CLASSIFICATION OF INVOLUTIONS ON GRADED-DIVISION SIMPLE REAL ALGEBRAS

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ABSTRACT. We classify, up to isomorphism and up to equivalence, involutions on graded-division finite-dimensional simple real (associative) algebras, when the grading group is abelian.

CONTENTS

1. INTRODUCTION

The study of gradings on various algebras has recently become an active research field — see the monograph [9] and the references therein for an overview of this topic. One of the milestone results in that monograph (following [5, 2, 8]) is the classification of gradings on classical simple Lie algebras over algebraically closed fields of characteristic different from 2. It was achieved by first reducing the problem to the classification of gradings on finite-dimensional simple associative algebras with involution (or, more generally, an antiautomorphism).

This was the main reason to write this article: ultimately, we want to classify gradings on real Lie algebras, and the first step in our approach is to study involutions on graded-division real associative algebras. In fact, we have already finished the classification of gradings on classical central simple real Lie algebras (except those of type D_4). The results are to appear in a separate article (see preprint [3]), in which some of the arguments rely on this paper. On the other hand, the classification of involutions (and related objects) may be of independent interest.

Involutions on graded-division finite-dimensional simple complex algebras are classified in [9, Propositions 2.51 and 2.53] (see also [4]). In this paper we solve the real case. As a prerequisite, we need to know the classification of division gradings on finite-dimensional simple real algebras (without involution). This classification has been done in [13], both up to isomorphism and up to equivalence, and independently in [6], up to equivalence (but note that one of the equivalence classes was overlooked). A classification up to equivalence has been obtained in [7] without assuming simplicity.

The main objective of this work is to classify, up to isomorphism and up to equivalence, involutions on graded-division simple real associative algebras of finite dimension, when the grading group is abelian. We consider only abelian grading groups here because of our intended applications: the support of a grading on a simple Lie algebra always generates an abelian subgroup of the grading group (see for example [9, Proposition 1.12]). Our main classification results are achieved in Sections 6, 7, 8 and 9.

The paper is structured as follows. We have collected the properties that characterize involutions on finite-dimensional simple real algebras in Section 2. Other preliminaries, such as the definitions of isomorphism, equivalence and division grading, can be found in Section 3, together with the rest of terminology related to gradings that we use in the paper. Our main classification results are presented in terms of quadratic forms on certain abelian groups and a similar kind of maps (which we call "nice maps"). These objects are introduced in Section 4.

All homogeneous components of finite-dimensional graded-division real algebras have the same dimension, which can be 1, 2 or 4, according to the identity component being the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ or the division algebra of quaternions $\mathbb{H}$. In the case of dimension 2, the identity component may or may not be contained in the center of the algebra. Consequently, our classification results are arranged into four sections. In Section 6, we classify involutions on graded-division algebras whose homogeneous components have dimension 1. In Section 8, we consider the case of dimension 2 where the identity component is contained in the center, or, equivalently, the center is $\mathbb{C}$ with the trivial grading; in this situation the algebra can be regarded as a graded algebra over $\mathbb{C}$. In Section 7, we also study the case of dimension 2, but the identity component is not contained in the center. Finally, the case of dimension 4 is reduced to the case of dimension 1 thanks to the Double Centralizer Theorem, as stated in Section 9. Note that these four sections are written as if they were very long theorems; we have made an effort to compile the classification to serve as a reference.

Section 5 is written in the same style, that is, as if it were a very long theorem, but its motivation is different. Instead of classifying involutions, we classify division gradings. Moreover, the underlying algebra is not necessarily simple here. The main goal of this section is to classify all quadratic forms that will appear in the following sections and, in particular, establish their existence. Thus, the logic of this section has the opposite direction as compared to the rest of the text.

As mentioned above, we use the results of this paper to classify gradings on classical real Lie algebras in [3]. There, in the case of outer gradings on special linear Lie algebras (which belong to series A), we have to deal with associative algebras that are not simple, but simple as algebras with involution. So, in Section 10 of this paper, we extend a part of the results of the previous sections to algebras whose center is isomorphic to $\mathbb{R} \times \mathbb{R}$.

Finally, in Section 11, we discuss involutions with special properties, which we call "distinguished involutions". We use them in our preprint [3], but they may also

be of independent interest. For example, in the situation of Section 8, they allow us to construct a special basis for a part of the graded-division algebra.

2. Background on involutions

In this section we review the basic properties of involutions on finite-dimensional simple real algebras. We will use [12] as a reference.

An *antiautomorphism* of an algebra $\mathcal{D}$ is a map $\varphi : \mathcal{D} \to \mathcal{D}$ which is an isomorphism of vector spaces and such that $\varphi(xy) = \varphi(y)\varphi(x)$ for all $x, y \in \mathcal{D}$. If it also satisfies $\varphi^2(x) = x$ for all $x \in \mathcal{D}$, φ is called an *involution*.

Let φ be an involution on a real algebra $\mathcal{D}$. The center $Z(\mathcal{D})$ is preserved under φ, so either the restriction of φ to $Z(\mathcal{D})$ is the identity and the involution is said to be *of the first kind*, or this restriction has order 2 and the involution is said to be *of the second kind*.

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let V be an $\mathbb{F}$-vector space of dimension n. An $\mathbb{F}$-bilinear form $b : V \times V \to \mathbb{F}$ is called *nonsingular* (or *nondegenerate*) if the only element $x \in V$ such that $b(x, y) = 0$ for all $y \in V$ is $x = 0$. The following is well known (see [12, p. 1]). First, given one such b, there exists a unique map $\sigma_b : \mathrm{End}_{\mathbb{F}}(V) \to \mathrm{End}_{\mathbb{F}}(V)$ that satisfies the equation

$$b(x, f(y)) = b(\sigma_b(f)(x), y)$$

for all $x, y \in V$ and $f \in \mathrm{End}_{\mathbb{F}}(V)$. Second, the map $b \mapsto \sigma_b$ induces a bijective correspondence between the classes of nonsingular $\mathbb{F}$-bilinear forms on V that are either symmetric or skew-symmetric, up to a factor in $\mathbb{F}^{\times}$, and involutions (of the first kind in the case $\mathbb{F} = \mathbb{C}$) on $\mathrm{End}_{\mathbb{F}}(V)$ ($\cong M_n(\mathbb{F})$). The involutions that are adjoint to symmetric bilinear forms are called *orthogonal*, while those that are adjoint to skew-symmetric bilinear forms are called *symplectic*.

Let φ be an orthogonal involution on $M_n(\mathbb{R})$, and take a nonsingular symmetric bilinear form b on a real vector space V such that φ corresponds to σ_b via some isomorphism $M_n(\mathbb{R}) \cong \mathrm{End}_{\mathbb{R}}(V)$. The number m_+ (respectively m_-) of positive (respectively negative) entries in a diagonalization of b does not depend on the choice of the orthogonal basis. Therefore, $|m_+ - m_-|$ is an invariant of φ, called its *signature*.

An involution φ on $M_n(\mathbb{H})$ is called *orthogonal* or *symplectic* if so is its complexification $\varphi \otimes_{\mathbb{R}} \mathrm{id}_{\mathbb{C}}$. We will use the following characterization ([12, Proposition 2.6]). Let $\mathcal{D}$ be a finite-dimensional simple real algebra, and let φ be an involution on $\mathcal{D}$ (of the first kind if $\mathcal{D} \cong M_n(\mathbb{C})$); then φ is orthogonal if and only if the dimension of $\{x \in \mathcal{D} \mid \varphi(x) = +x\}$ is greater than the dimension of $\{x \in \mathcal{D} \mid \varphi(x) = -x\}$, while it is symplectic if and only if it is smaller.

Let $\mathbb{D}$ be either $\mathbb{H}$ or $\mathbb{C}$, let V be a right $\mathbb{D}$-vector space of dimension n, and denote by $\overline{x}$ the conjugate of x in $\mathbb{D}$. A *hermitian form* on V is an $\mathbb{R}$-bilinear map $h : V \times V \to \mathbb{D}$ such that, for all $x, y \in V$ and $a, b \in \mathbb{D}$, we have: (1) $h(xa, yb) = \overline{a}h(x, y)b$ and (2) $h(y, x) = \overline{h(x, y)}$. The form is called *skew-hermitian* if condition (2) is replaced by: (2') $h(y, x) = -\overline{h(x, y)}$. Thus, these forms are *sesquilinear*: linear in the second variable and semilinear in the first. If we take $\mathbb{D} = \mathbb{R}$ (with $\overline{x} = x$) then we recover the definitions of symmetric and skew-symmetric forms.

A hermitian or skew-hermitian form h is called *nonsingular* if the only element $x \in V$ such that $h(x, y) = 0$ for all $y \in V$ is $x = 0$. It is well known (see [12, Proposition 4.1]) that, given one such h, there exists a unique map $\sigma_h : \mathrm{End}_{\mathbb{D}}(V) \to \mathrm{End}_{\mathbb{D}}(V)$ that satisfies the equation

$$h(x, f(y)) = h(\sigma_h(f)(x), y) \tag{1}$$

for all $x, y \in V$ and $f \in \mathrm{End}_{\mathbb{D}}(V)$. Also, by [12, Theorem 4.2], we have the following.

- In the case $\mathbb{D} = \mathbb{H}$, the map $h \mapsto \sigma_h$ defines a bijective correspondence between the classes of nonsingular hermitian (respectively skew-hermitian) forms on V, up to a factor in $\mathbb{R}^\times$, and symplectic (respectively orthogonal) involutions on $\mathrm{End}_{\mathbb{H}}(V)$ ($\cong M_n(\mathbb{H})$).
- In the case $\mathbb{D} = \mathbb{C}$, the map $h \mapsto \sigma_h$ defines a bijective correspondence between the classes of nonsingular hermitian forms on V, up to a factor in $\mathbb{R}^\times$, and involutions of the second kind on $\mathrm{End}_{\mathbb{C}}(V)$ ($\cong M_n(\mathbb{C})$).

For a symplectic involution on $M_n(\mathbb{H})$ or an involution of the second kind on $M_n(\mathbb{C})$, we define, in the same way as in the case of orthogonal involutions on $M_n(\mathbb{R})$, the *signature* as the absolute value of the difference between the number of positive and negative entries in any diagonalization of any adjoint hermitian form.

Finally, let us also state a couple of lemmas for future reference.

Lemma 1. *Let φ_1 be an orthogonal involution on $M_{n_1}(\mathbb{R})$. Let $\mathbb{D}$ be $\mathbb{R}$ (respectively $\mathbb{H}$, $\mathbb{C}$), and let φ_2 be an orthogonal (respectively symplectic, second kind) involution on $M_{n_2}(\mathbb{D})$. Then $\varphi_1 \otimes_{\mathbb{R}} \varphi_2$ is an orthogonal (respectively symplectic, second kind) involution on $M_{n_1}(\mathbb{R}) \otimes_{\mathbb{R}} M_{n_2}(\mathbb{D})$, and its signature is the product of the signatures of φ_1 and φ_2.*

Proof. Assume that φ_1 is adjoint to the bilinear form $b_1 : V_1 \times V_1 \to \mathbb{R}$ and φ_2 is adjoint to the hermitian form $h_2 : V_2 \times V_2 \to \mathbb{D}$. Note that we have the natural isomorphism of real algebras:

$$\mathrm{End}_{\mathbb{R}}(V_1) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{D}}(V_2) \cong \mathrm{End}_{\mathbb{D}}(V_1 \otimes_{\mathbb{R}} V_2).$$

Through these identifications, $b_1 \otimes_{\mathbb{R}} h_2$ is a hermitian form on $V_1 \otimes_{\mathbb{R}} V_2$ adjoint to $\varphi_1 \otimes_{\mathbb{R}} \varphi_2$. Picking orthogonal bases in V_1 and V_2, we reduce the proof to a straightforward combinatorial fact. $\square$

Lemma 2. *Let φ_1 and φ_2 be second kind involutions on $M_{n_1}(\mathbb{C})$ and $M_{n_2}(\mathbb{C})$. Then there is a unique second kind involution on $M_{n_1}(\mathbb{C}) \otimes_{\mathbb{C}} M_{n_2}(\mathbb{C})$ that sends $X_1 \otimes_{\mathbb{C}} X_2$ to $\varphi_1(X_1) \otimes_{\mathbb{C}} \varphi_2(X_2)$; we denote this map by $\varphi_1 \otimes_{\mathbb{C}} \varphi_2$. Moreover, its signature is the product of the signatures of φ_1 and φ_2.*

Proof. Let us just recall the well known construction of $\varphi_1 \otimes_{\mathbb{C}} \varphi_2$, because the rest of the proof is analogous to the proof of Lemma 1. We can consider the $\mathbb{C}$-vector space $\overline{M_{n_i}(\mathbb{C})}$, which has the same underlying abelian group as $M_{n_i}(\mathbb{C})$, but a twisted scalar multiplication $*$ given by $\alpha * X := \overline{\alpha} X$. If we denote by $\overline{\varphi_i}$ the map φ_i viewed as a map from $M_{n_i}(\mathbb{C})$ to $\overline{M_{n_i}(\mathbb{C})}$, then $\overline{\varphi_i}$ is $\mathbb{C}$-linear. Therefore, we have the $\mathbb{C}$-linear map:

$$\overline{\varphi_1} \otimes_{\mathbb{C}} \overline{\varphi_2} : M_{n_1}(\mathbb{C}) \otimes_{\mathbb{C}} M_{n_2}(\mathbb{C}) \longrightarrow \overline{M_{n_1}(\mathbb{C})} \otimes_{\mathbb{C}} \overline{M_{n_2}(\mathbb{C})}.$$

On the other hand, we have a natural $\mathbb{C}$-linear isomorphism:

$$\overline{M_{n_1}(\mathbb{C})} \otimes_{\mathbb{C}} \overline{M_{n_2}(\mathbb{C})} \longrightarrow \overline{M_{n_1}(\mathbb{C}) \otimes_{\mathbb{C}} M_{n_2}(\mathbb{C})}.$$

Finally, $\varphi_1 \otimes_{\mathbb{C}} \varphi_2$ is the $\mathbb{C}$-semilinear map corresponding to the composition of the two maps above, and it sends $X_1 \otimes_{\mathbb{C}} X_2$ to $\varphi_1(X_1) \otimes_{\mathbb{C}} \varphi_2(X_2)$. $\square$

3. Background on gradings

In this section we review, following [9], the basic definitions and properties of gradings that will be used in the rest of the paper. Here we only deal with associative algebras.

Definition 3. Let $\mathcal{D}$ be an algebra over a field $\mathbb{F}$, and let G be a group. A *G-grading* Γ on $\mathcal{D}$ is a decomposition of $\mathcal{D}$ into a direct sum of subspaces indexed by G,

$$\Gamma : \mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g,$$

such that, for all $g, h \in G$, we have

$$\mathcal{D}_g \mathcal{D}_h \subseteq \mathcal{D}_{gh}.$$

If such a decomposition is fixed, we refer to $\mathcal{D}$ as a *G-graded algebra*. The *support* of Γ (or of $\mathcal{D}$) is the set $\operatorname{supp}(\Gamma) := \{ g \in G \mid \mathcal{D}_g \neq 0 \}$. If $X \in \mathcal{D}_g$, then we say that X is *homogeneous of degree g*, and we write $\deg(X) = g$. The subspace $\mathcal{D}_g$ is called the *homogeneous component of degree g*.

Note that, if $\mathcal{D}$ is a G-graded algebra and $\mathcal{D}'$ is an H-graded algebra, then the tensor product $\mathcal{D} \otimes \mathcal{D}'$ has a natural $G \times H$-grading given by $(\mathcal{D} \otimes \mathcal{D}')_{(g,h)} = \mathcal{D}_g \otimes \mathcal{D}'_h$, for all $g \in G$, $h \in H$. This will be called the *product grading*.

A subspace $\mathcal{F}$ (in particular, a subalgebra or an ideal) of a G-graded algebra $\mathcal{D}$ is said to be *graded* if $\mathcal{F} = \bigoplus_{g \in G} (\mathcal{D}_g \cap \mathcal{F})$.

There are two natural ways to define an equivalence relation on group gradings, depending on whether the grading group plays a secondary role or is a part of the definition.

Definition 4. Let Γ be a G-grading on the algebra $\mathcal{D}$ and let Γ' be an H-grading on the algebra $\mathcal{D}'$. We say that Γ and Γ' are *equivalent* if there exist an isomorphism of algebras $\psi : \mathcal{D} \to \mathcal{D}'$ and a bijection $\alpha : \operatorname{supp}(\Gamma) \to \operatorname{supp}(\Gamma')$ such that $\psi(\mathcal{D}_t) = \mathcal{D}'_{\alpha(t)}$ for all $t \in \operatorname{supp}(\Gamma)$.

Definition 5. Let Γ and Γ' be G-gradings on the algebras $\mathcal{D}$ and $\mathcal{D}'$, respectively. We say that Γ and Γ' are *isomorphic* if there exists an isomorphism of algebras $\psi : \mathcal{D} \to \mathcal{D}'$ such that $\psi(\mathcal{D}_g) = \mathcal{D}'_g$ for all $g \in G$.

Definition 6. Given gradings on the same algebra, $\Gamma : \mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ and $\Gamma' : \mathcal{D} = \bigoplus_{h \in H} \mathcal{D}'_h$, we say that Γ' is a *coarsening* of Γ, or that Γ is a *refinement* of Γ', if, for any $g \in G$, there exists $h \in H$ such that $\mathcal{D}_g \subseteq \mathcal{D}'_h$. If, for some $g \in G$, this inclusion is strict, then we will speak of a *proper* refinement or coarsening. A grading is said to be *fine* if it does not admit a proper refinement.

Definition 7. A graded algebra is said to be a *graded division algebra* if it is unital and every nonzero homogeneous element has an inverse. In this case, the grading will be called a *division grading*.

If $\mathcal{D}$ is a G-graded division algebra, then $I \in \mathcal{D}_e$, where e is the identity element of G and I the unity of $\mathcal{D}$. Also, if $0 \neq X \in \mathcal{D}_g$, then $X^{-1} \in \mathcal{D}_{g^{-1}}$. Therefore, the support of $\mathcal{D}$ is a subgroup of G, since whenever $\mathcal{D}_g \neq 0$ and $\mathcal{D}_h \neq 0$, we also have $0 \neq \mathcal{D}_g \mathcal{D}_h \subseteq \mathcal{D}_{gh}$ and $\mathcal{D}_{g^{-1}} \neq 0$. This also shows that, in the situation of Definition 4, if Γ and Γ' are division gradings, then $\alpha : \operatorname{supp}(\Gamma) \to \operatorname{supp}(\Gamma')$ is a homomorphism of groups.

The identity component $\mathcal{D}_e$ of a graded division algebra $\mathcal{D}$ is a division algebra. Also, if $X_t \in \mathcal{D}_t$ is nonzero, then $\mathcal{D}_t = \mathcal{D}_e X_t$. Therefore, all the (nonzero) homogeneous components of the grading have the same dimension. In our case $\mathcal{D}$ will be finite-dimensional and the ground field will be $\mathbb{R}$, so this dimension must be 1, 2 or 4 depending on whether $\mathcal{D}_e$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Definition 8. Let $\mathcal{D}$ be a G-graded algebra. A map $\varphi : \mathcal{D} \to \mathcal{D}$ is said to be an *antiautomorphism of the G-graded algebra $\mathcal{D}$* if it is an isomorphism of vector spaces such that $\varphi(XY) = \varphi(Y)\varphi(X)$ for all $X, Y \in \mathcal{D}$ and $\varphi(\mathcal{D}_g) = \mathcal{D}_g$ for all

$g \in G$. If it also satisfies $\varphi^2(X) = X$ for all $X \in \mathcal{D}$, φ is called an *involution* of the G-graded algebra $\mathcal{D}$.

Definition 9. Let Γ and Γ' be gradings on the algebras $\mathcal{D}$ and $\mathcal{D}'$. Let $\varphi : \mathcal{D} \to \mathcal{D}$ and $\varphi' : \mathcal{D}' \to \mathcal{D}'$ be antiautomorphisms of graded algebras. We say that (Γ, φ) is *isomorphic* (respectively *equivalent*) to (Γ', φ') if there exists an isomorphism (respectively equivalence) of graded algebras $\psi : \mathcal{D} \to \mathcal{D}'$ such that $\varphi' = \psi \varphi \psi^{-1}$.

We will use the following result [8, Lemma 3.3].

Lemma 10. *Let $\mathcal{D}$ be an algebra endowed with a division grading by an abelian group G. Let $X \in \mathcal{D}^{\times}$ and consider the corresponding inner automorphism $\mathrm{Int}(X) : \mathcal{D} \to \mathcal{D}$ given by $\mathrm{Int}(X)(Y) = XYX^{-1}$. If $\mathrm{Int}(X)(\mathcal{D}_g) \subseteq \mathcal{D}_g$ for all $g \in G$, then there exists a nonzero homogeneous $X_0 \in \mathcal{D}$ such that $\mathrm{Int}(X) = \mathrm{Int}(X_0)$.*

Proof. Let $\psi := \mathrm{Int}(X)$ and note that $\psi(Y)X = XY$ for all $Y \in \mathcal{D}$. Write $X = X_0 + \ldots + X_n$ where the X_i are nonzero homogeneous elements of pairwise different degrees g_i. If $Y \in \mathcal{D}$ is homogeneous of degree g, so is $\psi(Y)$. Since G is abelian, if we consider the terms of degree gg_0 in the equation $\psi(Y)X = XY$, we get $\psi(Y)X_0 = X_0Y$. But X_0 is invertible because it is homogeneous, so $\psi(Y) = X_0 Y X_0^{-1}$ for all $Y \in \mathcal{D}_g$ and $g \in G$. □

4. Quadratic forms

In this section we introduce the necessary terminology concerning quadratic forms on certain abelian groups, mainly following [13, Section 4]. That article established a correspondence between isomorphism classes of division gradings and quadratic forms that are regular in the sense that usually appears in the literature. In this paper, however, we deal with quadratic forms that satisfy less restrictive conditions of regularity. The notation here is congruent with [13], but note that now we do not require quadratic forms and nice maps to be defined on elementary abelian 2-groups.

Definition 11. Let T be a finite abelian group. In this article, by an *alternating bicharacter* on T we will mean a map $\beta : T \times T \to \mathbb{R}^{\times}$ that satisfies $\beta(uv, w) = \beta(u, w)\beta(v, w)$, $\beta(u, vw) = \beta(u, v)\beta(u, w)$, and $\beta(u, u) = 1$ for all $u, v, w \in T$. (It follows that β takes values in $\{\pm 1\}$.) If we have to consider an alternating bicharacter β that takes values in $\mathbb{C}^{\times}$ instead of $\mathbb{R}^{\times}$, we will explicitly say that β is $\mathbb{C}$-valued. A *quadratic form* on T is a map $\mu : T \to \{\pm 1\}$ such that β_{μ} is an alternating bicharacter, where $\beta_{\mu} : T \times T \to \{\pm 1\}$, called the *polarization* of μ, is defined by

$$\beta_{\mu}(u, v) := \mu(uv)\mu(u)^{-1}\mu(v)^{-1}. \tag{2}$$

(The inverses above have no effect, but this way the equation is more similar to the usual definition of quadratic forms on a vector space.)

Definition 12. Let β be an alternating bicharacter on T, and consider its radical: $\mathrm{rad}(\beta) := \{t \in T \mid \beta(u, t) = 1, \forall u \in T\}$. We say that β has *type I* if $\mathrm{rad}(\beta) = \{e\}$, and that it has *type II* if $\mathrm{rad}(\beta) = \{e, f\}$ for some $f \in T$ (of order 2). In the latter case, as f is determined by β, we denote it by f_{β}.

We will say that a family $\{a_1, b_1, \ldots, a_m, b_m\}$ in T is *symplectic* if $\beta(a_i, b_i) = \beta(b_i, a_i) = -1$ $(i = 1, \ldots, m)$ and the value of β on all other pairs is $+1$. We will say that it is a *basis* if T is the direct product of the subgroups $\langle a_1 \rangle, \langle b_1 \rangle, \ldots, \langle a_m \rangle, \langle b_m \rangle$.

The following result [13, Proposition 9] describes alternating bicharacters satisfying Definition 12.

Proposition 13. *Let β be an alternating bicharacter on a finite abelian group T.*

(1) *If β has type I, then $T \cong (\mathbb{Z}_2^2)^m$ and there exists a symplectic basis $\{a_1, b_1, \ldots, a_m, b_m\}$ of T.*
(2) *If β has type II, then either $T \cong (\mathbb{Z}_2^2)^m \times \mathbb{Z}_2$ or $(\mathbb{Z}_2^2)^{m-1} \times (\mathbb{Z}_2 \times \mathbb{Z}_4)$, and there exists a symplectic family $\{a_1, b_1, \ldots, a_m, b_m\}$ such that:*
   (a) *In the first case, $\{a_1, b_1, \ldots, a_m, b_m, f_\beta\}$ is a basis of T.*
   (b) *In the second case, $b_m^2 = f_\beta$ and $\{a_1, b_1, \ldots, a_m, b_m\}$ is a basis of T.*
$\hfill\square$

Lemma 14. *Let μ and η be two different quadratic forms on a finite abelian group T such that $\beta_\mu = \beta_\eta$. Then $\{t \in T \mid \mu(t) = \eta(t)\}$ is a subgroup of T of index 2.*

Proof. Let $S := \{t \in T \mid \mu(t) = \eta(t)\}$. By Equation (2), we have $\mu(e) = 1 = \eta(e)$, so $e \in S$. Also, $u, v \in S$ implies $uv \in S$, hence S is a subgroup. Since μ and η take values in $\{\pm 1\}$, $u, v \in T \setminus S$ implies $\mu(u)\mu(v) = \eta(u)\eta(v)$, hence $uv \in S$ by Equation (2). Thus the quotient group T/S can have only two elements. $\hfill\square$

Lemma 15. *Let β be an alternating bicharacter of type I on a finite abelian group T. Then the following map is a bijection:*

$$T \longrightarrow \{S \mid S \text{ is a subgroup of } T, [T : S] \leq 2\}$$
$$u \longmapsto u^\perp = \{v \in T \mid \beta(u, v) = 1\}$$

Proof. It is enough to interpret β as a nondegenerate alternating bilinear form over the field of two elements. $\hfill\square$

Lemma 16. *Let β be an alternating bicharacter of type II on a finite abelian group T. Then the following map is a bijection:*

$$T/\langle f_\beta \rangle \longrightarrow \{S \mid S \text{ is a subgroup of } T, f_\beta \in S, [T : S] \leq 2\}$$
$$[u] \longmapsto u^\perp = \{v \in T \mid \beta(u, v) = 1\}$$

Proof. Consider the alternating bicharacter $\bar\beta$ on $T/\langle f_\beta \rangle$ such that $\beta = \bar\beta \circ (\pi \times \pi)$, where $\pi : T \to T/\langle f_\beta \rangle$ is the natural projection. Then $\bar\beta$ has type I and we can apply Lemma 15. $\hfill\square$

Notation 17. For any natural number n and abelian group T, we define $T_{[n]} = \{t \in T \mid t^n = e\}$ and $T^{[n]} = \{t^n \mid t \in T\}$.

(We will primarily need the case $n = 2$.)

Notation 18. Let β be an alternating bicharacter of type II on $T \cong \mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$. Then $T^{[2]}$ has order 2 and we denote by f_T its generator. By Proposition 13, $f_T = f_\beta$. We set $\mathrm{rad}'(\beta) := \mathrm{rad}(\beta|_{T_{[2]} \times T_{[2]}}) \setminus \mathrm{rad}(\beta)$ (which equals $\{a_m, a_m f_T\}$ with the notation of Proposition 13).

Remark 19. If η is a quadratic form on $T \cong \mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$ such that β_η has type II, then $\eta(f_T) = +1$. Indeed, if g is an element of T of order 4, then $f_T = g^2$, so $\eta(f_T) = \eta(g^2) = \eta(g)^2 \beta_\eta(g, g) = +1$. Also note that η takes the same value on the two elements of $\mathrm{rad}'(\beta_\eta)$, because one is the other multiplied by f_T. Finally, if β is an alternating bicharacter of type II on $T \cong \mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$, and μ is a quadratic form defined only on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$ and $\mu(f_T) = +1$, then there exist exactly two quadratic forms on T that extend μ and whose polarization is β.

Proposition 20. *Let T be a finite abelian group, K a subgroup of T of index 2, and $\nu : T \setminus K \to \{\pm 1\}$ a map. Consider the family of maps $\mu_g : K \to \{\pm 1\}$ defined by $\mu_g(k) := \nu(gk)\nu(g)^{-1}$, as g runs through $T \setminus K$. Then, if a member of this family is a quadratic form, so are the others, and all have the same polarization β. Moreover, if β has type II, the value $\mu_g(f_\beta)$ does not depend on the choice of $g \in T \setminus K$.*

Proof. Let $g, h \in T \setminus K$, and assume that μ_g is a quadratic form. Call β its polarization. The assertions follow from the following formula ($k \in K$):

$$\mu_h(k) = \frac{\nu(hk)}{\nu(h)} = \frac{\mu_g(g^{-1}hk)}{\mu_g(g^{-1}h)} = \mu_g(k)\beta(g^{-1}h, k). \tag{3}$$

Indeed, as β is multiplicative in k, μ_h is also a quadratic form with the same polarization as μ_g. And if β has type II, then $\mu_h(f_\beta) = \mu_g(f_\beta)\beta(g^{-1}h, f_\beta) = \mu_g(f_\beta)$. $\qquad\square$

Definition 21. In the situation of Proposition 20, we say that ν is a *nice map* on $T \setminus K$, and we denote by β_ν the common polarization β of the quadratic forms μ_g. If β has type II, we also define $\nu(f_\beta) := \mu_g(f_\beta)$, where g is any element of $T \setminus K$.

Lemma 22. *Under the conditions of Proposition 20, suppose that $T \cong \mathbb{Z}_2^{2m-2} \times \mathbb{Z}_4$, $K \cong \mathbb{Z}_2^{2m-3} \times \mathbb{Z}_4$, and $\beta = \beta_\nu$ has type II. Then the set $\mu_g(\mathrm{rad}'(\beta))$, where g is any element of $T \setminus K$ of order 2, does not depend on the choice of g.*

Proof. Let $g, h \in T \setminus K$ be elements of order 2 and let $a \in \mathrm{rad}'(\beta) \subseteq \mathrm{rad}(\beta|_{T_{[2]} \times T_{[2]}})$. By Equation (3), $\mu_h(a) = \mu_g(a)\beta(g^{-1}h, a) = \mu_g(a)$. $\qquad\square$

Notation 23. In the situation of Lemma 22, we define $\nu(\mathrm{rad}'(\beta)) := \mu_g(\mathrm{rad}'(\beta))$, where g is any element of $T \setminus K$ of order 2. Note that, by Remark 19 (applied to K), μ_g takes the same value on the two elements of $\mathrm{rad}'(\beta)$, so the set $\nu(\mathrm{rad}'(\beta))$ actually consists of one element.

Notation 24 (Arf invariant). Let T be a finite set and let $\mu : T \to \{\pm 1\}$ be a map. If the cardinality of $\mu^{-1}(+1)$ is greater than the cardinality of $\mu^{-1}(-1)$, we write $\mathrm{Arf}(\mu) = +1$. If it is smaller, we write $\mathrm{Arf}(\mu) = -1$. Finally, if both cardinalities are equal, $\mathrm{Arf}(\mu)$ is not defined.

5. Division gradings and quadratic forms

As mentioned in the Introduction, the main purpose of this section is to classify all quadratic forms whose polarization has type I or II. We establish a correspondence with gradings in order to prove their existence.

The following division gradings will be our building blocks.

Example 25. Two division gradings by the group $\mathbb{Z}_2$:

$$\mathbb{C} = \mathbb{R}1 \oplus \mathbb{R}i \quad \text{and} \quad \mathbb{R} \times \mathbb{R} = \mathbb{R}(1,1) \oplus \mathbb{R}(1,-1).$$

Two division gradings by the group $\mathbb{Z}_2^2$:

$$M_2(\mathbb{R}) = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k.$$

And the three division gradings by the group $\mathbb{Z}_2 \times \mathbb{Z}_4 = \{e, a; b, ab; b^2, ab^2; b^3, ab^3\}$ presented in Figure 1.

Let $\mathcal{D}$ be a finite-dimensional real (associative) algebra whose center $Z(\mathcal{D})$ has dimension 1 or 2. Let G be an abelian group and let Γ be a division G-grading on $\mathcal{D}$ with support T and homogeneous components of dimension 1. Note that $\mathcal{D}$ must be unital, but we do not assume that it is a simple algebra. By a generalization of Maschke's Theorem (see for example [11, Corollary 10.2.5 on p. 443]), $\mathcal{D}$ is necessarily semisimple, which implies that it is simple if $Z(\mathcal{D})$ is $\mathbb{R}$ or $\mathbb{C}$, and the direct product of two simple algebras if $Z(\mathcal{D})$ is $\mathbb{R} \times \mathbb{R}$.

$$M_2(\mathbb{C}) = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus$$

$$\mathbb{R}\begin{pmatrix} e^{\pi i/4} & 0 \\ 0 & -e^{\pi i/4} \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & -e^{\pi i/4} \\ e^{\pi i/4} & 0 \end{pmatrix} \oplus$$

$$\mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \oplus$$

$$\mathbb{R}\begin{pmatrix} e^{3\pi i/4} & 0 \\ 0 & -e^{3\pi i/4} \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & -e^{3\pi i/4} \\ e^{3\pi i/4} & 0 \end{pmatrix}$$

$$M_2(\mathbb{R} \times \mathbb{R}) = \mathbb{R}\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (-1,-1) \end{pmatrix} \oplus$$

$$\mathbb{R}\begin{pmatrix} (0,0) & (1,-1) \\ (1,1) & (0,0) \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} (0,0) & (1,-1) \\ (-1,-1) & (0,0) \end{pmatrix} \oplus$$

$$\mathbb{R}\begin{pmatrix} (1,-1) & (0,0) \\ (0,0) & (1,-1) \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} (1,-1) & (0,0) \\ (0,0) & (-1,1) \end{pmatrix} \oplus$$

$$\mathbb{R}\begin{pmatrix} (0,0) & (1,1) \\ (1,-1) & (0,0) \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} (0,0) & (1,1) \\ (-1,1) & (0,0) \end{pmatrix}$$

$$M_2(\mathbb{R}) \times \mathbb{H} = \mathbb{R}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \oplus \mathbb{R}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, i\right) \oplus$$

$$\mathbb{R}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j\right) \oplus \mathbb{R}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, k\right) \oplus$$

$$\mathbb{R}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1\right) \oplus \mathbb{R}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -i\right) \oplus$$

$$\mathbb{R}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -j\right) \oplus \mathbb{R}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -k\right)$$

FIGURE 1. Division gradings of Example 25; degrees in the group $\langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ are assigned line-by-line in the following order: $e, a; b, ab; b^2, ab^2; b^3, ab^3$.

We claim that the graded algebra $\mathcal{D}$ is equivalent to one, and only one, tensor product on the list below, equipped with the product grading where each factor is graded as in Example 25. The isomorphism classes are in bijective correspondence with the triples (T, β, μ), where β is an alternating bicharacter on T of type I or II, and μ is a quadratic form on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$. Namely, $\beta : T \times T \to \{\pm 1\}$ is defined by the commutation relations of homogeneous elements,

$$X_u X_v = \beta(u, v) X_v X_u \tag{4}$$

for all $X_u \in \mathcal{D}_u$ and $X_v \in \mathcal{D}_v$, and $\mu : T_{[2]} \to \{\pm 1\}$ is defined by the signs of the squares of homogeneous elements,

$$X_t^2 = \mu(t) I \tag{5}$$

for all $X_t \in \mathcal{D}_t$ such that $X_t^4 = I$, where I is the unity of $\mathcal{D}$ and $t \in T_{[2]}$. Conversely, given such (T, β, μ), we can construct $\mathcal{D}$ as the algebra with generators X_u, with u ranging over a basis of T, and defining relations given by Equations (4) for all basis elements u and v, Equations (5) for all basis elements t of order 2, and $X_t^4 = \mu(f_T) I$ for the basis element t of order 4 (if it is present). The grading on $\mathcal{D}$ is defined by declaring the generator X_u to be of degree u. Now we give the list of the

equivalence classes, and we also compile the classification up to isomorphism to serve as a reference:

(1-a) $M_n(\mathbb{R}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R})$, $n = 2^m \geq 1$ (if $n = 1$, $M_n(\mathbb{R}) = \mathbb{R}$ with the trivial grading). The grading Γ is determined up to isomorphism by (T, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m}$, and μ is a quadratic form on T such that β_μ has type I and $\mathrm{Arf}(\mu) = +1$.

(1-b) $M_{n/2}(\mathbb{H}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes \mathbb{H}$, $n = 2^m \geq 2$. The grading Γ is determined up to isomorphism by (T, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m}$, and μ is a quadratic form on T such that β_μ has type I and $\mathrm{Arf}(\mu) = -1$.

(1-c) $M_n(\mathbb{C}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes \mathbb{C}$, $n = 2^m \geq 1$. The grading Γ is determined up to isomorphism by (T, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m+1}$, and μ is a quadratic form on T such that $\beta := \beta_\mu$ has type II and $\mu(f_\beta) = -1$.

(1-d) $M_n(\mathbb{C}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{C})$, $n = 2^m \geq 2$. The grading Γ is determined up to isomorphism by (T, β, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$, β is an alternating bicharacter on T of type II, and μ is a quadratic form on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$ and $\mu(f_T) = -1$.

(1-e) $M_n(\mathbb{R}) \times M_n(\mathbb{R}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes [\mathbb{R} \times \mathbb{R}]$, $n = 2^m \geq 1$. The grading Γ is determined up to isomorphism by (T, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m+1}$, and μ is a quadratic form on T such that $\beta := \beta_\mu$ has type II, $\mu(f_\beta) = +1$ and $\mathrm{Arf}(\mu) = +1$.

(1-f) $M_{n/2}(\mathbb{H}) \times M_{n/2}(\mathbb{H}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \otimes [\mathbb{R} \times \mathbb{R}]$, $n = 2^m \geq 2$. The grading Γ is determined up to isomorphism by (T, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m+1}$, and μ is a quadratic form on T such that $\beta := \beta_\mu$ has type II, $\mu(f_\beta) = +1$ and $\mathrm{Arf}(\mu) = -1$.

(1-g) $M_n(\mathbb{R}) \times M_n(\mathbb{R}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R} \times \mathbb{R})$, $n = 2^m \geq 2$. The grading Γ is determined up to isomorphism by (T, β, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$, β is an alternating bicharacter on T of type II, and μ is a quadratic form on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$, $\mu(f_T) = +1$, $\mu(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\mu) = +1$.

(1-h) $M_{n/2}(\mathbb{H}) \times M_{n/2}(\mathbb{H}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \otimes M_2(\mathbb{R} \times \mathbb{R})$, $n = 2^m \geq 4$. The grading Γ is determined up to isomorphism by (T, β, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$, β is an alternating bicharacter on T of type II, and μ is a quadratic form on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$, $\mu(f_T) = +1$, $\mu(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\mu) = -1$.

(1-i) $M_n(\mathbb{R}) \times M_{n/2}(\mathbb{H}) \cong M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes [M_2(\mathbb{R}) \times \mathbb{H}]$, $n = 2^m \geq 2$. The grading Γ is determined up to isomorphism by (T, β, μ), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$, β is an alternating bicharacter on T of type II, and μ is a quadratic form on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$, $\mu(f_T) = +1$ and $\mu(\mathrm{rad}'(\beta)) = \{-1\}$.

Proof. Denote by I the unity of $\mathcal{D}$. If $Z(\mathcal{D})$ has dimension 2, denote $Z = iI$ in the case $Z(\mathcal{D}) = \mathbb{C}$ (where i is the imaginary unit) and $Z = (1, -1)I$ in the case $Z(\mathcal{D}) = \mathbb{R} \times \mathbb{R}$. Then $Z(\mathcal{D})$ is either $\mathbb{R}I$ or $\mathbb{R}I \oplus \mathbb{R}Z$, and it is easy to show that Z is homogeneous and its degree f is an element of order 2 (see for example [13, Lemma 14]).

Now the classification is obtained by repeating the arguments of [13, Theorems 15 and 16]. Remark that $f_\beta = f$ and that now the case $\mu(f_\beta) = +1$ is possible.

As an example, let us recall the construction of a graded division algebra $\mathcal{D}$ for a given datum (T, μ) in the case (1-c), that is, $T \cong \mathbb{Z}_2^{2m+1}$, β_μ of type II and $\mu(f_\beta) = -1$. We pick a basis $\{a_1, b_1, \ldots, a_m, b_m, f_\beta\}$ of T as in Proposition 13. Changing if necessary a_j or b_j to $a_j f_\beta$ or $b_j f_\beta$, respectively, we can get $\mu(a_j) = \mu(b_j) = +1$

$(j = 1, \ldots, m)$. Hence we can choose as the graded algebra $M_2(\mathbb{R}) \otimes \ldots \otimes M_2(\mathbb{R}) \otimes \mathbb{C}$, as it is stated in the list, where the support of the j-th factor is $\langle a_j, b_j \rangle$ $(j = 1, \ldots, m)$ and the support of the last factor is $\langle f_\beta \rangle$.

The only new argument is the fact that $M_2(\mathbb{R}) \otimes [M_2(\mathbb{R}) \times \mathbb{H}]$ and $\mathbb{H} \otimes [M_2(\mathbb{R}) \times \mathbb{H}]$ are in the same equivalence class, (1-i). Indeed, consider the symplectic basis $\{a_1, b_1, a_2, b_2\}$ of $\mathbb{Z}_2^2 \times (\mathbb{Z}_2 \times \mathbb{Z}_4)$ relative to the grading of $\mathbb{H} \otimes [M_2(\mathbb{R}) \times \mathbb{H}]$, that is, a_1 and b_1 generate the support of the grading on $\mathbb{H}$, while a_2 and b_2 generate the support of the grading on $M_2(\mathbb{R}) \times \mathbb{H}$ playing the roles of a and b in Example 25, so $b_2^2 \neq e$. The quadratic form is determined by $\mu(a_1) = \mu(b_1) = -1$ and $\mu(a_2) = -1$, $\mu(b_2^2) = +1$. Then $a_1' = a_1 a_2$, $b_1' = b_1 a_2$, $a_2' = a_2$, $b_2' = a_1 b_1 b_2$ form another symplectic basis, but now $\mu(a_1') = \mu(b_1') = +1$ and still $\mu(a_2') = -1$, $\mu((b_2')^2) = +1$. Therefore, we can rewrite $\mathbb{H} \otimes [M_2(\mathbb{R}) \times \mathbb{H}]$ as $M_2(\mathbb{R}) \otimes [M_2(\mathbb{R}) \times \mathbb{H}]$ by renaming the elements of the group, so these graded algebras are equivalent. $\square$

Remark 26. If $Z(\mathcal{D})$ is $\mathbb{R} \times \mathbb{R}$ then it must be nontrivially graded, so it is isomorphic to the group algebra of a subgroup of G of order 2, namely, $\{e, f\}$ where $f = f_\beta$. Hence, $\mathcal{D}$ with its G-grading can be obtained from a simple algebra with a grading by the quotient group $G/\langle f \rangle$ by means of the *loop construction* (see [1]): (1-e) and (1-g) from (1-a); (1-f) and (1-h) from (1-b); and (1-i) from either (1-a) or (1-b).

6. Classification in the one-dimensional case

Example 27. We define involutions on $M_2(\mathbb{R})$, $\mathbb{H}$, $\mathbb{C}$ and $M_2(\mathbb{C})$ that respect the gradings of Example 25. The notation with subscripts will make sense later on, when the classification of this section is stated.

- Let $\varphi_{(1\text{-a-}1)}$ be the matrix transpose on $M_2(\mathbb{R})$. It is an orthogonal involution with signature 2.
- Let $\varphi_{(1\text{-a-}2)}$ be the involution on $M_2(\mathbb{R})$ given by $\varphi_{(1\text{-a-}2)}(X) = A^{-1} X^T A$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

  It is an orthogonal involution with signature 0.
- Let $\varphi_{(1\text{-a-}3)}$ be the involution on $M_2(\mathbb{R})$ that acts as minus the identity on the matrices of trace zero, and acts as the identity on the center of $M_2(\mathbb{R})$. It is a symplectic involution.
- Let $\varphi_{(1\text{-b-}1)}$ be the standard conjugation on $\mathbb{H}$. It is a symplectic involution with signature 1.
- Let $\varphi_{(1\text{-b-}3)}$ be the involution on $\mathbb{H}$ that acts as the identity on 1, i and j, and acts as minus the identity on k. It is an orthogonal involution.
- Let $\varphi_{(1\text{-c-}1)}$ be the conjugation on $\mathbb{C}$. It is an involution of the second kind and has signature 1.
- Let $\varphi_{(1\text{-c-}3)}$ be the identity on $\mathbb{C}$. It is an involution of the first kind and orthogonal.
- Let $\varphi_{(1\text{-d-}1)}$ be the matrix transpose on $M_2(\mathbb{C})$. It is an involution of the first kind and orthogonal.
- Let $\varphi_{(1\text{-d-}3)}$ be the involution on $M_2(\mathbb{C})$ given by $\varphi_{(1\text{-a-}2)}(X) = A^{-1} X^T A$, with A as in Equation (6). It is an involution of the first kind and orthogonal.
- Let $\varphi_{(1\text{-d-}4)}$ be the involution on $M_2(\mathbb{C})$ that acts as minus the identity on the matrices of trace zero, and acts as the identity on the center of $M_2(\mathbb{C})$. It is an involution of the first kind and symplectic.

The involution $\varphi_{(1\text{-a-}1)}$ will occur most frequently, so we will abbreviate it as φ_*.

Let G be an abelian group, $\mathcal{D}$ a finite-dimensional simple real (associative) algebra, and Γ a division G-grading on $\mathcal{D}$ with homogeneous components of dimension 1. These gradings are classified in [13, Theorems 15 and 16]; there are four families of equivalence classes: (1-a), (1-b), (1-c) and (1-d). We keep the same notation, so let T be the support of Γ, and let $\beta : T \times T \to \{\pm 1\}$ be the alternating bicharacter given by the commutation relations.

Then, any antiautomorphism φ of the G-graded algebra $\mathcal{D}$ is an involution. We want to classify the pairs (Γ, φ), up to isomorphism and up to equivalence. The isomorphism classes are in bijective correspondence with the quadratic forms η on T such that $\beta_\eta = \beta$. Namely, the correspondence is given by the equation

$$\varphi(X_t) = \eta(t)X_t \tag{7}$$

for all $X_t \in \mathcal{D}_t$. Now we give a list of the equivalence classes together with a representative of every class, and we also compile the classification up to isomorphism to serve as a reference:

(1-a) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{R})$ ($n = 2^m \geq 1$) was determined up to isomorphism by (T, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m}$, and μ was a quadratic form on T such that $\beta := \beta_\mu$ had type I and $\mathrm{Arf}(\mu) = +1$. Now (Γ, φ) is determined up to isomorphism by (T, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following three equivalence classes:

  (1) $\eta = \mu$ ($n = 2^m \geq 1$).
      The involution φ is orthogonal with signature n.
      A representative is $\varphi_* \otimes \cdots \otimes \varphi_*$ (if $n = 1$, φ is just the identity on $\mathbb{R}$).
  (2) $\mathrm{Arf}(\eta) = +1$ but $\eta \neq \mu$ ($n = 2^m \geq 2$).
      The involution φ is orthogonal with signature 0.
      A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-2})}$.
  (3) $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 2$).
      The involution φ is symplectic.
      A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-3})}$.

(1-b) The grading Γ on $\mathcal{D} \cong M_{n/2}(\mathbb{H})$ ($n = 2^m \geq 2$) was determined up to isomorphism by (T, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m}$, and μ was a quadratic form on T such that $\beta := \beta_\mu$ had type I and $\mathrm{Arf}(\mu) = -1$. Now (Γ, φ) is determined up to isomorphism by (T, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following three equivalence classes:

  (1) $\eta = \mu$ ($n = 2^m \geq 2$).
      The involution φ is symplectic with signature $n/2$.
      A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-b-1})}$.
  (2) $\mathrm{Arf}(\eta) = -1$ but $\eta \neq \mu$ ($n = 2^m \geq 4$).
      The involution φ is symplectic with signature 0.
      A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-2})} \otimes \varphi_{(1\text{-b-1})}$.
  (3) $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 2$).
      The involution φ is orthogonal.
      A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-b-3})}$.

(1-c) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ ($n = 2^m \geq 1$) was determined up to isomorphism by (T, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m+1}$, and μ was a quadratic form on T such that $\beta := \beta_\mu$ had type II and $\mu(f_\beta) = -1$. Now (Γ, φ) is determined up to isomorphism by (T, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following four equivalence classes:

(1) $\eta = \mu$ $(n = 2^m \geq 1)$.
   The involution φ is of the second kind and has signature n.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}c\text{-}1)}$.
(2) $\eta(f_\beta) = -1$ but $\eta \neq \mu$ $(n = 2^m \geq 2)$.
   The involution φ is of the second kind and has signature 0.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}2)} \otimes \varphi_{(1\text{-}c\text{-}1)}$.
(3) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = +1$ $(n = 2^m \geq 1)$.
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}c\text{-}3)}$.
(4) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = -1$ $(n = 2^m \geq 2)$.
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}3)} \otimes \varphi_{(1\text{-}c\text{-}3)}$.

(1-d) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ $(n = 2^m \geq 2)$ was determined up to isomorphism by (T, β, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1} \times \mathbb{Z}_4$, β was an alternating bicharacter on T of type II, and μ was a quadratic form on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$ and $\mu(f_T) = -1$. Now (Γ, φ) is determined up to isomorphism by (T, β, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta$ (so $\eta(f_T) = +1$ by Remark 19). These isomorphism classes belong to one of the following four equivalence classes:

(1) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = +1$ $(n = 2^m \geq 2)$.
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}d\text{-}1)}$.
(2) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = -1$ $(n = 2^m \geq 4)$.
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}3)} \otimes \varphi_{(1\text{-}d\text{-}1)}$.
(3) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = +1$ $(n = 2^m \geq 2)$.
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}d\text{-}3)}$.
(4) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = -1$ $(n = 2^m \geq 2)$.
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}d\text{-}4)}$.

Proof. Define $\eta : T \to \mathbb{R}^\times$ by Equation (7). For all $t \in T$, we can pick X_t so that $X_t^8 = +I$, hence $\eta(t)^8 = +1$; therefore η takes values in $\{\pm 1\}$ and φ is an involution. The fact that φ reverses the order of the product is equivalent to η being a quadratic form with $\beta_\eta = \beta$; indeed:

$$\eta(uv)X_v X_u = \varphi(X_v X_u) = \varphi(X_u)\varphi(X_v) = \eta(u)\eta(v)\beta(u,v)X_v X_u.$$

Thus we have a bijective correspondence between the isomorphism classes of pairs (Γ, φ) and the quadratic forms η on T such that $\beta_\eta = \beta$.

Involutions belonging to (1-a-1) or (1-a-2) are not equivalent to those in (1-a-3), because of the Arf invariant. The involution (1-a-1) is determined by Equation (5), so it is not equivalent to the involutions that belong to (1-a-2), in other words, it is a distinguished involution of the grading. Considering also $\eta(f_\beta)$ and $\eta(\mathrm{rad}'(\beta))$, we see that the rest of the equivalence classes of the list do not overlap.

We know that there exist quadratic forms η for the indicated values of n because of Section 5.

The tricky point is to prove that involutions that lie in the same item of the list are equivalent. The idea is to write any φ in a given equivalence class as the representative that we indicated in the list. Let us start with the case (1-a-2), so assume that $T \cong \mathbb{Z}_2^{2m}$, β has type I, $\mathrm{Arf}(\mu) = \mathrm{Arf}(\eta) = +1$, and $\mu \neq \eta$. By Lemmas 14 and 15, there exists $b_1 \in T$ such that $b_1^\perp = \{t \in T \mid \mu(t) = \eta(t)\}$. We

are going to prove that $\mu(b_1) = +1$. Take $c \in T \setminus b_1^\perp$, so $T \cong \langle c \rangle \times b_1^\perp$. Because of Equation (2), μ takes the value -1 either once or three times on $\langle c, b_1 \rangle$. Since η has the same Arf invariant as μ and $T \cong \langle c, b_1 \rangle \times \langle c, b_1 \rangle^\perp$, η takes the value -1 on $\langle c, b_1 \rangle$ as many times as μ. This number cannot be three, because $\mu(c) \neq \eta(c)$, so $\mu(b_1) = \eta(b_1) = +1$.

We can take $a_1 \in T$, and then inductively $a_2, b_2, \ldots, a_m, b_m \in T$ so that $\{a_1, b_1, \ldots, a_m, b_m\}$ is a symplectic basis as defined before Proposition 13 (follow, for example, the arguments in [9, Equation (2.6) on p. 36]). Moreover, since $\mathrm{Arf}(\mu) = +1$ and $\mu(b_1) = +1$, we can argue as in the last paragraph of [13, proof of Theorems 15 and 16] and assume that our symplectic basis satisfies

$$\mu(a_1) = \mu(b_1) = \ldots = \mu(a_m) = \mu(b_m) = +1.$$

By construction, this implies

$$\eta(a_1) = -1 \quad \text{and} \quad \eta(b_1) = \ldots = \eta(a_m) = \eta(b_m) = +1.$$

We have shown that any φ in (1-a-2) can be written as $\varphi_{(1\text{-}a\text{-}2)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$, thus they are all equivalent.

The same reasoning works for (1-a-3), but note that now $\mu(b_1) = \eta(b_1) = -1$. Analogously for (1-b-2), (1-b-3) and (1-c-2), but in this last case we use Lemma 16 instead of Lemma 15, and we may replace b_1 by $b_1 f_\beta$ so that $\mu(b_1) = +1$. In the cases (1-c-3), (1-c-4), (1-d-1), (1-d-2), (1-d-3) and (1-d-4), we cannot apply Lemma 16, but in fact they are easier, because $\mu(f_\beta) = -1$ whereas $\eta(f_\beta) = +1$. We can first pick $a_1, b_1, \ldots, a_m, b_m \in T$ so that η takes the values that we want on them. Then, changing, if necessary, the a_i and b_j to $a_i f_\beta$ and $b_j f_\beta$, we can also select the values taken by μ. For example, in the case (1-c-3), η is a quadratic form on $T \cong \mathbb{Z}_2^{2m+1}$ such that β_η has type II, $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = +1$. This means that η belongs to the item (1-e) of the list of Section 5, hence there exists a basis $\{a_1, b_1, \ldots, a_m, b_m, f_\beta\}$ of T as in Proposition 13 such that $\eta(a_1) = \eta(b_1) = \cdots = \eta(a_m) = \eta(b_m) = 1$ (and $\eta(f_\beta) = 1$). We can assume without loss of generality that also $\mu(a_1) = \mu(b_1) = \cdots = \mu(a_m) = \mu(b_m) = 1$ (and $\mu(f_\beta) = -1$). Therefore any φ can be written as $\varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}c\text{-}3)}$.

Finally, in order to compute the signatures it is enough to apply Lemma 1 to the representative of every equivalence class, since we already calculated the signature of each factor in Example 27. □

Remark 28. Recall from Lemma 10 that, given an involution φ on the graded algebra $\mathcal{D}$, we can obtain the rest of the involutions (of the same kind, if $\mathcal{D} \cong M_n(\mathbb{C})$) as $\mathrm{Int}(X_u) \circ \varphi$, where u runs through T. If η is the quadratic form on T corresponding to φ, then the quadratic form $\eta_u : T \to \{\pm 1\}$ corresponding to $\mathrm{Int}(X_u) \circ \varphi$ is given by $\eta_u(v) = \beta(u, v)\eta(v) = \eta(uv)\eta(u)$. In particular, $\mathrm{Arf}(\eta_u) = \mathrm{Arf}(\eta)\eta(u)$ if the Arf invariant is defined.

7. Classification in the two-dimensional non-complex case

Example 29. Consider the division grading by the group $\mathbb{Z}_2$ on the algebra $M_2(\mathbb{R})$ obtained by coarsening of the grading of Example 25:

$$M_2(\mathbb{R}) = \left[\mathbb{R} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \oplus \left[\mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right].$$

When $M_2(\mathbb{R})$ is endowed with this grading, we denote the involutions of Example 27 as:

- $\varphi_{(2\text{-}a\text{-}1)} = \varphi_{(1\text{-}a\text{-}2)}$;
- $\varphi_{(2\text{-}a\text{-}3)} = \varphi_{(1\text{-}a\text{-}1)}$;
- $\varphi_{(2\text{-}a\text{-}5)} = \varphi_{(1\text{-}a\text{-}3)}$.

Analogously, if $\mathbb{H}$ is $\mathbb{Z}_2$-graded as $\mathbb{H} = [\mathbb{R}1 \oplus \mathbb{R}i] \oplus [\mathbb{R}j \oplus \mathbb{R}k]$, we denote:

- $\varphi_{(2\text{-b-}2)} = \varphi_{(1\text{-b-}3)}$;
- $\varphi_{(2\text{-b-}3)} = \varphi_{(1\text{-b-}1)}$;
- we also denote $\varphi_{(2\text{-b-}5)}$ the involution on $\mathbb{H}$ that acts as the identity on 1, j and k, and acts as minus the identity on i; it is an orthogonal involution.

Finally, consider the division grading by the group $\mathbb{Z}_4$ on the algebra $M_2(\mathbb{C})$ obtained by the coarsening of the grading of Figure 1 that joins, for all $t \in \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, the homogeneous component of degree t to the homogeneous component of degree ab^2t. When $M_2(\mathbb{C})$ is endowed with this grading, we denote the involutions of Example 27 as:

- $\varphi_{(2\text{-e-}1)} = \varphi_{(1\text{-d-}1)}$;
- $\varphi_{(2\text{-e-}3)} = \varphi_{(1\text{-d-}3)}$;
- $\varphi_{(2\text{-e-}4)} = \varphi_{(1\text{-d-}4)}$.

Let G be an abelian group, $\mathcal{D}$ a finite-dimensional simple real (associative) algebra, and Γ a division G-grading on $\mathcal{D}$ with homogeneous components of dimension 2 such that the identity component does not coincide with the center of $\mathcal{D}$. These gradings are classified in [13, Theorems 22 and 23]; there are five families of equivalence classes: (2-a), (2-b), (2-c), (2-d) and (2-e). We keep the same notation, so write $\mathcal{D}_e = \mathbb{R}I \oplus \mathbb{R}J$ ($\cong \mathbb{C}$), where I is the unity of $\mathcal{D}$ and $J^2 = -I$; and let T be the support of Γ, K the support of the centralizer of the identity component, and $\beta : K \times K \to \{\pm 1\}$ the alternating bicharacter given by the commutation relations in the centralizer of the identity component.

Then, for any antiautomorphism φ of the G-graded algebra $\mathcal{D}$, either $\varphi(J) = +J$ or $\varphi(J) = -J$. We want to classify the pairs (Γ, φ), up to isomorphism and up to equivalence, when φ is an involution. In the case $\varphi(J) = +J$, any antiautomorphism is an involution, and there is exactly one proper refinement of Γ compatible with a given involution; the isomorphism classes are in bijective correspondence with the quadratic forms η on K such that $\beta_\eta = \beta$ (and, in the case (2-e), $\eta(f_T) = +1$) by means of the equation:

$$\varphi(X_k) = \eta(k)X_k \tag{8}$$

for all $k \in K$ and $X_k \in \mathcal{D}_k$. In the case $\varphi(J) = -J$, there are antiautomorphisms that are not involutions, but any proper refinement of Γ is compatible with a given involution; the isomorphism classes of involutions are in bijective correspondence with the nice maps ω on $T \backslash K$ such that $\beta_\omega = \beta$ (and, in the case (2-e), $\omega(f_T) = +1$) by means of the equation:

$$\varphi(X_t) = \omega(t)X_t \tag{9}$$

for all $t \in T \setminus K$ and $X_t \in \mathcal{D}_t$. Now we give a list of the equivalence classes together with a representative of every class, and we also compile the classification up to isomorphism to serve as a reference:

(2-a) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{R})$ ($n = 2^m \geq 2$) was determined up to isomorphism by (T, K, ν), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1}$, K was a subgroup of T of index 2, and ν was a nice map on $T \setminus K$ such that $\beta := \beta_\nu$ had type I and $\mathrm{Arf}(\nu) = +1$. Now, in the case $\varphi(J) = +J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, η), where η is a quadratic form on K such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following two equivalence classes:

(1) $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 2$).
   The involution φ is orthogonal with signature 0.
   A representative is $\varphi_{(2\text{-a-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.
(2) $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
   The involution φ is symplectic.
   A representative is $\varphi_{(2\text{-a-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}3)}$.

On the other hand, in the case $\varphi(J) = -J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, ω), where ω is a nice map on $T \backslash K$ such that $\beta_\omega = \beta$. These isomorphism classes belong to one of the following four equivalence classes:

(3) $\omega = \nu$ $(n = 2^m \geq 2)$.
    The involution φ is orthogonal with signature n.
    A representative is $\varphi_{(2\text{-a-}3)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.
(4) $\mathrm{Arf}(\omega) = +1$ but $\omega \neq \nu$ $(n = 2^m \geq 4)$.
    The involution φ is orthogonal with signature 0.
    A representative is $\varphi_{(2\text{-a-}3)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}2)}$.
(5) $\omega = -\nu$ $(n = 2^m \geq 2)$.
    The involution φ is symplectic.
    A representative is $\varphi_{(2\text{-a-}5)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.
(6) $\mathrm{Arf}(\omega) = -1$ but $\omega \neq -\nu$ $(n = 2^m \geq 4)$.
    The involution φ is symplectic.
    A representative is $\varphi_{(2\text{-a-}5)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}2)}$.

(2-b) The grading Γ on $\mathcal{D} \cong M_{n/2}(\mathbb{H})$ $(n = 2^m \geq 2)$ was determined up to isomorphism by (T, K, ν), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1}$, K was a subgroup of T of index 2, and ν was a nice map on $T \setminus K$ such that $\beta := \beta_\nu$ had type I and $\mathrm{Arf}(\nu) = -1$. Now, in the case $\varphi(J) = +J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, η), where η is a quadratic form on K such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following two equivalence classes:

(1) $\mathrm{Arf}(\eta) = -1$ $(n = 2^m \geq 4)$.
    The involution φ is symplectic with signature 0.
    A representative is $\varphi_{(2\text{-b-}2)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}3)}$.
(2) $\mathrm{Arf}(\eta) = +1$ $(n = 2^m \geq 2)$.
    The involution φ is orthogonal.
    A representative is $\varphi_{(2\text{-b-}2)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.

On the other hand, in the case $\varphi(J) = -J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, ω), where ω is a nice map on $T \backslash K$ such that $\beta_\omega = \beta$. These isomorphism classes belong to one of the following four equivalence classes:

(3) $\omega = \nu$ $(n = 2^m \geq 2)$.
    The involution φ is symplectic with signature $n/2$.
    A representative is $\varphi_{(2\text{-b-}3)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.
(4) $\mathrm{Arf}(\omega) = -1$ but $\omega \neq \nu$ $(n = 2^m \geq 4)$.
    The involution φ is symplectic with signature 0.
    A representative is $\varphi_{(2\text{-b-}3)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}2)}$.
(5) $\omega = -\nu$ $(n = 2^m \geq 2)$.
    The involution φ is orthogonal.
    A representative is $\varphi_{(2\text{-b-}5)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.
(6) $\mathrm{Arf}(\omega) = +1$ but $\omega \neq -\nu$ $(n = 2^m \geq 4)$.
    The involution φ is orthogonal.
    A representative is $\varphi_{(2\text{-b-}5)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}2)}$.

(2-c) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ $(n = 2^m \geq 2)$ was determined up to isomorphism by (T, K, ν), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m}$, K was a subgroup of T of index 2, and ν was a nice map on $T \setminus K$ such that $\beta := \beta_\nu$ had type II and $\nu(f_\beta) = -1$. Now, in the case $\varphi(J) = +J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, η), where η is a quadratic form on K such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following three equivalence classes:

(1) $\eta(f_\beta) = -1$ $(n = 2^m \geq 2)$.
   The involution φ is of the second kind and has signature 0.
   A representative is $\varphi_{(2\text{-a-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-1})}$.
(2) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = +1$ $(n = 2^m \geq 2)$.
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(2\text{-a-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-3})}$.
(3) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = -1$ $(n = 2^m \geq 4)$.
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(2\text{-a-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-3})} \otimes \varphi_{(1\text{-c-3})}$.

On the other hand, in the case $\varphi(J) = -J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, ω), where ω is a nice map on $T \backslash K$ such that $\beta_\omega = \beta$. These isomorphism classes belong to one of the following five equivalence classes:

(4) $\omega = \nu$ $(n = 2^m \geq 2)$.
   The involution φ is of the second kind and has signature n.
   A representative is $\varphi_{(2\text{-a-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-1})}$.
(5) $\omega = -\nu$ $(n = 2^m \geq 2)$.
   The involution φ is of the second kind and has signature 0.
   A representative is $\varphi_{(2\text{-a-5})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-1})}$.
(6) $\omega(f_\beta) = -1$ but $\omega \neq \pm\nu$ $(n = 2^m \geq 4)$.
   The involution φ is of the second kind and has signature 0.
   A representative is $\varphi_{(2\text{-a-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-2})} \otimes \varphi_{(1\text{-c-1})}$.
(7) $\omega(f_\beta) = +1$ and $\mathrm{Arf}(\omega) = +1$ $(n = 2^m \geq 2)$.
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(2\text{-a-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-3})}$.
(8) $\omega(f_\beta) = +1$ and $\mathrm{Arf}(\omega) = -1$ $(n = 2^m \geq 2)$.
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(2\text{-a-5})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-3})}$.

(2-d) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ $(n = 2^m \geq 4)$ was determined up to isomorphism by (T, K, β, ν), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-2} \times \mathbb{Z}_4$, K was a subgroup of T of index 2 but different from $T_{[2]}$, β was an alternating bicharacter on K of type II, and ν was a nice map on $T_{[2]} \setminus K_{[2]}$ such that $\beta_\nu = \beta|_{K_{[2]} \times K_{[2]}}$ and $\nu(f_T) = -1$. Now, in the case $\varphi(J) = +J$, (Γ, φ) is determined up to isomorphism by (T, K, β, ν, η), where η is a quadratic form on K such that $\beta_\eta = \beta$ (so $\eta(f_T) = +1$). These isomorphism classes belong to one of the following four equivalence classes:

(1) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = +1$ $(n = 2^m \geq 4)$.
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(2\text{-a-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-1})}$.
(2) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = -1$ $(n = 2^m \geq 8)$.
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(2\text{-a-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-3})} \otimes \varphi_{(1\text{-d-1})}$.
(3) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = +1$ $(n = 2^m \geq 4)$.
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(2\text{-a-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-3})}$.
(4) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = -1$ $(n = 2^m \geq 4)$.
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(2\text{-a-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-4})}$.

On the other hand, in the case $\varphi(J) = -J$, (Γ, φ) is determined up to isomorphism by $(T, K, \beta, \nu, \omega)$, where ω is a nice map on $T \backslash K$ such that $\beta_\omega = \beta$ (so $\omega(f_T) = +1$). These isomorphism classes belong to one of the following four equivalence classes:

(5) $\omega(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\omega) = +1$ ($n = 2^m \geq 4$).
The involution φ is of the first kind and orthogonal.
A representative is $\varphi_{(2\text{-a-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-1})}$.

(6) $\omega(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\omega) = -1$ ($n = 2^m \geq 8$).
The involution φ is of the first kind and symplectic.
A representative is $\varphi_{(2\text{-a-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-3})} \otimes \varphi_{(1\text{-d-1})}$.

(7) $\omega(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\omega) = +1$ ($n = 2^m \geq 4$).
The involution φ is of the first kind and orthogonal.
A representative is $\varphi_{(2\text{-a-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-3})}$.

(8) $\omega(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\omega) = -1$ ($n = 2^m \geq 4$).
The involution φ is of the first kind and symplectic.
A representative is $\varphi_{(2\text{-a-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-4})}$.

(2-e) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ ($n = 2^m \geq 2$) was determined up to isomorphism by $(T, [\nu])$, where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-2} \times \mathbb{Z}_4$ ($K = T_{[2]}$), and $[\nu]$ was an equivalence class of nice maps ν on $T \setminus T_{[2]}$ such that $\beta := \beta_\nu$ had type II, $f_\beta = f_T$ and $\nu(f_T) = -1$, with the equivalence relation $\nu \sim \nu'$ if either $\nu' = \nu$ or $\nu' = -\nu$. Now, in the case $\varphi(J) = +J$, (Γ, φ) is determined up to isomorphism by $(T, [\nu], \eta)$, where η is a quadratic form on $T_{[2]}$ such that $\beta_\eta = \beta$ and $\eta(f_T) = +1$. These isomorphism classes belong to one of the following two equivalence classes:

(1) $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 2$).
The involution φ is of the first kind and orthogonal.
A representative is $\varphi_{(2\text{-e-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.

(2) $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
The involution φ is of the first kind and symplectic.
A representative is $\varphi_{(2\text{-e-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-3})}$.

On the other hand, in the case $\varphi(J) = -J$, (Γ, φ) is determined up to isomorphism by $(T, [\nu], \omega)$, where ω is a nice map on $T \setminus T_{[2]}$ such that $\beta_\omega = \beta$ and $\omega(f_T) = +1$. These isomorphism classes belong to one of the following two equivalence classes:

(3) $\mathrm{Arf}(\omega) = +1$ ($n = 2^m \geq 2$).
The involution φ is of the first kind and orthogonal.
A representative is $\varphi_{(2\text{-e-3})} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.

(4) $\mathrm{Arf}(\omega) = -1$ ($n = 2^m \geq 2$).
The involution φ is of the first kind and symplectic.
A representative is $\varphi_{(2\text{-e-4})} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.

Proof. Let $g \in T \setminus K$. We start with the case (2-a). We know from [13, Theorem 22] that we can write $\mathcal{D}$ as follows:

$$(\mathcal{D}_e \oplus \mathcal{D}_g) \otimes_{\mathbb{R}} C_{\mathcal{D}}(\mathcal{D}_e \oplus \mathcal{D}_g). \tag{10}$$

Recall that $C_{\mathcal{D}}(\mathcal{D}_e \oplus \mathcal{D}_g)$ is a subalgebra isomorphic to $M_{n/2}(\mathbb{R})$ or $M_{n/4}(\mathbb{H})$, endowed with a division grading whose homogeneous components have dimension 1. Since $\varphi(\mathcal{D}_e \oplus \mathcal{D}_g) = \mathcal{D}_e \oplus \mathcal{D}_g$, also $\varphi(C_{\mathcal{D}}(\mathcal{D}_e \oplus \mathcal{D}_g)) = C_{\mathcal{D}}(\mathcal{D}_e \oplus \mathcal{D}_g)$. Therefore we have reduced the problem to the study of antiautomorphisms on $\mathcal{D}_e \oplus \mathcal{D}_g$, which is isomorphic either to $M_2(\mathbb{R})$ if $\nu(g) = +1$, or to $\mathbb{H}$ if $\nu(g) = -1$.

If $\varphi(J) = +J$, then φ is $\mathcal{D}_e$-semilinear on $\mathcal{D}_g$, hence there exists $X \in \mathcal{D}_g$ such that $\varphi(X) = X$ (and $\varphi(JX) = -JX$). Therefore, φ is an involution and it is only compatible with the proper refinement that splits $\mathcal{D}_g$ as $\mathbb{R}X \oplus \mathbb{R}JX$. It is straightforward to check the assertions about the isomorphisms classes. Let us see that involutions that lie in the same item of the list are equivalent. If $n \geq 4$, we can always choose g such that $\nu(g) = +1$ and the quadratic form $\mu_g(k) := \nu(gk)\nu(g)^{-1}$ is different from η, so we can write any involution in (2-a-1) (respectively (2-a-2))

as the tensor product of an involution on $M_2(\mathbb{R})$ and an involution on $M_{n/2}(\mathbb{R})$ that lies in (1-a-2) (respectively (1-a-3)), hence they are all equivalent.

If $\varphi(J) = -J$, then $\varphi|_{\mathcal{D}_g} = \lambda \, \mathrm{id}_{\mathcal{D}_g}$, where $\lambda \in \mathcal{D}_e$. Therefore, φ is an involution if and only if $\lambda = \pm 1$ and, in that case, every refinement is compatible with φ. Again, we can always choose g such that $\nu(g) = +1$ and $\omega(g) = +1$ (respectively $\omega(g) = -1$), so we can write any involution in (2-a-4) (respectively (2-a-6)) as the tensor product of an involution on $M_2(\mathbb{R})$ and an involution on $M_{n/2}(\mathbb{R})$ that lies in (1-a-2), hence they are all equivalent.

The same arguments work for (2-b) and (2-c), and also for the case (2-d), which is, in fact, easier because there is no distinguished involution.

Let us now consider the remaining case (2-e). Any proper refinement of the grading has to split $\mathcal{D}_k$, for all $k \in T_{[2]}$, as $\mathcal{D}_k^+ \oplus \mathcal{D}_k^-$, where the squares of the elements in $\mathcal{D}_k^+$ (respectively $\mathcal{D}_k^-$) are positive (respectively negative) multiples of I. Also recall from [13, Remark 21] that, if $X \in \mathcal{D}_g$, then there exists a proper refinement of the grading such that the element X is still homogeneous; this implies that $\mathcal{D}_{gk}$ splits as $X\mathcal{D}_k^+ \oplus X\mathcal{D}_k^-$ for all $k \in T_{[2]}$.

We have $\varphi(\mathcal{D}_k^+) = \mathcal{D}_k^+$ and $\varphi(\mathcal{D}_k^-) = \mathcal{D}_k^-$ for all $k \in T_{[2]}$. Assume that $\varphi(J) = +J$. As before, φ is $\mathcal{D}_e$-semilinear on $\mathcal{D}_g$ and there exists $X \in \mathcal{D}_g$ such that $\varphi(X) = X$ (and $\varphi(JX) = -JX$). This implies that $\varphi(X\mathcal{D}_k^+) = X\mathcal{D}_k^+$ and $\varphi(X\mathcal{D}_k^-) = X\mathcal{D}_k^-$ for all $k \in T_{[2]}$, that is, φ is an involution and there is exactly one proper refinement compatible with φ. Assume that $\varphi(J) = -J$. Then $\varphi|_{\mathcal{D}_g} = \lambda \, \mathrm{id}_{\mathcal{D}_g}$, where $\lambda \in \mathcal{D}_e$, thus $\varphi|_{\mathcal{D}_{gk}} = \pm\lambda \, \mathrm{id}_{\mathcal{D}_{gk}}$ for all $k \in T_{[2]}$. Therefore φ is an involution if and only if $\lambda = \pm 1$, and, in that case, every refinement is compatible with φ.

Now that we know that every involution is compatible with at least one proper refinement, we can use this fact to prove the rest of the assertions of the theorem (see Remark 19). Unlike in the previous cases, in (2-e), if ψ is any isomorphism or equivalence between two refinements with supports $\langle h_1 \rangle \times T_1$ and $\langle h_2 \rangle \times T_2$, then ψ will continue to be an isomorphism or equivalence with respect to the original gradings, with supports T_1 and T_2. Indeed, ψ has to send h_1 to h_2, because they are distinguished elements.

Finally, the computation of signature of φ can be done similarly to Section 6 or, alternatively, we can take a compatible refinement and see the signature of the corresponding isomorphism class already on the list of Section 6. $\qquad \square$

8. Classification in the two-dimensional complex case

Example 30. Let $\varepsilon = e^{2\pi i/l} \in \mathbb{C}$ and consider the generalized Pauli Matrices $X_a, X_b \in M_l(\mathbb{C})$ of Figure 2. Note that

$$X_a X_b = \varepsilon X_b X_a \quad \text{and} \quad X_a^l = X_b^l = I.$$

Therefore, we can construct a division grading on $M_l(\mathbb{C})$ by the group $\mathbb{Z}_l \times \mathbb{Z}_l$ if we define the homogeneous component of degree $(\bar{j}, \bar{k})$ to be $\mathbb{C}X_a^j X_b^k$. Let φ_A and φ_B be the second kind antiautomorphisms on $M_l(\mathbb{C})$ given by $\varphi_A(X) = A^{-1}X^*A$ and $\varphi_B(X) = B^{-1}X^*B$, where $X^* := \overline{X^T}$ and the matrices $A, B \in M_l(\mathbb{C})$ are those of Figure 2. Since $A^* = A$ and $B^* = B$, both φ_A and φ_B are involutions. The signatures of φ_A and φ_B are, respectively, 2 and 0 if l is even, and 1 and 1 if l is odd. Both involutions respect the grading because:

$$\varphi_A(X_a) = X_a, \quad \varphi_A(X_b) = X_b; \qquad \varphi_B(X_a) = \varepsilon X_a, \quad \varphi_B(X_b) = X_b.$$

We will write:

- $\varphi_l = \varphi_A$,
- $\phi_l = \varphi_B$.

$$X_a = \begin{pmatrix} \varepsilon^{l-1} & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon^{l-2} & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^{l-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \qquad X_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

FIGURE 2. Matrices in $M_l(\mathbb{C})$ of Example 30; $\varepsilon = e^{2\pi i/l}$.

Example 31. We introduce two involutions on $M_2(\mathbb{C})$ that respect the division grading induced by the Pauli matrices in the case $l = 2$:

- Let $\varphi_{(2\text{-f-1-1})}$ be the matrix transpose on $M_2(\mathbb{C})$. It is an involution of the first kind and orthogonal.
- Let $\varphi_{(2\text{-f-1-2})}$ be the involution on $M_2(\mathbb{C})$ that acts as minus the identity on the matrices of trace zero, and acts as the identity on the center of $M_2(\mathbb{C})$. It is an involution of the first kind and symplectic.

Let G be an abelian group, $\mathcal{D}$ a real algebra isomorphic to $M_n(\mathbb{C})$, and Γ a division G-grading on $\mathcal{D}$ with homogeneous components of dimension 2 such that the identity component coincides with the center of $\mathcal{D}$. Γ can be regarded as a grading of the complex algebra $M_n(\mathbb{C})$, and these gradings are classified in [9, Theorem 2.15]; there is one family of equivalence classes: (2-f). The isomorphism and equivalence classes in this classification remain the same over $\mathbb{R}$, because the invariants that differentiate them, namely, the pair (T, β) and the isomorphism class of T respectively, are also preserved by isomorphisms of real algebras. As always, T is the support of Γ and β is the alternating bicharacter given by the commutation relations, $X_u X_v = \beta(u, v) X_v X_u$ (where $0 \neq X_t \in \mathcal{D}_t$ for all $t \in T$), but, in contrast with Section 6, β is now $\mathbb{C}$-valued.

Any antiautomorphism φ of the G-graded algebra $\mathcal{D}$ is an involution, and satisfies either $\varphi(iI) = +iI$ or $\varphi(iI) = -iI$ (where I is the unity of $\mathcal{D}$). We classify the pairs (Γ, φ), up to isomorphism and up to equivalence, and we give a representative of every equivalence class:

(2-f) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ $(n \geq 1)$ was determined up to isomorphism by (T, β), where T was a subgroup of G isomorphic to $\mathbb{Z}_{l_1}^2 \times \ldots \times \mathbb{Z}_{l_r}^2$ $(l_1 \cdots l_r = n)$, and β was a $\mathbb{C}$-valued alternating bicharacter on T such that $\operatorname{rad}(\beta) = \{e\}$. The equivalence class of the grading Γ was determined by the isomorphism class of the group T, which we fix henceforth. Now, in the case $\varphi(iI) = +iI$, we have $l_1 = \ldots = l_r = 2$ (so β takes values in $\{\pm 1\} \subseteq \mathbb{R}^\times$) and (Γ, φ) is determined up to isomorphism by (T, β, η), where η is a quadratic form on T such that $\beta_\eta = \beta$, and η is defined by the equation

$$\varphi(X_t) = \eta(t) X_t \tag{11}$$

for all $X_t \in \mathcal{D}_t$. These isomorphism classes belong to one of the following two equivalence classes:

(1-1) $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 1$).

The involution φ is of the first kind and orthogonal.

A representative is $\varphi_{(\text{2-f-1-1})} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \varphi_{(\text{2-f-1-1})}$ (if $n = 1$, we take $\varphi = \mathrm{id}_{\mathbb{C}}$).

(1-2) $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 2$).

The involution φ is of the first kind and symplectic.

A representative is $\varphi_{(\text{2-f-1-1})} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \varphi_{(\text{2-f-1-1})} \otimes_{\mathbb{C}} \varphi_{(\text{2-f-1-2})}$.

On the other hand, in the case $\varphi(iI) = -iI$, (Γ, φ) is determined up to isomorphism by (T, β, S), where S is a subgroup of $T_{[2]}$ of index 1 or 2, and S is defined as

$$S := \{t \in T_{[2]} \mid \exists X \in \mathcal{D}_t \text{ such that } X^2 = +I \text{ and } \varphi(X) = X\}. \qquad (12)$$

These isomorphism classes belong to one of the following equivalence classes:

(2-0) $S = T_{[2]}$ ($n \geq 1$).

The involution φ is of the second kind and has signature $\sqrt{|T_{[2]}|}$.

A representative is $\varphi_{l_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \varphi_{l_r}$.

(2-p) $S \neq T_{[2]}$, and any $t \in T$ of order 2^{p+1} satisfies $t^{2^p} \in S$, but there exists $t \in T$ of order 2^p such that $t^{2^{p-1}} \in T_{[2]} \setminus S$ ($p \geq 1$).

The involution φ is of the second kind and has signature 0.

A representative is $\phi_{l_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \phi_{l_s} \otimes_{\mathbb{C}} \varphi_{l_{s+1}} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \varphi_{l_r}$, provided that $l_1 \mid l_2 \mid \cdots \mid l_r$ and 2^{p+1} divides l_{s+1} but does not divide l_s.

Proof. The case $\varphi(iI) = +iI$ was proved in [9, Propositions 2.51 and 2.53]. So assume that $\varphi(iI) = -iI$. For any element $t \in T$, denote its order by $o(t)$ and define:

$$\mathcal{D}_t^{[+]} := \{X \in \mathcal{D}_t \mid X^{o(t)} = +I\} \quad \text{and} \quad \mathcal{D}_t^{[-]} := \{X \in \mathcal{D}_t \mid X^{o(t)} = -I\}. \qquad (13)$$

Note that, if $X \in \mathcal{D}_t^{[+]}$ and $\varepsilon \in \mathbb{C}$ is a primitive $o(t)$-th root of unity, then $\mathcal{D}_t^{[+]} = \{X, \varepsilon X, \ldots, \varepsilon^{o(t)-1} X\}$, and similarly for $\mathcal{D}_t^{[-]}$. Besides, $\varphi(\mathcal{D}_t^{[+]}) = \mathcal{D}_t^{[+]}$ and $\varphi(\mathcal{D}_t^{[-]}) = \mathcal{D}_t^{[-]}$; in particular, φ is an involution. We define the following subsets of T, which are invariants of the isomorphism class of (Γ, φ):

$$S' := \{t \in T \mid \exists X \in \mathcal{D}_t^{[+]} \text{ such that } \varphi(X) = X\} \quad \text{and} \quad S := S' \cap T_{[2]}. \qquad (14)$$

If $o(t)$ is odd, then $t \in S'$, while if $o(t)$ is even, then $t \in T \setminus S'$ if and only if there exists $X \in \mathcal{D}_t^{[-]}$ such that $\varphi(X) = X$.

Write T as $U \times V$, where U is the subgroup of T formed by the elements whose order is a power of 2, and V is the subgroup of T formed by the elements of odd order. We know that $V \subseteq S'$. Moreover, if $u \in U$ and $v \in V$, then $u \in S'$ if and only if $uv \in S'$, because $\beta(u, v) = 1$. Finally, if $u \in U \setminus T_{[2]}$, then $u \in S'$ if and only if $u^2 \in S'$. Therefore, S determines S'.

The restriction of β to $T_{[2]} \times T_{[2]}$ takes values in $\{\pm 1\}$. Hence, $u, v \in S$ implies $uv \in S$, and also $u, v \in T_{[2]} \setminus S$ implies $uv \in S$. Therefore, S is a subgroup of $T_{[2]}$ of index 1 or 2.

We know, for example from [9, Equation (2.6)], that we can write T as follows:

$$T = \langle a_1 \rangle \times \langle b_1 \rangle \times \ldots \times \langle a_r \rangle \times \langle b_r \rangle, \qquad (15)$$

where $a_i, b_i \in T$, $\langle a_i \rangle \times \langle b_i \rangle \cong \mathbb{Z}_{l_i}^2$, l_i is a power of a prime, $\beta(a_i, b_i) = \beta(b_i, a_i)^{-1} = e^{2\pi i/l_i}$, and the value of β on all other pairs is 1. We claim that (T, β, S) determines (Γ, φ) up to isomorphism. We can pick $X_{a_i} \in \mathcal{D}_{a_i}$ such that $\varphi(X_{a_i}) = X_{a_i}$, and either $X_{a_i} \in \mathcal{D}_{a_i}^{[+]}$ if $a_i \in S'$, or $X_{a_i} \in \mathcal{D}_{a_i}^{[-]}$ if $a_i \in T \setminus S'$. We pick $X_{b_i} \in \mathcal{D}_{b_i}$ in the same way. The elements X_{a_i}, X_{b_i} generate $\mathcal{D}$, with defining relations of two kinds: the commutation relations are determined by β and the power relations are determined by S through Equation (13). This proves the claim; in fact, the isomorphism can be chosen to be an isomorphism of complex algebras. Conversely,

let us find an involution φ for a given subgroup S of $T_{[2]}$ of index 1 or 2. Thanks to Lemma 2, it is enough to construct it for every factor $\langle a_i \rangle \times \langle b_i \rangle$ of T, but we have already done it in Example 30.

Let us see that, for a fixed $p \geq 1$, all the involutions that lie in (2-f-2-p) are equivalent. In fact, we will show that $a_1, b_1, \ldots, a_r, b_r$ in Equation (15) may be chosen so that they also satisfy: $a_i, b_i \in S'$ for all i, except in the case $l_i = 2^{m_i} \leq 2^p$, when $a_i \in S'$ but $b_i \in T \setminus S'$. We can follow the same induction process as the one leading to [9, Equation (2.6)], until we arrive to a situation in which T is a 2-group and there are elements in T of maximal order, 2^p, that do not belong to S'. Rearranging, we may assume that $l_1 = 2^{m_1} \geq l_2 = 2^{m_2} \geq \ldots \geq l_r = 2^{m_r}$. If $r = 1$, the statement is clear, so suppose that $r \geq 2$. Then we choose the next a, b in the following way.

We want to take $a, b \in T$ such that $o(a) = o(b) = 2^{m_1}$, $\beta(a, b) = e^{2\pi i / l_1}$, $a \in S'$, $b \in T \setminus S'$, and such that there are elements in $\langle a, b \rangle^\perp$ of maximal order, 2^{m_2}, that do not belong to S', because then $T = \langle a, b \rangle \times \langle a, b \rangle^\perp$ and we will be able to continue the induction process with $\langle a, b \rangle^\perp$. We know the existence of a decomposition $T = \langle \tilde{a}_1 \rangle \times \langle \tilde{b}_1 \rangle \times \ldots \times \langle \tilde{a}_r \rangle \times \langle \tilde{b}_r \rangle$ as in Equation (15), but we cannot assure that $\tilde{a}_i \in S'$ and $\tilde{b}_i \in T \setminus S'$. Without loss of generality, $\tilde{a}_1, \tilde{a}_2 \in S'$ and $\tilde{b}_1 \in T \setminus S'$, hence $\tilde{a}_1 \tilde{a}_2 \in S'$. If $\tilde{b}_2 \in T \setminus S'$, simply take $a = \tilde{a}_1$ and $b = \tilde{b}_1$. If $\tilde{b}_2 \in S'$, take $a = \tilde{a}_1 \tilde{a}_2$ and $b = \tilde{b}_1$, and note that $\tilde{b}_1^{l_1/l_2} \tilde{b}_2^{-1}$ has order 2^{m_2} and belongs both to $\langle a, b \rangle^\perp$ and to $T \setminus S'$.

Finally, the computation of signature is analogous to Section 6, but using Example 30 instead of Example 25 and Lemma 2 instead of Lemma 1. For involutions in (2-f-2-p), we pick up a zero factor. For involutions in (2-f-2-0), we may assume that $l_1, \ldots, l_s$ are even and $l_{s+1}, \ldots, l_r$ are odd, then s is the number of factors 2, so the signature equals $2^s = \sqrt{|T_{[2]}|}$. $\qquad\square$

Remark 32. Consider an involution φ of the second kind on the graded algebra $\mathcal{D}$. By Lemma 10, all such involutions can be obtained as $\mathrm{Int}(X_u) \circ \varphi$ where u runs through T. Since β is nondegenerate, it is easy to see that $u \in T^{[2]}$ (recall Notation 17) if and only if $\beta(u, v) = 1$ for all $v \in T_{[2]}$. Therefore, $\mathrm{Int}(X_u) \circ \varphi$ and φ are in the same isomorphism class if and only if $u \in T^{[2]}$, because $(\mathrm{Int}(X_u) \circ \varphi)(X_v) = \beta(u, v)\varphi(X_v)$. Now assume that φ lies in (2-f-2-0). We have just shown that $\mathrm{Int}(X_u) \circ \varphi$ lies in (2-f-2-0) if and only if u is a square in T. Now we claim that $\mathrm{Int}(X_u) \circ \varphi$ lies in (2-f-2-p) ($p \geq 1$) if and only if $uT_{[2^p]}$ is a square in $T/T_{[2^p]}$ but $uT_{[2^{p-1}]}$ is not a square in $T/T_{[2^{p-1}]}$. Indeed, using the nondegeneracy of β (or explicitly using its values on the pairs of generators in Equation (15)), it is straightforward to show that $uT_{[2^p]}$ is a square in $T/T_{[2^p]}$ if and only if $\beta(u, v^{2^p}) = 1$ for all $v \in T_{[2^{p+1}]}$.

9. Classification in the four-dimensional case

Let G be an abelian group, $\mathcal{D}$ a finite-dimensional simple real (associative) algebra, and Γ a division G-grading on $\mathcal{D}$ with homogeneous components of dimension 4. We can apply the Double Centralizer Theorem (see for example [10, Theorem 4.7]) to the identity component $\mathcal{D}_e$, which is isomorphic to $\mathbb{H}$, to conclude that $\mathcal{D}$ is isomorphic, as a graded algebra, to $\mathcal{D}_e \otimes_\mathbb{R} C_\mathcal{D}(\mathcal{D}_e)$ (see [13, Theorem 19] for more details). Note that $C_\mathcal{D}(\mathcal{D}_e)$ is again a finite-dimensional simple real graded-division algebra, but with homogeneous components of dimension 1. Any antiautomorphism φ of the G-graded algebra $\mathcal{D}$ is the tensor product of its restrictions to $\mathcal{D}_e$ and to $C_\mathcal{D}(\mathcal{D}_e)$.

The following result is well known and easily follows from Skolem–Noether Theorem.

Proposition 33. *Any antiautomorphism φ of the real algebra $\mathbb{H}$ can be written as $\varphi(X) = A^{-1}\overline{X}A$, for some $A = a + bi + cj + dk \in \mathbb{H}^{\times}$. So φ is an involution if and only if either $b = c = d = 0$ or $a = 0$. In the first case φ is symplectic with signature 1 (φ is the conjugation), while in the second φ is orthogonal.* $\qquad\square$

For us, this means the following: if Γ is the trivial grading on $\mathbb{H}$, then there are exactly two isomorphism classes of pairs (Γ, φ), where φ is an involution, and they coincide with the equivalence classes. When $\mathbb{H}$ is endowed with the trivial grading, we denote:

- Let $\varphi_{(3\text{-b-1})}$ be the conjugation on $\mathbb{H}$, that is, $\varphi_{(3\text{-b-1})} = \varphi_{(1\text{-b-1})}$.
- Let $\varphi_{(3\text{-b-4})}$ be a representative of the orthogonal equivalence class, for example, $\varphi_{(3\text{-b-4})} = \varphi_{(1\text{-b-3})}$.

Now, the classification of pairs (Γ, φ), where Γ is a division grading on $\mathcal{D}$ as above and φ is an involution, is easily obtained from Proposition 33 and Section 6. Therefore, the isomorphism classes are in two-to-one correspondence with the quadratic forms η on T such that $\beta_\eta = \beta$, where T is the support of Γ and β is the alternating bicharacter on T given by the commutation relations in the centralizer of the identity component. Each quadratic form η corresponds to two isomorphism classes, one for every class of involutions on $\mathcal{D}_e \cong \mathbb{H}$, by means of the equation:

$$\varphi(X_t) = \eta(t)X_t \tag{16}$$

for all $X_t \in \mathcal{D}_t \cap C_{\mathcal{D}}(\mathcal{D}_e)$. Note that, if we have to compute the signature of φ, we take a compatible refinement and check the signature of the corresponding isomorphism class in the list of Section 6. We compile the classification, up to isomorphism and up to equivalence, and we give a representative of every equivalence class, to serve as a reference:

(3-a) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{R})$ ($n = 2^m \geq 4$) was determined up to isomorphism by (T, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-2}$, and μ was a quadratic form on T such that β_μ had type I and $\text{Arf}(\mu) = -1$. Now, if φ is the conjugation on $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta_\mu$. These isomorphism classes belong to one of the following three equivalence classes:

(1) $\eta = \mu$ ($n = 2^m \geq 4$).
    The involution φ is orthogonal with signature n.
    A representative is $\varphi_{(3\text{-b-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-b-1})}$.
(2) $\text{Arf}(\eta) = -1$ but $\eta \neq \mu$ ($n = 2^m \geq 8$).
    The involution φ is orthogonal with signature 0.
    A representative is $\varphi_{(3\text{-b-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-2})} \otimes \varphi_{(1\text{-b-1})}$.
(3) $\text{Arf}(\eta) = +1$ ($n = 2^m \geq 4$).
    The involution φ is symplectic.
    A representative is $\varphi_{(3\text{-b-1})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-b-3})}$.

On the other hand, if φ is orthogonal on $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, μ, η), where again η is a quadratic form on T such that $\beta_\eta = \beta_\mu$. These isomorphism classes belong to one of the following three equivalence classes:

(4) $\eta = \mu$ ($n = 2^m \geq 4$).
    The involution φ is symplectic.
    A representative is $\varphi_{(3\text{-b-4})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-b-1})}$.
(5) $\text{Arf}(\eta) = -1$ but $\eta \neq \mu$ ($n = 2^m \geq 8$).
    The involution φ is symplectic.
    A representative is $\varphi_{(3\text{-b-4})} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-2})} \otimes \varphi_{(1\text{-b-1})}$.

(6) $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 4$).
   The involution φ is orthogonal with signature 0.
   A representative is $\varphi_{(3\text{-}b\text{-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}b\text{-}3)}$.

(3-b) The grading Γ on $\mathcal{D} \cong M_{n/2}(\mathbb{H})$ ($n = 2^m \geq 2$) was determined up to isomorphism by (T, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-2}$, and μ was a quadratic form on T such that β_μ had type I and $\mathrm{Arf}(\mu) = +1$. Now, if φ is the conjugation on $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta_\mu$. These isomorphism classes belong to one of the following three equivalence classes:

(1) $\eta = \mu$ ($n = 2^m \geq 2$).
   The involution φ is symplectic with signature $n/2$.
   A representative is $\varphi_{(3\text{-}b\text{-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.
(2) $\mathrm{Arf}(\eta) = +1$ but $\eta \neq \mu$ ($n = 2^m \geq 4$).
   The involution φ is symplectic with signature 0.
   A representative is $\varphi_{(3\text{-}b\text{-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}2)}$.
(3) $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
   The involution φ is orthogonal.
   A representative is $\varphi_{(3\text{-}b\text{-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}3)}$.

On the other hand, if φ is orthogonal on $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, μ, η), where again η is a quadratic form on T such that $\beta_\eta = \beta_\mu$. These isomorphism classes belong to one of the following three equivalence classes:

(4) $\eta = \mu$ ($n = 2^m \geq 2$).
   The involution φ is orthogonal.
   A representative is $\varphi_{(3\text{-}b\text{-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_*$.
(5) $\mathrm{Arf}(\eta) = +1$ but $\eta \neq \mu$ ($n = 2^m \geq 4$).
   The involution φ is orthogonal.
   A representative is $\varphi_{(3\text{-}b\text{-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}2)}$.
(6) $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
   The involution φ is symplectic with signature 0.
   A representative is $\varphi_{(3\text{-}b\text{-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}3)}$.

(3-c) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ ($n = 2^m \geq 2$) was determined up to isomorphism by (T, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-1}$, and μ was a quadratic form on T such that $\beta := \beta_\mu$ had type II and $\mu(f_\beta) = -1$. Now, if φ is the conjugation on $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following four equivalence classes:

(1) $\eta = \mu$ ($n = 2^m \geq 2$).
   The involution φ is of the second kind and has signature n.
   A representative is $\varphi_{(3\text{-}b\text{-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}c\text{-}1)}$.
(2) $\eta(f_\beta) = -1$ but $\eta \neq \mu$ ($n = 2^m \geq 4$).
   The involution φ is of the second kind and has signature 0.
   A representative is $\varphi_{(3\text{-}b\text{-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}2)} \otimes \varphi_{(1\text{-}c\text{-}1)}$.
(3) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 2$).
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(3\text{-}b\text{-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}c\text{-}3)}$.
(4) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(3\text{-}b\text{-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-}a\text{-}3)} \otimes \varphi_{(1\text{-}c\text{-}3)}$.

On the other hand, if φ is orthogonal in $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, μ, η), where again η is a quadratic form on T such that $\beta_\eta = \beta$. These isomorphism classes belong to one of the following four equivalence classes:

(5) $\eta = \mu$ ($n = 2^m \geq 2$).
   The involution φ is of the second kind and has signature 0.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-}1)}$.

(6) $\eta(f_\beta) = -1$ but $\eta \neq \mu$ ($n = 2^m \geq 4$).
   The involution φ is of the second kind and has signature 0.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}2)} \otimes \varphi_{(1\text{-c-}1)}$.

(7) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 2$).
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-c-}3)}$.

(8) $\eta(f_\beta) = +1$ and $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}3)} \otimes \varphi_{(1\text{-c-}3)}$.

(3-d) The grading Γ on $\mathcal{D} \cong M_n(\mathbb{C})$ ($n = 2^m \geq 4$) was determined up to isomorphism by (T, β, μ), where T was a subgroup of G isomorphic to $\mathbb{Z}_2^{2m-3} \times \mathbb{Z}_4$, β was an alternating bicharacter on T of type II, and μ was a quadratic form on $T_{[2]}$ such that $\beta_\mu = \beta|_{T_{[2]} \times T_{[2]}}$ and $\mu(f_T) = -1$. Now, if φ is the conjugation on $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, β, μ, η), where η is a quadratic form on T such that $\beta_\eta = \beta$ (so $\eta(f_T) = +1$). These isomorphism classes belong to one of the following four equivalence classes:

(1) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(3\text{-b-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-}1)}$.

(2) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 8$).
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(3\text{-b-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}3)} \otimes \varphi_{(1\text{-d-}1)}$.

(3) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(3\text{-b-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-}3)}$.

(4) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(3\text{-b-}1)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-}4)}$.

On the other hand, if φ is orthogonal on $\mathcal{D}_e$, (Γ, φ) is determined up to isomorphism by (T, β, μ, η), where again η is a quadratic form on T such that $\beta_\eta = \beta$ (so $\eta(f_T) = +1$). These isomorphism classes belong to one of the following four equivalence classes:

(5) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-}1)}$.

(6) $\eta(\mathrm{rad}'(\beta)) = \{+1\}$ and $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 8$).
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-a-}3)} \otimes \varphi_{(1\text{-d-}1)}$.

(7) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = +1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and orthogonal.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-}3)}$.

(8) $\eta(\mathrm{rad}'(\beta)) = \{-1\}$ and $\mathrm{Arf}(\eta) = -1$ ($n = 2^m \geq 4$).
   The involution φ is of the first kind and symplectic.
   A representative is $\varphi_{(3\text{-b-}4)} \otimes \varphi_* \otimes \cdots \otimes \varphi_* \otimes \varphi_{(1\text{-d-}4)}$.

10. Semisimple algebras with involution

The results of the previous sections can be extended to finite-dimensional real algebras that are not necessarily simple, but do not have nontrivial ideals preserved by the involution. As mentioned in the Introduction, our purpose is to apply these results for the classification of gradings on classical real Lie algebras in a forthcoming article [3], so here we restrict ourselves to the situation relevant for that application.

Let G be an abelian group, $\mathcal{D}$ a finite-dimensional *non*-simple real (associative) algebra whose center has dimension 2, and Γ a division G-grading on $\mathcal{D}$. Recall from Section 5 that this implies that $\mathcal{D}$ is the direct product of two central simple algebras over $\mathbb{R}$. Let φ be a second kind involution on the G-graded algebra $\mathcal{D}$. We want to classify the pairs (Γ, φ), up to isomorphism (but not up to equivalence). In fact, we can repeat the arguments in [13] and in the previous sections, because they do not depend on the simplicity of the underlying algebra.

Let us start by considering the grading Γ and disregarding the involution φ. As always, we denote by T the support of Γ, by K the support of the centralizer of the identity component, and by $\beta : K \times K \to \{\pm 1\}$ the alternating bicharacter given by the commutation relations in the centralizer of the identity component. Also, if the homogeneous components have dimension 2, we write $\mathcal{D}_e = \mathbb{R}I \oplus \mathbb{R}J$ ($\cong \mathbb{C}$), where I is the unity of $\mathcal{D}$ and $J^2 = -I$. By [13, Proposition 20], if the homogeneous components have dimension 2 or 4, then there exists a proper refinement of the grading.

The existence of a second kind involution φ prevents T from having a factor $\mathbb{Z}_4$, in other words, T is an elementary abelian 2-group. Indeed, Remark 19 can be invoked if the homogeneous components have dimension 1. As in Section 9, the case of dimension 4 reduces to dimension 1 using the Double Centralizer Theorem (note that [10, Theorem 4.7] does not require the ambient algebra to be simple). Finally, in the case where the homogeneous components have dimension 2, if there existed an element $g \in T \setminus K$ of order 4, then, by [13, Remark 21], any $0 \neq X, X' \in \mathcal{D}_g$ would satisfy $X^2 \in Z(\mathcal{D})$ and $(X')^2 \in \mathbb{R}_{>0}X^2$, so $\varphi(X^2) = \varphi(X)^2 \in \mathbb{R}_{>0}X^2$ would give us a contradiction with φ being of the second kind.

Looking at the list in Section 5, we see that $\mathcal{D}$ must be isomorphic to either $M_n(\mathbb{R}) \times M_n(\mathbb{R})$ ($n = 2^m \geq 1$) or $M_{n/2}(\mathbb{H}) \times M_{n/2}(\mathbb{H})$ ($n = 2^m \geq 2$), both with a grading whose support is an elementary 2-group of rank $2m + 1$, $2m$ or $2m - 1$, according to the homogeneous components being of dimension 1, 2 or 4 respectively.

- If the homogeneous components have dimension 1, then (Γ, φ) is determined up to isomorphism by (T, μ, η), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m+1}$, μ is a quadratic form on T such that $\beta := \beta_\mu$ has type II and $\mu(f_\beta) = +1$, and η is a quadratic form on T such that $\beta_\eta = \beta$ and $\eta(f_\beta) = -1$.
  If $\mathrm{Arf}(\mu) = +1$, then $\mathcal{D} \cong M_n(\mathbb{R}) \times M_n(\mathbb{R})$ ($n = 2^m \geq 1$), whereas if $\mathrm{Arf}(\mu) = -1$, then $\mathcal{D} \cong M_{n/2}(\mathbb{H}) \times M_{n/2}(\mathbb{H})$ ($n = 2^m \geq 2$).
- If the homogeneous components have dimension 4, then the classification is again reduced to the case of dimension 1 (see Section 9).
- If the homogeneous components have dimension 2, then the grading Γ is determined up to isomorphism by (T, K, ν), where T is a subgroup of G isomorphic to $\mathbb{Z}_2^{2m}$, K is a subgroup of T of index 2, and ν is a nice map on $T \setminus K$ such that $\beta := \beta_\nu$ has type II and $\nu(f_\beta) = +1$.
  If $\mathrm{Arf}(\nu) = +1$, then $\mathcal{D} \cong M_n(\mathbb{R}) \times M_n(\mathbb{R})$ ($n = 2^m \geq 2$), whereas if $\mathrm{Arf}(\nu) = -1$, then $\mathcal{D} \cong M_{n/2}(\mathbb{H}) \times M_{n/2}(\mathbb{H})$ ($n = 2^m \geq 2$).
  Now, in the case $\varphi(J) = +J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, η), where η is a quadratic form on K such that $\beta_\eta = \beta$ and $\eta(f_\beta) =$

-1. On the other hand, in the case $\varphi(J) = -J$, (Γ, φ) is determined up to isomorphism by (T, K, ν, ω), where ω is a nice map on $T \setminus K$ such that $\beta_\omega = \beta$ and $\omega(f_\beta) = -1$.

11. Distinguished involutions

Let $\mathcal{D}$ be as in Sections 6, 7, 8 or 9, that is, a finite-dimensional simple real algebra with a division grading Γ by an abelian group G such that $\mathcal{D}$ admits an involution as a graded algebra. Let T be the support of Γ.

We already observed (see Remarks 28 and 32) that, given one such involution φ, we can obtain all involutions (of the same kind in the case $Z(\mathcal{D}) = \mathbb{C}$) as $\mathrm{Int}(X) \circ \varphi$, where X runs through nonzero homogeneous elements of $\mathcal{D}$. Over an algebraically closed field such as $\mathbb{C}$, which appears in this paper in Section 8 when φ is of the first kind, there is no special choice of φ. Over the field $\mathbb{R}$, however, we conclude from our results that there is often a special choice, which we refer to as a *distinguished involution*. Here we collect some of the properties of these involutions.

First assume that T is an elementary 2-group and, if the identity component $\mathcal{D}_e$ has dimension 2, it does not coincide with $Z(\mathcal{D})$. Then, looking at the lists in Sections 6, 7 and 9, we can see that there is a unique involution φ characterized by any of the following equivalent properties:

 (i) φ has a nonzero signature;
 (ii) φ has the maximal possible signature;
 (iii) $X\varphi(X) \in \mathbb{R}_{>0}$ for all nonzero homogeneous $X \in \mathcal{D}$.

This distinguished involution appears in items (1-a-1), (1-b-1), (1-c-1), (2-a-3), (2-b-3), (2-c-4), (3-a-1), (3-b-1) and (3-c-1).

Let us now turn to the case of Section 8, that is, $\mathcal{D} \cong M_n(\mathbb{C})$ and Γ is a division grading on $\mathcal{D}$ as a complex algebra, and consider involutions of the second kind. Then there is a unique isomorphism class, (2-f-2-0), of distinguished involutions φ characterized by any of the following equivalent properties:

 (i′) φ has a nonzero signature;
 (ii′) φ has signature $\sqrt{|T_{[2]}|}$;
 (iii′) for any $t \in T$ of even order $o(t)$, we have that $\varphi(X) = X$ implies $X^{o(t)} \in \mathbb{R}_{>0}$ for all nonzero $X \in \mathcal{D}_t$.

Note that the signature of distinguished involutions reaches the maximal possible value, n, if and only if T is an elementary 2-group. This latter condition is also necessary and sufficient for the uniqueness of a distinguished involution (see Remark 32). Moreover, if it is satisfied, then property (iii') is equivalent to property (iii).

If T is not an elementary 2-group then the presence of a (fixed) distinguished involution φ allows us to construct a special basis in the graded subalgebra

$$\mathcal{D}^{[2]} := \bigoplus_{s \in T^{[2]}} \mathcal{D}_s.$$

(If T is an elementary 2-group then $\mathcal{D}^{[2]} = \mathcal{D}_e = \mathbb{C}$.) The construction is as follows.

In each component $\mathcal{D}_t$, $t \in T$, we can find a nonzero element X_t such that $\varphi(X_t) = X_t$, and this element is determined up to multiplication by a real scalar. If $o(t)$ is odd, then we can scale X_t so that $X_t^{o(t)} = 1$, and this determines the element X_t uniquely. If $o(t)$ is even, then we can also scale X_t so that $X_t^{o(t)} = 1$ because φ is distinguished, but such an element X_t is unique only up to sign. For $t \in T^{[2]}$, we have a way to choose the sign, which is given by the following result.

Lemma 34. *Fix an isomorphism $\mathcal{D} \cong \mathrm{End}_{\mathbb{C}}(V)$ and a hermitian form h on V that defines φ, that is, $\varphi = \sigma_h$ as in Equation (1). For any $X \in \mathcal{D}$, set $h_X(v, w) := h(v, Xw)$ for all $v, w \in V$. Then, for any $s \in T^{[2]}$, we have:*

(1) *If $o(s)$ is odd, then (a) for any $t \in T$, $t^2 = s$ implies $X_t^2 = X_s$ and (b) the signature of h_{X_s} equals the signature of h.*
(2) *If $o(s)$ is even, then there exists $\epsilon \in \{\pm 1\}$ such that (a) for any $t \in T$, $t^2 = s$ implies $X_t^2 = \epsilon X_s$ and (b) the signature of h_{X_s} equals the signature of ϵh.*

Proof. Suppose $t^2 = s$. Since X_t^2 belongs to $\mathcal{D}_s$ and satisfies $\varphi(X_t^2) = X_t^2$ and $(X_t^2)^{o(s)} = 1$, we have $X_t^2 = \epsilon X_s$ where $\epsilon = 1$ if $o(s)$ is odd and $\epsilon \in \{\pm 1\}$ if $o(s)$ is even. Next, since $\varphi(X_t) = X_t$, we can write $h_{X_s}(v, w) = h(v, X_s w) = \epsilon h(X_t v, X_t w)$, which shows that the hermitian forms ϵh and h_{X_s} are isometric. $\square$

Note that, since ϵ is determined by each of the conditions (a) and (b), it depends only on X_s, and neither on the choice of $t \in T$ satisfying $t^2 = s$ nor on the choice of the isomorphism $\mathcal{D} \cong \mathrm{End}_{\mathbb{C}}(V)$ and hermitian form h. If $\epsilon = -1$, we replace X_s by $-X_s$. We have proved the existence and uniqueness of a basis $\{X_s \mid s \in T^{[2]}\}$ for the complex algebra $\mathcal{D}^{[2]}$ with the following properties: $\varphi(X_s) = X_s$, $X_s^{o(s)} = 1$, and, for any $t \in T$ with $t^2 = s$, we have that $\varphi(X) = X$ implies $X^2 \in \mathbb{R}_{>0} X_s$ for all nonzero $X \in \mathcal{D}_t$. We will refer to it as the *distinguished basis*. The following result gives explicit formulas for the products of the elements of the distinguished basis of $\mathcal{D}^{[2]}$ and also for the quadratic Jordan operators of the basis elements of $\mathcal{D}$ acting on the distinguished basis of $\mathcal{D}^{[2]}$.

Proposition 35. *Let $\{X_s \mid s \in T^{[2]}\}$ be the distinguished basis of $\mathcal{D}^{[2]}$. Then:*
(1) $X_{u^2} X_{v^2} = \beta(u, v)^2 X_{u^2 v^2}$ *for all $u, v \in T$.*
(2) *For any $t \in T$, we have that $\varphi(X) = X$ implies $X X_s X \in \mathbb{R}_{>0} X_{st^2}$ for all $s \in T^{[2]}$ and nonzero $X \in \mathcal{D}_t$. In particular, if X is scaled to satisfy $X^{o(t)} = 1$ then $X X_s X = X_{st^2}$.*

Proof. Recall that we chose X_t for all $t \in T$ such that $\varphi(X_t) = X_t$ and $X_t^{o(t)} = 1$. Then $X_t^2 = X_{t^2}$ for all $t \in T$.

We have $X_u X_v = \lambda X_{uv}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Applying φ to both sides of this equation, we get $X_v X_u = \bar{\lambda} X_{uv}$ and hence $\beta(u, v) = \lambda^2$. Then, on the one hand, $(X_u X_v)^2 = (\lambda X_{uv})^2 = \lambda^2 X_{u^2 v^2} = \beta(u, v) X_{u^2 v^2}$ and, on the other hand, $(X_u X_v)^2 = X_u X_v X_u X_v = \beta(v, u) X_u^2 X_v^2 = \beta(v, u) X_{u^2} X_{v^2}$. This proves (1).

For (2), it is necessary and sufficient to prove that $X_t X_s X_t = X_{st^2}$. Indeed, pick $u \in T$ such that $u^2 = s$ and compute:

$$X_t X_s X_t = \beta(t, s) X_s X_t^2 = \beta(t, s) X_s X_{t^2} = \beta(t, s) \beta(u, t)^2 X_{st^2} = X_{st^2},$$

where in the second last step we have used (1). $\square$

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