zeta \in \mathbb{G}'_{24}: \chi_R(g_R) = \zeta^{15} \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{-\zeta^{12}} \circ^{\zeta^5}, \zeta \in \mathbb{G}'_{15}: \chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}.
\end{aligned}$$

(31): the unique diagram is $\circ \xrightarrow{\zeta^3} \circ^{-1}, \zeta \in \mathbb{G}'_9$, and $\chi_R(g_R) = -\zeta^6 \neq q_{ii}, q_{jj}$.

(32): we consider each possible generalized Dynkin diagram.

$$\begin{aligned}
& \circ \xrightarrow{\zeta^2} \circ^{-1}, \zeta \in \mathbb{G}'_5: \chi_R(g_R) = 1 \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^{17}} \circ^{-1}, \zeta \in \mathbb{G}'_{20}: \chi_R(g_R) = \zeta^5 \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^{11}} \circ^{-1}, \zeta \in \mathbb{G}'_{20}: \chi_R(g_R) = \zeta^{15} \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^3} \circ^{-\zeta^{11}}, \zeta \in \mathbb{G}'_{15}: \chi_R(g_R) = \zeta \neq q_{ii}, q_{jj}. \\
& \circ \xrightarrow{\zeta^5} \circ^{-1}, \zeta \in \mathbb{G}'_{15}: \chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}.
\end{aligned}$$

(33): the unique diagram is $\circ \xrightarrow{-\zeta^2} \circ^{-1}, \zeta \in \mathbb{G}'_9$, and $\chi_R(g_R) = \zeta^9 \neq q_{ii}, q_{jj}$. \square

Theorem 6.3. *Suppose V is an object in ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}\mathcal{YD}$ such that its Nichols algebra has a finite root system. Then $\text{Hom}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}(\mathbb{k}\mathcal{R}_V, V) = 0$.*

Proof. If $f \in \text{Hom}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}(\mathbb{k}\mathcal{R}_V, V)$ and $R \in \mathcal{R}_V$, then $f(R) \in V_{g_R}^{\chi_R}$. By Proposition 6.2, $V_{g_R}^{\chi_R} = 0$ for each $R \in \mathcal{R}_V$, so $f = 0$. \square

Theorem 6.4. *If $\mathcal{B}(V)$ is a Nichols algebra of diagonal type with finite root system then $\mathcal{B}(V)$ does not admit nontrivial graded deformations as a braided bialgebra.*

Proof. We fix a realization of (V, c) in ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}\mathcal{YD}$ where Γ is an abelian group. Without loss of generality, we may assume that the g_i 's generate Γ and the χ_i 's generate $\widehat{\Gamma}$. By Theorem 6.3

and Remark 4.2, the conditions needed to invoke Theorem 5.3 are satisfied, so $\mathcal{B}(V)$ does not admit nontrivial graded deformations in ${}_{\mathbb{k}\Gamma}^{\Gamma}\mathcal{YD}$. But our choice of realization ensures that any graded deformation of $\mathcal{B}(V)$ is in ${}_{\mathbb{k}\Gamma}^{\Gamma}\mathcal{YD}$ and hence must be trivial. \square

7. EXAMPLES

7.1. Positive parts of quantum groups. It is well known that, in the generic case, the positive part of a quantized enveloping algebra is a Nichols algebra of diagonal type. By Theorem 6.4, these positive parts are rigid. More generally, this applies to the “diagram” of the pointed Hopf algebra $U(\mathcal{D})$ associated to a generic datum \mathcal{D} of finite Cartan type — see [2], where it is shown that any pointed Hopf algebra whose group-like elements form a finitely generated abelian group is isomorphic to some $U(\mathcal{D})$ if it is a domain with finite Gelfand-Kirillov dimension and its infinitesimal braiding is positive.

7.2. Distinguished pre-Nichols algebras. These are infinite-dimensional braided Hopf algebras projecting onto the corresponding finite-dimensional Nichols algebras. They were formally defined in [5, Definition 3.1] generalizing the situation with quantum groups at roots of unity and the corresponding small quantum groups. Let V be a braided vector space of diagonal type such that $\mathcal{B}(V)$ is finite-dimensional. Then the distinguished pre-Nichols algebra $\tilde{\mathcal{B}}(V)$ is the quotient of $T(V)$ by the relations in Theorem 6.1 except the powers of root vectors (6). As a consequence of Theorem 6.3, we have:

Theorem 7.1. *Let (V, c) be a braided vector space of diagonal type such that $\mathcal{B}(V)$ is finite-dimensional. Then $\tilde{\mathcal{B}}(V)$ does not admit nontrivial graded deformations as a braided bialgebra.* \square

7.3. Nichols algebras over dihedral groups. Let D_m denote the dihedral group of order $2m$. For odd m , it is not known whether the category of Yetter-Drinfeld modules over D_m has any finite-dimension Nichols algebras. For even $m \geq 4$, the only known finite-dimensional Nichols algebras have a symmetric braiding [13], so Theorem 3.3 applies.

7.4. Nichols algebras over symmetric groups. Let $n \geq 3$. The quadratic algebra \mathcal{FK}_n , introduced by Fomin and Kirillov [14], is presented by generators $x_{(ij)}$, $1 \leq i < j \leq n$, and relations

$$\begin{aligned} x_{(ij)}^2 &= 0, & 1 \leq i < j \leq n, \\ x_{(ij)}x_{(jk)} &= x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, & 1 \leq i < j < k \leq n, \\ x_{(jk)}x_{(ij)} &= x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, & 1 \leq i < j < k \leq n, \\ x_{(ij)}x_{(kl)} &= x_{(kl)}x_{(ij)}, & \#\{i, j, k, l\} = 4. \end{aligned}$$

Milinski and Schneider [26] showed how to make \mathcal{FK}_n a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group S_n . As an algebra, it is generated by the vector space V_n with basis $\{x_{(ij)} \mid 1 \leq i < j \leq n\}$. Identifying (ij) with the corresponding transposition in S_n , we can make V_n a Yetter-Drinfeld module where the coaction is defined by declaring x_σ a homogeneous element of degree σ , and the action is

the conjugation twisted by the sign. The corresponding braiding on V_n is given by

$$c(x_\sigma \otimes x_\tau) = \chi(\sigma, \tau) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma, \quad \chi(\sigma, \tau) = \begin{cases} 1 & \sigma(i) < \sigma(j), \tau = (ij), i < j, \\ -1 & \text{otherwise,} \end{cases}$$

where σ and τ are transpositions. Then the above relations generate a biideal in the (braided) tensor bialgebra $T(V_n)$.

It is easy to see that \mathcal{FK}_n projects onto the Nichols algebra $\mathcal{B}(V_n)$. For $n = 3, 4, 5$, it is known that $\mathcal{FK}_n = \mathcal{B}(V_n)$ and has dimension, respectively, 12, 576 and 8294400 (see [26] for $n = 3, 4$ and [17] for $n = 5$). Milinski and Schneider conjectured that \mathcal{FK}_n coincides with $\mathcal{B}(V_n)$ for all n . Moreover, it has been conjectured that $\dim \mathcal{FK}_n = \infty$ for $n \geq 6$ [14].

Theorem 7.2. *Let $n \geq 3$. Then \mathcal{FK}_n does not admit nontrivial graded deformations as a braided bialgebra.*

Proof. All relations are in degree 2 and cannot have coaction given by transposition. As the only primitives in degrees smaller than 2 are in degree 1 and have coaction given by transpositions, the assumption of Theorem 5.3 is satisfied and these algebras are rigid. \square

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