GRADINGS ON FINITE-DIMENSIONAL SIMPLE LIE ALGEBRAS

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ABSTRACT. In this survey paper we present recent classification results for gradings by arbitrary groups on finite-dimensional simple Lie algebras over an algebraically closed field of characteristic different from 2. We also describe the main tools that were used to obtain these results (in particular, the classification of group gradings on matrix algebras).

1. Introduction

Let A be an algebra (not necessarily associative) over a field $\mathbb{F}$ and let G be a group. We will usually use multiplicative notation for G, but for abelian groups we will sometimes switch to additive notation.

Definition 1.1. A *G-grading* on A is a vector space decomposition

$$A = \bigoplus_{g \in G} A_g$$

such that

$$A_g A_h \subset A_{gh} \quad \text{for all} \quad g, h \in G.$$

A_g is called the *homogeneous component* of degree g. The *support* of the G-grading is the set

$$S = \{g \in G \mid A_g \neq 0\}.$$

Example 1.2. The matrix algebra $R = M_n(\mathbb{F})$ has a $\mathbb{Z}_2$-grading associated to each block decomposition $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, with $A \in M_l(\mathbb{F})$, $D \in M_{n-l}(\mathbb{F})$:

$$R_0 = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right] \right\} \text{ and } R_1 = \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right] \right\}.$$

More generally, if $(g_1, \ldots, g_n)$ is an n-tuple of elements in G, then we obtain a G-grading on $R = M_n(\mathbb{F})$ by setting

$$R_g = \mathrm{Span}\,\{E_{ij} \mid g_i^{-1} g_j = g\}, \tag{1}$$

where E_{ij} are the matrix units.

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Example 1.3. There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $R = M_2(\mathbb{C})$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Namely, we set

$$R_{(0,0)} = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \right\}, \qquad R_{(1,0)} = \left\{ \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix} \right\},$$
$$R_{(0,1)} = \left\{ \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} \right\}, \qquad R_{(1,1)} = \left\{ \begin{bmatrix} 0 & \delta \\ -\delta & 0 \end{bmatrix} \right\}. \tag{2}$$

If $\mathbb{F}$ contains a primitive n-th root of unity ε, then we can define the following $n \times n$ matrices that generalize $-\sigma_3$ and σ_1:

$$X_a = \begin{bmatrix} \varepsilon^{n-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \varepsilon^{n-2} & 0 & \dots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \dots & \varepsilon & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \text{ and } X_b = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{3}$$

Since $X_a X_b = \varepsilon X_b X_a$ and $X_a^n = X_b^n = I$, the following is a $\mathbb{Z}_n \times \mathbb{Z}_n$-grading on $R = M_n(\mathbb{F})$:

$$R_{(k,l)} = \mathrm{Span}\,\{X_a^k X_b^l\}. \tag{4}$$

It turns out that any grading on $M_n(\mathbb{F})$ by an abelian group G can be obtained by combining gradings of the form (1) and (4) — see Section 5.

Example 1.4. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra. Then the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$$

can be viewed as a grading by the root lattice $\langle \Phi \rangle \cong \mathbb{Z}^n$, $n = \dim \mathfrak{h}$ (Cartan grading).

If G is a torsion-free abelian group, then any G-grading on $\mathfrak{g}$ can be obtained from the Cartan grading by joining some of its components (see Corollary 4.7). Such gradings have been extensively studied and find numerous applications (see e.g. [31]). If G has nontrivial torsion, then G-gradings on finite-dimensional semisimple Lie algebras are much more abundant. For instance, there are gradings arizing from automorphisms of finite order as follows.

Example 1.5. Let A be an algebra and φ an automorphism of A with $\varphi^m = id$. Suppose $\mathbb{F}$ contains a primitive m-th root of unity ξ. Set

$$A_k = \{x \in A \mid \varphi(x) = \xi^k x\}.$$

Then $A = \bigoplus_{k \in \mathbb{Z}_m} A_k$ is a $\mathbb{Z}_m$-grading on A. Conversely, any $\mathbb{Z}_m$-grading on A gives rise to an automorphism φ of A with $\varphi^m = id$ as follows. Define

$$\varphi(x) = \xi^k x \text{ for all } x \in A_k,\ k \in \mathbb{Z}_m,$$

and extend to A by linearity.

All possible gradings by finite cyclic groups on finite-dimensional semisimple Lie algebras in characteristic zero were classified by V. Kac in the 1960's [32] and used in the theory of Kac–Moody algebras [33]. Gradings by other finite groups arise in the study of generalized symmetric spaces in differential geometry (see e.g. [32] and many more references in [4]) and in the classification of infinite-dimensional simple Lie algebras endowed with a finite grading by a torsion-free group [41]. The knowledge of all possible gradings on simple Lie superalgebras can also be used to obtain a classification of simple Lie colour algebras via the "colouration–discolouration" process (see Section 2).

The purpose of this paper is to give a survey of recent classification results for gradings by arbitrary groups on finite-dimensional simple Lie algebras over an algebraically closed field of characteristic zero or $p \neq 2$. For results concerning gradings on real Lie algebras the reader is referred to [29] and to the recent survey [39].

The paper is structured as follows. In Sections 2 we briefly mention some of the applications of gradings on Lie algebras. Section 3 is devoted to the basics on gradings and to clarifying some terminological inconsistencies that developed in the literature. In Section 4 we briefly explain the duality between gradings and actions that is extensively used in the study of gradings. In Section 5 we introduce a classification of gradings on full matrix algebras (with or without involution), which play a key role in the classification of gradings on the simple Lie algebras of types $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Sections 6 and 7 are devoted to the known classification results for gradings on finite-dimensional simple Lie algebras in characteristic 0 and p, respectively.

2. Some applications of gradings on Lie algebras

2.1. **Symmetric spaces.** Let $\mathfrak{G}$ be a Lie group and G a finite abelian group that acts on $\mathfrak{G}$ by automorphisms. Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$ such that

$$(\mathfrak{G}^G)^\circ \subset \mathfrak{H} \subset \mathfrak{G}^G$$

where $\mathfrak{G}^G$ is the subgroup of fixed points in $\mathfrak{G}$ and $(\mathfrak{G}^G)^\circ$ is the connected component of identity in $\mathfrak{G}^G$. Then the homogeneous space $\mathfrak{G}/\mathfrak{H}$ is a *G-symmetric space.*

An important tool in studying such spaces is the associated $\widehat{G}$-grading on the tangent Lie algebra of $\mathfrak{G}$: $\mathfrak{g} = \bigoplus_{\chi\in\widehat{G}} \mathfrak{g}_\chi$.

2.2. **Loop and multiloop algebras.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ endowed with a $\mathbb{Z}_m$-grading:

$$\Gamma:\ \mathfrak{g} = \bigoplus_{\bar{k}\in\mathbb{Z}_m} \mathfrak{g}_{\bar{k}}.$$

The *loop algebra* $\mathcal{L}(\mathfrak{g},\Gamma)$ is the subalgebra of $\mathfrak{g}\otimes\mathbb{C}[z,z^{-1}]$ defined by

$$\mathcal{L}(\mathfrak{g},\Gamma) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\bar{k}} \otimes z^k,$$

where $\bar{k}$ denotes $k \mod m$.

Theorem 2.1 (V. Kac)**.** *Let L be an affine Kac–Moody Lie algebra. Then there exists a simple finite-dimensional Lie algebra $\mathfrak{g}$ and a $\mathbb{Z}_m$-grading Γ on $\mathfrak{g}$ such that*

$$[L,L]/Z(L) \cong \mathcal{L}(\mathfrak{g},\Gamma).$$

Gradings by non-cyclic abelian groups can be used to construct multiloop algebras.

2.3. **Deformations.**

Definition 2.2. Given a G-graded algebra L over a field $\mathbb{F}$ and a map $\sigma : G\times G \to \mathbb{F}$, we can define a new operation on L by setting

$$[x,y]^\sigma = \sigma(g,h)[x,y] \text{ for all } x\in A_g,\, y\in A_h.$$

We will denote by L^σ the vector space L endowed with this new operation. The algebra L^σ is referred to as a "twist" or "deformation" or "graded contraction" of L.

In the so-called "generic case", we have $\sigma(g,h)\neq 0$ for all $g,h\in G$, so $\sigma : G\times G\to\mathbb{F}^\times$. Then L can be recovered from L^σ by applying σ^{-1}.

If L is a Lie (super)algebra and σ is a symmetric 2-cocycle, then L^σ is again a Lie (super)algebra. The passage from L to L^σ finds numerous applications in the study of Lie algebras and in theoretical physics (see e.g. the works of J. Patera and co-authors).

If σ is not symmetric, then L^σ satisfies the antisymmetry and Jacobi identities that are twisted by a "commutation factor"

$$\beta(g,h) = \frac{\sigma(h,g)}{\sigma(g,h)}.$$

This leads to Lie colour algebras ("colouration").

2.4. **Lie colour algebras.** Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F}\neq 2,3$. Let G be an abelian group and $\beta : G\times G\to\mathbb{F}^\times$ a skew-symmetric bicharacter, i.e.,

$$\begin{aligned}\beta(ab,c) &= \beta(a,c)\beta(b,c),\\ \beta(c,ab) &= \beta(c,a)\beta(c,b),\\ \beta(a,b)^{-1} &= \beta(b,a),\end{aligned}$$

for all $a,b,c\in G$.

Definition 2.3. A *Lie colour algebra* with commutation factor β is a G-graded algebra $L=\bigoplus_{g\in G} L_g$ whose operation $[\,,\,]$ satisfies β-anticommutativity:

$$[x,y]+\beta(a,b)[y,x]=0$$

and β-Jacobi identity:

$$[[x,y],z]+\beta(ab,c)\,[[z,x],y]+\beta(a,bc)\,[[y,z],x]=0$$

for all $x\in L_a$, $y\in L_b$, $z\in L_c$.

Example 2.4. If G is trivial, then we recover the usual definition of a Lie algebra. If $G=\mathbb{Z}_2$ and β is given by $\beta(0,0)=\beta(0,1)=\beta(1,0)=1$ and $\beta(1,1)=-1$ (the only nontrivial bicharacter on $\mathbb{Z}_2$), then we obtain a *Lie superalgebra*.

Example 2.5. Let A be an associative algebra and $A=\bigoplus_{g\in G} A_g$ a G-grading. Define the β-commutator by

$$[x,y]_\beta = xy-\beta(g,h)yx \text{ for all } x\in A_g,\, y\in A_h.$$

Then $(A,[\,,\,]_\beta)$ is a Lie colour algebra with commutation factor β.

The following result was obtained in [38] for finitely generated G and then generalized in [7].

Theorem 2.6 ("discolouration")**.** *Let G be an abelian group and $\beta : G \times G \to \mathbb{F}^\times$ a skew-symmetric bicharacter. Then there exists a 2-cocycle $\sigma : G \times G \to \mathbb{F}^\times$ such that, for any Lie colour algebra $L = \bigoplus_{g\in G} L_g$ over $\mathbb{F}$ with commutation factor β, the twist L^σ is a Lie superalgebra.*

3. Generalities on gradings

3.1. **General gradings and group gradings.** Let A be a nonassociative algebra over a field $\mathbb{F}$. The most general concept of grading on A is a decomposition of A into a direct sum of (nonzero) subspaces such that the product of any two subspaces is either zero or contained in a third subspace:

$$\Gamma :\ A = \bigoplus_{s\in S} A_s \text{ where } A_s \neq 0 \text{ for any } s \in S \tag{5}$$

such that for any $s_1, s_2 \in S$ either $A_{s_1}A_{s_2} = 0$ or there is a unique $s_3 \in S$ with $A_{s_1}A_{s_2} \subset A_{s_3}$. Thus the indexing set S is equipped with a partially defined (nonassociative) binary operation $s_1 \cdot s_2 := s_3$. The grading Γ is *nontrivial* if S consists of more than one element. Γ is said to be a *(semi)group grading* if $(S, \cdot)$ can be imbedded into a (semi)group G. In this case S will be regarded as a subset of G, called the *support* of Γ and denoted $S = \operatorname{Supp}\Gamma$. Setting $A_g = 0$ for $g \in G \setminus S$, we recover Definition 1.1. Replacing G with a sub(semi)group if necessary, we can assume that G is generated by S.

Definition 3.1. We will say that a grading Γ as in (5) is *realized* as a G-grading if G is a (semi)group containing S, the subspaces $A_g := \begin{cases} A_s & \text{if } g = s \in S; \\ 0 & \text{if } g \notin S; \end{cases}$ form a G-grading on A, and S generates G.

One can ask whether or not all gradings on a certain class of algebras can be realized in this way. It was asserted in [37, Theorem 1(d)] that any grading on a Lie algebra is a semigroup grading, but later a counterexample was discovered by A. Elduque [24]. In that example the grading is on a nilpotent Lie algebra of dimension 16. The same author [25] has recently found much easier examples of nonsemigroup gradings on $\mathfrak{sl}_2(\mathbb{F}) \times \mathfrak{sl}_2(\mathbb{F})$ and on a metabelian Lie algebra of dimension 4. He has also shown that no examples exist in dimensions ≤ 3. The direct product of the 2-dimensional non-abelian and 1-dimensional abelian Lie algebras admits a semigroup grading that is not a group grading [25]; this is obviously an example of minimal possible dimension. The following still remains open:

Question 3.2 (A. Elduque)**.** *Is any grading on a finite-dimensional simple Lie algebra over $\mathbb{C}$ a group grading?*

If we assume from the start that the grading is a semigroup grading, then the answer is positive. In fact, we have the following result.

Proposition 3.3. *Let L be a simple Lie algebra. If G is a semigroup and $L = \bigoplus_{g\in G} L_g$ is a G-grading with support S where G is generated by S, then G is an abelian group.*

Proof. First we prove that for any $g \in S$, the multiplication maps, $l_g : G \to G : x \mapsto gx$ and $r_g : G \to G : x \mapsto xg$, are surjective. Indeed, fix $s \in S$, $s \neq g$. Since L is simple and $L_g \neq 0$, the ideal generated by L_g is the entire L. It follows that there exist $s_1, \ldots, s_n \in S$ ($n \geq 1$) such that $0 \neq [[L_g, L_{s_1}], \ldots, L_{s_n}] \subset L_s$. Hence $gs_1 \cdots s_n = s$ and $s \in l_g(G)$. We have proved that $S \setminus \{g\} \subset l_g(G)$. Now if $h \in G$, $h \neq g$, then we can write $h = h_1 \cdots h_k$ with $h_i \in S$. If $h_1 = g$, then $k > 1$ and hence $h \in l_g(G)$. If $h_1 \neq g$, then $h_1 \in l_g(G)$ and hence $h \in l_g(G)$. We have proved that $G \setminus \{g\} \subset l_g(G)$. It remains to show that $g \in l_g(G)$. Since $[L, L] = L$, we have $S \subset SS$ and hence $g = xy$ for some $x, y \in S$. If $x = g$, then $g \in l_g(G)$; otherwise $x \in l_g(G)$ and hence $g \in l_g(G)$. The proof for r_g is similar.

Since S generates G, it follows that l_g and r_g are surjective for all $g \in G$. It is easy to see that any semigroup with this property is a group [1].

Now we can finish the proof as in [14, Lemma 2.1] (another proof is given in [19, Proposition 1]). Namely, we show by induction on $n \geq 2$ that $[[L_{g_1}, L_{g_2}], \ldots, L_{g_n}] \neq 0$ implies that g_i commute pairwise. (This property holds for an arbitrary Lie algebra L.) Indeed, for $n = 2$ we obtain by anticommutativity that $0 \neq [L_{g_1}, L_{g_2}] \subset L_{g_1 g_2} \cap L_{g_2 g_1}$, so $g_1 g_2 = g_2 g_1$. For $n \geq 3$, we know by induction that $g_1, \ldots, g_{n-1}$ commute pairwise, so it remains to consider g_n. By Jacobi identity, at least one of the subspaces $[[[L_{g_1}, L_{g_2}], \ldots, L_{g_{n-2}}], L_{g_n}]$ and $[[[L_{g_1}, L_{g_2}], \ldots, L_{g_{n-2}}], [L_{g_{n-1}}, L_{g_n}]]$ is nonzero, so by induction at least one of the elements g_n and $g_{n-1}g_n$ commutes with all of $g_1, \ldots, g_{n-2}$. In either case it follows that g_n commutes with $g_1, \ldots, g_{n-1}$, as desired. Finally, for any $g, h \in S$, using the simplicity of L as before, we can find $g_1, \ldots, g_n$ such that $0 \neq [[L_g, L_{g_1}], \ldots, L_{g_n}] \subset L_h$. It follows that $gg_1 \cdots g_n = h$ and hence h commutes with g. □

3.2. **Equivalences of group gradings.** Given such a grading Γ, there are, in general, many groups G such that Γ can be realized as a G-grading.

Example 3.4 ([19])**.** Let $L = \mathfrak{sl}_2(\mathbb{F}) \times \mathfrak{sl}_2(\mathbb{F})$, with standard bases $\{e_i, f_i, h_i\}$ in each component ($i = 1, 2$). Consider $\Gamma : \ L = L_{s_1} \oplus L_{s_2} \oplus L_{s_3} \oplus L_{s_4}$ where $L_{s_1} = \mathrm{Span}\,\{h_1, h_2\}$, $L_{s_2} = \mathrm{Span}\,\{e_2, f_2\}$, $L_{s_3} = \mathrm{Span}\,\{e_1\}$, $L_{s_4} = \mathrm{Span}\,\{f_1\}$. Then Γ can be realized as a grading by the cyclic group $\langle g \rangle_6$ with $s_1 = 1$, $s_2 = g^3$, $s_3 = g^2$, $s_4 = g^4$ and also as a grading by the symmetric group S_3 with $s_1 = 1$, $s_2 = (12)$, $s_3 = (123)$, $s_4 = (132)$.

There are two natural equivalence relations on group gradings that appear in the literature. Both are referred to as "equivalence" by different authors. We will use the term "group-equivalence" for the stronger of the two relations. Let

$$\Gamma : \ A = \bigoplus_{s \in S} A_s \text{ and } \Gamma' : \ A = \bigoplus_{t \in T} A'_t \tag{6}$$

be two gradings on the same algebra, with supports S and T, respectively.

Definition 3.5. We say that Γ and Γ' are *equivalent* if there exist an algebra automorphism $\varphi : A \to A$ and a bijection $\alpha : S \to T$ such that

$$\varphi(A_s) = A'_{\alpha(s)} \text{ for all } s \in S.$$

Thus the two gradings can be obtained from one another by the action of $\mathrm{Aut}\,(A)$ and relabeling the components. A simple, but important invariant of a grading is obtained by looking at the dimensions of the components: the *type* of Γ is the

[1] The above proof was communicated to the author by C. Draper.

sequence of numbers $(n_1, n_2, \ldots)$ where n_1 is the number of 1-dimensional components, n_2 is the number of 2-dimensional components, etc.

If Γ and Γ' are group gradings, i.e., there exist groups G and H such that Γ is realized as a G-grading and Γ' is realized as an H-grading, then one can further require that the bijection between S and T be the restriction of a group isomorphism:

Definition 3.6. We say that a G-grading, $A = \bigoplus_{g\in G} A_g$, and an H-grading, $A = \bigoplus_{h\in H} A'_h$, are *group-equivalent* if there exist an algebra automorphism $\varphi : A \to A$ and a group isomorphism $\alpha : G \to H$ such that

$$\varphi(A_g) = A'_{\alpha(g)} \quad \text{for all} \quad g \in G.$$

It is easy to construct examples of a grading that has two realizations as a G-grading and an H-grading where G and H are not isomorphic — see, e.g., the above example with $\mathfrak{sl}_2(\mathbb{F}) \times \mathfrak{sl}_2(\mathbb{F})$. This shows that group-equivalence is indeed a stronger relation than equivalence. There is, however, an important case when the two relations coincide. This happens when we consider *universal groups* of the gradings Γ and Γ'.

3.3. The universal group of a grading.

Proposition 3.7. *Let Γ be a grading on an algebra A as in (5). Assume that Γ is a group grading, i.e., there is a realization of Γ as a G-grading for some group G. Then there exists a universal realization of Γ. Namely, there exists a group $U(\Gamma)$ and a realization of Γ as a $U(\Gamma)$-grading such that for any other realization of Γ as a G-grading, there exists a unique homomorphism $U(\Gamma) \to G$ that restricts to identity on $\operatorname{Supp}\Gamma$. Two group gradings, Γ and Γ', are equivalent if and only if the corresponding $U(\Gamma)$- and $U(\Gamma')$-gradings are group-equivalent (so, in particular, $U(\Gamma)$ and $U(\Gamma')$ are isomorphic).*

Proof. The idea is contained in [37]. We define $U(\Gamma)$ to be the group with generating set $S = \operatorname{Supp}\Gamma$ and relations $s_1 s_2 = s_3$ for all $0 \neq A_{s_1} A_{s_2} \subset A_{s_3}$. Then for any realization of Γ as a G-grading, we have a unique homomorphism $U(\Gamma) \to G$ induced by the identity map on S. Since S is imbedded in G, the canonical map $S \to U(\Gamma)$ is also an imbedding. The last statement of the proposition follows from the universal property of $U(\Gamma)$. □

Corollary 3.8. *For a given group grading Γ and a group G, the realizations of Γ as a G-grading are in one-to-one correspondence with the epimorphisms $U(\Gamma) \to G$ that are injective on $\operatorname{Supp}\Gamma$.*

From Proposition 3.3, we immediately obtain the following result.

Corollary 3.9. *Let Γ be a group grading on a simple Lie algebra. Then $U(\Gamma)$ is an abelian group.*

Remark 3.10. For any group grading Γ, we can define the universal *abelian* group $U_{ab}(\Gamma)$ by the same generators and relations as in the proof above. The canonical map $S \to U_{ab}(\Gamma)$ is an imbedding if and only if Γ can be realized as a grading by an abelian group.

3.4. **The automorphism group and the diagonal group of a grading.**

Definition 3.11 ([37])**.** Let Γ be a grading on an algebra A as in (5). The *automorphism group of* Γ, denoted $\mathrm{Aut}\,(\Gamma)$, is the subgroup of $\mathrm{Aut}\,(A)$ consisting of all automorphisms that permute the components of Γ, i.e., an automorphism φ is in $\mathrm{Aut}\,(\Gamma)$ iff there exists a (unique) bijection $\alpha = \alpha(\varphi) : S \to S$ such that $\varphi(A_s) = A_{\alpha(s)}$ for all $s \in S$. The *diagonal group of* Γ, denoted $\mathrm{Diag}\,(\Gamma)$, consists of all automorphisms φ such that the restriction of φ to any component of Γ is the multiplication by a (nonzero) scalar.

It follows from Proposition 3.7 that any $\varphi \in \mathrm{Aut}\,(\Gamma)$ gives rise to a unique automorphism $U(\varphi)$ of $U(\Gamma)$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\alpha(\varphi)"'] & U(\Gamma) \arrow[d, "U(\varphi)"] \\ S \arrow[r] & U(\Gamma) \end{tikzcd}$$

where the horizontal arrows are the canonical imbeddings.

Now it follows from Definition 3.6 that two realizations of Γ, one as a G-grading and the other as an H-grading, are group-equivalent iff there exist $\varphi \in \mathrm{Aut}\,(\Gamma)$ and an isomorphism $\beta : G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} U(\Gamma) \arrow[r] \arrow[d, "U(\varphi)"'] & G \arrow[d, "\beta"] \\ U(\Gamma) \arrow[r] & H \end{tikzcd}$$

where the horizontal arrows are the epimorphisms coming from the universal property of $U(\Gamma)$. Hence we obtain the following result.

Corollary 3.12. *For a given group grading* Γ*, the group-equivalence classes of the realizations of* Γ *are in one-to-one correspondence with the* $\mathrm{Aut}\,(\Gamma)$*-orbits in the set of all normal subgroups* N *of* $U(\Gamma)$ *such that the quotient map* $U(\Gamma) \to U(\Gamma)/N$ *is injective on* $\mathrm{Supp}\,\Gamma$.

3.5. **Categorical approach.** The algebras graded by a fixed group G form a category where the morphisms are the *G-graded algebra maps*, i.e., homomorphisms of algebras $\varphi : A \to B$ such that $\varphi(A_g) \subset B_g$ for all $g \in G$, where $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ are G-graded algebras (here we do not assume that the supports of the gradings generate G). In particular, we have the following "categorical" notion of isomorphism between gradings on the same algebra A:

Definition 3.13. We say that two G-gradings, $A = \bigoplus_{g\in G} A_g$ and $A = \bigoplus_{g\in G} A'_g$, are *isomorphic* if there exists an algebra automorphism $\varphi : A \to A$ such that

$$\varphi(A_g) = A'_g \ \text{ for all } \ g \in G.$$

Now if $\alpha : G \to H$ is a homomorphism of groups, then we have a functor from the category of G-graded algebras to the category of H-graded algebras as follows. If $\Gamma :\ A = \bigoplus_{g\in G} A_g$ is a G-grading on A, then ${}^{\alpha}\Gamma :\ A = \bigoplus_{h\in H} A'_h$ defined by

$$A'_h = \bigoplus_{g\in G\,:\,\alpha(g)=h} A_g$$

is an H-grading on A. The functor sends A with grading Γ to A with grading ${}^{\alpha}\Gamma$; it is the identity map on morphisms.

It is well-known (see, e.g., [35]) that a G-grading is equivalent to the structure of an $\mathbb{F}G$-comodule, where the group algebra $\mathbb{F}G$ is regarded as a Hopf algebra with comultiplication $\Delta(g) = g \otimes g$, counit $\varepsilon(g) = 1$, and antipode $S(g) = g^{-1}$, for all $g \in G$ (extended by linearity to the entire $\mathbb{F}G$). Namely, if $\Gamma: \ A = \bigoplus_{g\in G} A_g$ is a G-grading on A, then the corresponding structure of a right $\mathbb{F}G$-comodule algebra is given by the homomorphism of algebras $\rho_\Gamma : A \to A \otimes \mathbb{F}G$ where

$$\rho_\Gamma(x) = x \otimes g \ \text{ for all } \ x \in A_g,\, g \in G. \tag{7}$$

From this point of view, the above functor induced by a homomorphism of groups $\alpha : G \to H$ is analogous to the base change functor for categories of modules.

Clearly, a G-grading Γ and an H-grading Γ' on the same algebra A are group-equivalent (in the sense of Definition 3.6) iff Γ' is isomorphic to ${}^{\alpha}\Gamma$ for some isomorphism $\alpha : G \to H$. If we apply an arbitrary homomorphism $\alpha : G \to H$ to a G-grading Γ, then some components of Γ may coalesce in ${}^{\alpha}\Gamma$.

3.6. **Refinements and coarsenings.**

Definition 3.14. Let Γ and Γ' be two gradings on A as in (6). We will say that Γ is a *refinement* of Γ' or Γ' is a *coarsening* of Γ, and write $\Gamma \le \Gamma'$, if for any $s \in S$ there exists $t \in T$ such that $A_s \subset A'_t$. If, for some $s \in S$, this inclusion is strict, then we speak of a *proper* refinement or coarsening.

Clearly, $\le$ is a partial order on the set of all gradings on A (if we regard all relabelings as one grading). The trivial grading is the unique maximal element. If A is finite-dimensional, then there also exist minimal elements, which are called *fine gradings*. It should be pointed out that the notion of fine grading depends on the class of gradings one is working with. For example, grading (1) is fine in the class of group gradings if $(g_i^{-1}g_j)^2 \ne e$ for all $i \ne j$, but for $n \ge 2$ it admits a proper refinement in the class of semigroup gradings: namely, take the 1-dimensional subspaces $\mathrm{Span}\,\{E_{ij}\}$ as the components. It is remarkable that, by virtue of Proposition 3.3, the notions of fine semigroup gradings, fine group gradings and fine abelian group gradings are all equivalent for simple Lie algebras.

The element $t \in T$ in Definition 3.14 is uniquely determined by $s \in S$, so $s \mapsto t$ defines a mapping $\pi : S \to T$. Clearly, this mapping is surjective, and we have $A'_t = \bigoplus_{s\in S\,:\,\pi(s)=t} A_s$.

If $\Gamma: \ A = \bigoplus_{g\in G} A_g$ is a G-grading and $\alpha : G \to H$ is a homomorphism of groups, then the H-grading ${}^{\alpha}\Gamma$ is a coarsening of Γ (not necessarily proper). However, it is not true in general that all coarsenings of Γ arise in this way. In fact, the example of a non-group grading on $\mathfrak{sl}_2(\mathbb{F}) \times \mathfrak{sl}_2(\mathbb{F})$ in [25] shows that a coarsening of a group grading is not necessarily a group grading. The following result shows what can still be salvaged in this situation.

Proposition 3.15. *Let Γ be a grading on an algebra A as in (5). Assume that Γ is a group grading and $G = U(\Gamma)$ is its universal group. If Γ' is a coarsening of Γ which is itself a group grading, then for any realization of Γ' as an H-grading for some group H, there exists a unique epimorphism $\alpha : G \to H$ such that $\Gamma' = {}^{\alpha}\Gamma$. Moreover, if $S = \mathrm{Supp}\,\Gamma$, $T = \mathrm{Supp}\,\Gamma'$ and $\pi : S \to T$ is the map associated to the coarsening, then $U(\Gamma')$ is the quotient of G by the normal subgroup generated by the elements $s_1 s_2^{-1}$ for all $s_1, s_2 \in S$ with $\pi(s_1) = \pi(s_2)$.*

Proof. Since $0 \neq A_{s_1}A_{s_2} \subset A_{s_3}$ implies $A'_{\pi(s_1)}A'_{\pi(s_2)} \cap A'_{\pi(s_3)} \neq 0$, we conclude that $\pi(s_1)\pi(s_2) = \pi(s_3)$ in any realization of Γ' as an H-grading. It follows that π induces an epimorphism $U(\Gamma) \to H$. The uniqueness is obvious.

Now let N be the normal subgroup of G stated above. Then for any realization of Γ' as an H-grading, the epimorphism $G \to H$ factors through $G/N \to H$. Hence T is imbedded into G/N, and G/N satisfies the universal property of $U(\Gamma')$. $\square$

Corollary 3.16. *For a given group grading* Γ*, the coarsenings which are themselves group gradings are obtained by taking* $U(\Gamma)$ *modulo a normal subgroup generated by some elements of the form* $s_1s_2^{-1}$ *where* $s_1, s_2 \in \operatorname{Supp}\Gamma$*. The normal subgroups belonging to one* $\operatorname{Aut}(\Gamma)$*-orbit result in equivalent coarsenings.*

3.7. **Classification of gradings.** One can classify group gradings on a given algebra up to equivalence or up to group-equivalence. In the categorical framework one is also interested in classifying G-gradings (with a fixed G) up to isomorphism.

In order to classify all group gradings on a given finite-dimensional algebra A up to equivalence, one can adopt the following approach: 1) find all equivalence classes of fine gradings on A as well as their universal groups and 2) apply Corollary 3.16 to each class of fine gradings to obtain its coarsenings. Note that it is not an easy task to produce an irredundant list, as non-equivalent fine gradings can have equivalent coarsenings. Comparing the types of the gradings may be helpful for this purpose.

Step 1) was carried out in [28] for the classical simple Lie algebras (except of type $\mathcal{D}_4$) over an algebraically closed field of characteristic zero. It should be noted, however, that the answer is given in terms of the so-called "MAD subgroups" of the group of automorphisms (explained in Section 4) and also that the parametrization given for these subgroups is redundant: the same conjugacy class can appear many times. An explicit (and irredundant) description of the corresponding fine gradings was later given for some classical simple Lie algebras of small rank — see [39] and references therein. Such descriptions are also known for the octonions [23], the exceptional simple Jordan algebra (the Albert algebra) [20], and the exceptional Lie algebras of types $\mathcal{G}_2$ [10, 19] and $\mathcal{F}_4$ [20]; they were also announced for $\mathcal{D}_4$ [22, 21] and $\mathcal{E}_6$ (conference presentation). Step 2) was carried out for the Lie algebra of type $\mathcal{G}_2$ in [19]. An irredundant list of all gradings is also known for the octonions [23] (over an arbitrary field). Irredundant lists of non-toral gradings (see Section 4) are known for the Albert algebra and for the simple Lie algebra of type $\mathcal{F}_4$ [20].

If one wants to classify gradings up to group-equivalence, one can take one more step: 3) apply Corollary 3.12 to each equivalence class of gradings to obtain all its realizations as G-gradings for various groups G.

Since producing irredundant lists of gradings — say, using steps 2) and 3) — appears to be impractical in all but very small examples, one can restrict oneself to a construction that gives all possible G-gradings for a given G, albeit in a redundant way. Then it is sufficient to know the fine gradings (up to equivalence) and their universal groups. Indeed, by Proposition 3.15, any G-grading is isomorphic to ${}^{\alpha}\Gamma$ for some fine grading Γ and a homomorphism $\alpha : U(\Gamma) \to G$.

Another, more direct approach to G-gradings on the classical Lie algebras (except of type $\mathcal{D}_4$) in any characteristic different from 2 was adopted in [9, 14, 6, 2, 5]. It exploits a close connection between the gradings on these Lie algebras and the gradings on the full matrix algebras (with involution). The latter were classified in [8, 13, 12, 3, 11]. This approach yields a (redundant) description of all possible G-gradings up to isomorphism.

We will return to classification results for gradings on finite-dimensional simple Lie algebras in Sections 6 and 7.

In the rest of this paper we will consider only group gradings.

4. Duality of gradings and actions

Throughout this section we assume that all grading groups G are *abelian* (which is always the case for simple Lie algebras by Proposition 3.3) and that the ground field $\mathbb{F}$ is *algebraically closed*. We want to reformulate G-gradings on a given (nonassociative) algebra A in the language of actions of a suitable object on A. We will assume that A is *finite-dimensional* and that the *support of a G-grading generates* G (otherwise one can replace G with a smaller group). Hence G is a finitely generated abelian group.

4.1. **Characteristic zero.** Let $\widehat{G}$ be the group of characters $G \to \mathbb{F}^\times$. Given a G-grading $A = \bigoplus_{g\in G} A_g$, we can define a $\widehat{G}$-action on A by setting

$$\chi \cdot x = \chi(g)x \;\text{ for all }\; x \in A_g\, g \in G \text{ and } \chi \in \widehat{G}. \tag{8}$$

Thus we obtain a homomorphism $\widehat{G} \to \mathrm{Aut}\,(A)$. This homomorphism is injective, because the support of the grading generates G. Furthermore, both $\widehat{G}$ and $\mathrm{Aut}\,(A)$ are algebraic groups, and $\widehat{G} \to \mathrm{Aut}\,(A)$ is a homomorphism of algebraic groups. Writing $G \cong \mathbb{Z}^n \times G_f$ where G_f is the torsion subgroup of G, we see that $\widehat{G} \cong (\mathbb{F}^\times)^n \times \widehat{G_f}$. Thus $\widehat{G}$ is isomorphic (as an algebraic group) to the product of a torus, $(\mathbb{F}^\times)^n$, and a finite group, $\widehat{G_f}$. Such algebraic groups are called *quasitori*. They are characterized by the property that all their representations are diagonalizable. Hence any quasitorus $Q \subset \mathrm{Aut}\,(A)$ gives rise to a grading by a finitely generated abelian group — namely, the group of characters $\mathfrak{X}(Q)$. This discussion implies the following result.

Proposition 4.1. *The gradings on A by a finitely generated abelian group G are in one-to-one correspondence with the imbeddings of the algebraic group $\widehat{G}$ into* $\mathrm{Aut}\,(A)$. *Two G-gradings are isomorphic if and only if the corresponding imbeddings are conjugate by an element of* $\mathrm{Aut}\,(A)$. *The group-equivalence classes of gradings on A are in one-to-one correspondence with the conjugacy classes of quasitori in* $\mathrm{Aut}\,(A)$.

Note that everything boils down to the structure of the algebraic group $\mathrm{Aut}\,(G)$. So if two algebras share the same automorphism group, then they have the same classification of gradings up to group-equivalence. This fact was used in [10] for the Lie algebra of type $\mathcal{G}_2$ and the algebra of octonions $\mathbb{O}$.

The question when two gradings on A are equivalent can also be answered in this language. However, the answer depends not only on $\mathrm{Aut}\,(A)$, but also on A itself. Let Γ be a grading on A as in (5), which is assumed to come from an abelian group. Let U be the abelian universal group of Γ as in Remark 3.10. Then by Proposition 4.1 we have an imbedding $\widehat{U} \to \mathrm{Aut}\,(A)$. Denote by Q the image of this imbedding. Clearly, $Q \subset \mathrm{Diag}\,(\Gamma)$ (see Definition 3.11). Looking at the defining relations of U, we see that in fact $Q = \mathrm{Diag}\,(\Gamma)$. Also note that $\mathrm{Aut}\,(\Gamma)$ is the normalizer of $\mathrm{Diag}\,(\Gamma)$ in $\mathrm{Aut}\,(A)$.

Definition 4.2. Let $Q \subset \mathrm{Aut}\,(A)$ be a quasitorus. Let Γ be the eigenspace decomposition of A induced by Q. Then the quasitorus $\mathrm{Diag}\,(\Gamma)$ in $\mathrm{Aut}\,(A)$ will be called the *saturation* of Q. Clearly, $Q \subset \mathrm{Diag}\,(\Gamma)$. We will say that Q is *saturated* if $Q = \mathrm{Diag}\,(\Gamma)$.

Combining Proposition 3.7 (modified to the case of abelian groups) and Proposition 4.1, we obtain the following result.

Proposition 4.3. *The equivalence classes of gradings on A are in one-to-one correspondence with the conjugacy classes of saturated quasitori in* $\mathrm{Aut}\,(A)$*.*

Given a grading Γ on A, Corollary 3.12 describes all possible realizations of Γ as a grading by an abelian group. It translates to the dual language as follows:

Corollary 4.4. *For a given grading Γ on A, the group-equivalence classes of the realizations of Γ are in one-to-one correspondence with the* $\mathrm{Aut}\,(\Gamma)$*-orbits in the set of all quasitori $Q \subset \mathrm{Diag}\,(\Gamma)$ whose saturation equals* $\mathrm{Diag}\,(\Gamma)$*.*

Clearly, any maximal quasitorus in $\mathrm{Aut}\,(A)$ is saturated. Maximal quasitori are called "MAD subgroups" in [37] (maximal abelian diagonalizable). They correspond to fine gradings on A. A maximal torus in $\mathrm{Aut}\,(A)$ gives rise to a grading Γ_0. Since all maximal tori are conjugate, there is only one such grading up to equivalence. If $\mathrm{Aut}\,(A)$ is connected, then Γ_0 is a fine grading.

Definition 4.5. A grading Γ on A is said to be *toral* if it can be realized as a G-grading such that the image of $\widehat{G}$ in $\mathrm{Aut}\,(A)$ is contained in a torus.

If $\mathrm{Aut}\,(A)$ is connected, then the toral gradings are precisely the gradings equivalent to a coarsening of Γ_0. For semisimple Lie algebras, Γ_0 is the Cartan decomposition. It is known that in this case any automorphism in $\mathrm{Diag}\,(\Gamma_0)$ is inner. It follows that Γ_0 is a fine grading and its universal (abelian) group is the root lattice.

Proposition 4.6 ([19])**.** *Let L be a semisimple Lie algebra and Γ is a grading on L by an abelian group. Then the following conditions are equivalent:*

(1) Γ *is toral;*
(2) Γ *is equivalent to a coarsening of the Cartan decomposition;*
(3) *the identity component of Γ contains a Cartan subalgebra of L.*

Corollary 4.7. *Let L be a semisimple Lie algebra. Then any grading on L by a torsion-free abelian group is equivalent to a coarsening of the Cartan decomposition.*

It turns out that in some cases non-toral fine gradings can be explicitly constructed as refinements of toral gradings [19, 20].

4.2. **Prime characteristic.** Let $\mathrm{char}\,\mathbb{F} = p > 0$. If G has no p-torsion, then one can proceed in the same way as in characteristic zero. Otherwise the character group $\widehat{G}$ will be insufficient, as it does not detect the p-torsion. One way to fix this problem is to replace the algebraic group $\widehat{G}$ by the *algebraic group scheme* G^D, the *Cartier dual* of G.

For general information on algebraic group schemes the reader is referred to [40]. An affine group scheme $\mathbf{G}$ over $\mathbb{F}$ is determined by a commutative Hopf algebra $\mathbb{F}[\mathbf{G}]$. A homomorphism of group schemes $\mathbf{G}_1 \to \mathbf{G}_2$ is determined by a Hopf algebra map $\mathbb{F}[\mathbf{G}_2] \to \mathbb{F}[\mathbf{G}_1]$. An affine group scheme $\mathbf{G}$ is said to be *algebraic* if $\mathbb{F}[\mathbf{G}]$ is a finitely generated algebra; it is said to be *smooth* if $\mathbb{F}[\mathbf{G}]$ does not have nilpotent elements. (By a well-known result of P. Cartier, all affine group schemes over a field of characteristic zero are smooth.) A smooth algebraic group scheme $\mathbf{G}$ can be identified with the algebraic group $\mathrm{Alg}\,(\mathbb{F}[\mathbf{G}],\mathbb{F})$ of all algebra maps from $\mathbb{F}[\mathbf{G}]$ to $\mathbb{F}$; then $\mathbb{F}[\mathbf{G}]$ is the algebra of polynomial functions on $\mathrm{Alg}\,(\mathbb{F}[\mathbf{G}],\mathbb{F})$. Given a finite-dimensional algebra A, one can define the algebraic group scheme $\mathbf{Aut}\,(A)$ [40, Section 7.6], which contains the algebraic group $\mathrm{Aut}\,(A)$ as the largest smooth sub-group-scheme.

The Cartier dual of a finitely generated abelian group G is the algebraic group scheme G^D such that $\mathbb{F}[G^D]=\mathbb{F}G$, the group algebra of G (which is a commutative Hopf algebra). Writing $G\cong G_{p'}\times G_p$ where G_p is a p-group and $G_{p'}$ has no p-torsion, we obtain: $G^D\cong\widehat{G_{p'}}\times(G_p)^D$ where the first factor is smooth and the second factor is "infinitesimal".

A G-grading Γ on A can be encoded as a comodule structure ρ_Γ defined by (7). This can be interpreted as an imbedding $G^D\to\mathbf{Aut}\,(A)$, and we have the following analogue of Proposition 4.1.

Proposition 4.8. *The gradings on A by a finitely generated abelian group G are in one-to-one correspondence with the imbeddings of the algebraic group scheme G^D into $\mathbf{Aut}\,(A)$. Two G-gradings are isomorphic if and only if the corresponding imbeddings are conjugate by an element of* $\mathrm{Aut}\,(A)$.

If one wants to translate a G-grading on A to an action of certain operators on A, then one can consider the dual Hopf algebra $K=(\mathbb{F}G)^\circ$ — see [35, Chapter 9]. Assume for simplicity that G is finite. Then K consists of all functions $G\to\mathbb{F}$, with point-wise multiplication, and acts on A by extension of (8):

$$f\cdot x=f(g)x\ \text{ for all }\ x\in A_g,\ g\in G \text{ and } f\in K.$$

With respect to this action A becomes a K-module algebra, i.e.,

$$f\cdot(xy)=\sum_i(f_i'\cdot x)(f_i''\cdot y)\ \text{ for all }\ f\in K,\ x,y\in A,$$

where we wrote the coproduct as $\Delta(f)=\sum_i f_i'\otimes f_i''$.

If $f\in K$ is a *group-like element*, i.e.,

$$\Delta(f)=f\otimes f\ \text{ and }\ f\neq 0,$$

then f acts on A as an automorphism:

$$f\cdot(xy)=(f\cdot x)(f\cdot y)\ \text{ for all }\ x,y\in A.$$

The group-like elements of K are in one-to-one correspondence with the multiplicative characters of G. If p does not divide $|G|$, then $K\cong\mathbb{F}\widehat{G}$ as Hopf algebras. In this case the G-gradings on an algebra A are equivalent to the $\widehat{G}$-actions on A by automorphisms.

If p divides $|G|$, then K is no longer a group algebra. In particular, it contains nonzero *primitive elements*, i.e., $f\in K$ with

$$\Delta(f)=f\otimes 1+1\otimes f.$$

They act on A as derivations:

$$f\cdot(xy)=(f\cdot x)y+x(f\cdot y)\ \text{ for all }\ x,y\in A.$$

The primitive elements of K are in one-to-one correspondence with the additive characters of G. Unless the p-torsion subgroup of G has period p, K will not be generated by group-like and primitive elements and one has to study elements $f \in K$ with more complicated expansion formulas for $f \cdot (xy)$ [5]. The action of K on A can be put into the context of *formal groups* [6].

5. Gradings on matrix algebras

5.1. **Without involution.** We consider gradings by a group G on the matrix algebra $R = M_n(\mathbb{F})$ over an algebraically closed field $\mathbb{F}$ of any characteristic.

Let V be a vector space of dimension n and let $V = V_{h_1} \oplus \cdots \oplus V_{h_s}$ be a direct sum decomposition labeled by some elements $h_1, \ldots, h_s \in G$. This decomposition induces a G-grading on $R = \mathrm{Hom}\,(V, V)$ as follows: $\varphi \in R$ is homogeneous of degree g if $\varphi(V_h) \subset V_{gh}$. Clearly, this grading is given by (1) with a suitably chosen basis of matrix units $\{E_{ij}\}$ and the n-tuple $(g_1, \ldots, g_n)$ as follows: first $\dim V_{h_1}$ elements g_i equal to h_1^{-1}, the second $\dim V_{h_2}$ elements g_i equal to h_2^{-1}, etc. The permutations of $(g_1, \ldots, g_n)$ give rise to isomorphic gradings.

Definition 5.1 ([8]). A G-grading on $R = M_n(F)$ is called *"elementary"* if it is induced from a decomposition of an n-dimensional vector space as described above.

Note that, for $n \geq 2$, the identity component R_e always has dimension greater than 1.

Definition 5.2 ([8]). A G-grading on $R = M_n(F)$ is called *"fine"* if $\dim R_g \leq 1$ for all $g \in G$.

It should be noted that these gradings are indeed fine in the sense of Section 3, i.e., they have no proper refinements. However, elementary gradings can also be fine in that sense (in the class of group gradings). We will use quotation marks when talking about "fine" gradings in the sense of Definition 5.2. The gradings (4) constructed using generalized Pauli matrices (3) are examples of "fine" gradings. Note that for these gradings $G = \mathbb{Z}_n \times \mathbb{Z}_n$ and $\dim R_g = 1$ for all $g \in G$. It turns out [13] that the support of any "fine" grading is a subgroup $H \subset G$, and R is isomorphic to the twisted group algebra $\mathbb{F}^\sigma H$ for some 2-cocycle $\sigma : H \times H \to \mathbb{F}^\times$. Namely, there exists a basis $\{X_h \,|\, h \in H\}$ of R such that $X_{h_1} X_{h_2} = \sigma(h_1, h_2) X_{h_1 h_2}$. In other words, H admits an irreducible projective representation of dimension $n = \sqrt{|H|}$. If $\mathrm{char}\,\mathbb{F} = 0$, this implies that H is solvable [34].

Example 5.3. Let $H = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Then $R = M_n(\mathbb{F})$ with the grading (4) can be written as above by setting $X_{a^i b^j} = X_a^i X_b^j$.

If R is represented as the tensor product $C \otimes D$ where $C \cong M_k(\mathbb{F})$ has the "elementary" grading associated to $(g_1, \ldots, g_k) \in G^k$ and D has any G-grading, then the following defines a G-grading on R:

$$R_g = \mathrm{Span}\,\{E_{ij} \otimes d \mid d \in D_h, g_i^{-1} h g_j = g\}. \tag{9}$$

The following result was first obtained in [8] for abelian G and $\mathrm{char}\,\mathbb{F} = 0$, and then extended in [13, 12] to arbitrary G and arbitrary characteristic.

Theorem 5.4. *For any G-grading on $R = M_n(\mathbb{F})$, there exists a decomposition $R = C \otimes D$ where $C \cong M_k(\mathbb{F})$ has an "elementary" grading and $D \cong M_l(\mathbb{F})$ has a "fine" grading such that the G-grading on R is given by (9). Moreover, the support of the grading on D is a subgroup of G whose intersection with the support of the grading on C is $\{e\}$.*

In view of applications to classical simple Lie algebras, we are especially interested in the case of abelian G. Then any "fine" grading is a tensor product of gradings (4) by generalized Pauli matrices, as shown in [8] for $\operatorname{char}\mathbb{F} = 0$ and in [12, Theorem 8] for arbitrary characteristic:

Theorem 5.5. *Let H be an abelian group. Then for any "fine" grading on $D = M_l(\mathbb{F})$ with support H, there exists a decomposition $H = H_1 \times \cdots \times H_t$ such that $H_i \cong \mathbb{Z}_{l_i} \times \mathbb{Z}_{l_i}$, $i = 1, \ldots, t$ and $D \cong M_{l_1}(\mathbb{F}) \otimes \cdots \otimes M_{l_t}(\mathbb{F})$ as H-graded algebras where each $M_{l_i}(\mathbb{F})$ is graded as in (4) for some l_i-th primitive root of unity, $i = 1, \ldots, t$. In particular, no "fine" gradings exist if $\operatorname{char}\mathbb{F}$ divides l.*

Corollary 5.6. *If $\operatorname{char}\mathbb{F} = p$ and the torsion subgroup of G is a p-group, then any G-grading on $M_n(\mathbb{F})$ is "elementary".*

In view of duality described in Section 4, it is instructive to compare Theorems 5.4 and 5.5 with the following classification of maximal quasitori in $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{Aut}\,(M_n(\mathbb{C}))$ [28, Theorem 3.2]. Let $\mathcal{P}(n)$ be the subgroup of $\mathrm{PGL}_n(\mathbb{C})$ generated by the (images of) generalized Pauli matrices (3) and let $\mathcal{D}(n)$ be the subgroup of all diagonal matrices in $\mathrm{PGL}_n(\mathbb{C})$. Let $Q(k, l_1, \ldots, l_t)$ be the image of the imbedding

$$\mathcal{D}(k) \times \mathcal{P}(l_1) \times \cdots \times \mathcal{P}(l_t) \hookrightarrow \mathrm{PGL}_n(\mathbb{C})$$

defined by the conjugation action on $M_k(\mathbb{F}) \otimes M_{l_1}(\mathbb{F}) \otimes \cdots \otimes M_{l_t}(\mathbb{F}) = M_n(\mathbb{F})$ where $n = kl_1 \ldots l_t$.

Theorem 5.7. *All $Q(k, l_1, \ldots, l_t)$ are maximal quasitori. Any maximal quasitorus in $\mathrm{PGL}_n(\mathbb{C})$ is conjugate to one and only one of the $Q(k, l_1, \ldots, l_t)$ with $n = kl_1 \ldots l_t$ and $l_1, \ldots, l_t$ powers of primes.*

The uniqueness statement in this theorem follows from the fact that for $Q = Q(k, l_1, \ldots, l_t)$, the character group $\mathfrak{X}(Q)$ — which is the universal abelian group of the corresponding grading — is isomorphic to $\mathbb{Z}^{k-1} \times (\mathbb{Z}_{l_1} \times \mathbb{Z}_{l_1}) \times \cdots \times (\mathbb{Z}_{l_t} \times \mathbb{Z}_{l_t})$. Using the terminology of Section 4 for the case $\operatorname{char}\mathbb{F} = 0$, the "elementary" gradings on $M_n(\mathbb{F})$ are precisely the toral gradings.

5.2. **With involution present.** In view of applications to gradings on classical simple Lie algebras, we have to consider ($\mathbb{F}$-linear) involutions on the algebra $R = M_n(\mathbb{F})$, which is graded by an abelian group G. Throughout this subsection we assume that $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} \neq 2$.

Definition 5.8. An *involution* on a G-graded algebra $A = \bigoplus_{g\in G} A_g$ is an anti-isomorphism $* : A \to A$ of G-graded algebras, denoted $x \mapsto x^*$, such that $(x^*)^* = x$ for all $x \in A$. Then the subspaces of $*$-symmetric and $*$-skew elements are also G-graded. They are denoted $\mathcal{H}(A, *)$ and $\mathcal{K}(A, *)$, respectively.

If G is abelian, then $\mathcal{H}(A, *)$ is a G-graded Jordan algebra with multiplication $x \circ y = xy + yx$, and $\mathcal{K}(A, *)$ is a G-graded Lie algebra with multiplication $[x, y] = xy - yx$. It is shown in [9] that if $R = M_n(\mathbb{F})$ is G-graded and admits an involution, then the support of the grading generates an abelian subgroup in G.

It follows from the Noether–Skolem Theorem that any involution on $R = M_n(\mathbb{F})$ has the form:

$$X^* = \Phi^{-1} X^T \Phi \quad \text{for all} \quad X \in R$$

where Φ is a nondegenerate matrix such that either $\Phi^T = \Phi$ or $\Phi^T = -\Phi$. Involutions with $\Phi^T = \Phi$ are called *transpose involutions*, and those with $\Phi^T = -\Phi$ are called *symplectic involutions*. If we conjugate $*$ by an inner automorphism, then Φ is transformed as a matrix of a bilinear form. Hence, after a suitable conjugation, we can make Φ either I or $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. However, in the presence of a G-grading we have to use only those conjugations that are automorphisms of G-graded algebras. The following lemmas give the canonical forms of involutions when R has an "elementary" grading [9, 3].

Lemma 5.9. *Let G be an abelian group. Let $R = M_n(\mathbb{F})$ be equipped with an "elementary" G-grading. Suppose the G-graded algebra R admits an involution $*$ which is determined by a symmetric Φ. Then there exist $m, k \geq 0$ with $n = m + 2k$ such that, after conjugation by a suitable matrix, the G-grading is defined by an n-tuple $(g_1, \ldots, g_n)$ with*

$$g_1^2 = \ldots = g_m^2 = g_{m+1}g_{m+k+1} = \ldots = g_{m+k}g_{m+2k},$$

and the matrix Φ is given by

$$\Phi = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_k & 0 \end{bmatrix}.$$

The spaces of $*$-symmetric and $*$-skew elements have the following block form (corresponding to the blocks in Φ):

$$\mathcal{H}(R, *) = \left\{ \begin{bmatrix} S & P & Q \\ Q^T & A & B \\ P^T & C & A^T \end{bmatrix} \text{ where } S^T = S, B^T = B, C^T = C \right\}$$

and

$$\mathcal{K}(R, *) = \left\{ \begin{bmatrix} S & P & Q \\ -Q^T & A & B \\ -P^T & C & -A^T \end{bmatrix} \text{ where } S^T = -S, B^T = -B, C^T = -C \right\}.$$

Lemma 5.10. *Let G be an abelian group. Let $R = M_n(\mathbb{F})$ be equipped with an "elementary" G-grading. Suppose the G-graded algebra R admits an involution $*$ which is determined by a skew-symmetric Φ. Then $n = 2k$ and, after conjugation by a suitable matrix, the G-grading is defined by an n-tuple $(g_1, \ldots, g_n)$ with*

$$g_1 g_{k+1} = \ldots = g_k g_{2k},$$

and the matrix Φ is given by

$$\Phi = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}.$$

The spaces of $*$-symmetric and $*$-skew elements have the following block form (corresponding to the blocks in Φ):

$$\mathcal{H}(R, *) = \left\{ \begin{bmatrix} A & B \\ C & A^T \end{bmatrix} \text{ where } B^T = -B, C^T = -C \right\}$$

and

$$\mathcal{K}(R, *) = \left\{ \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \text{ where } B^T = B, C^T = C \right\}.$$

It turns out [9, 3] that if we consider a "fine" grading on $R = M_n(\mathbb{F})$ with support H, then R admits an involution only in the case $n = 2^t$ and abelian H. By Theorem 5.5, we have $H = H_1 \times \cdots \times H_t$ where $H_k = \langle a_k \rangle \times \langle b_k \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and $R \cong M_2(\mathbb{F}) \otimes \cdots \otimes M_2(\mathbb{F})$ (t factors) as H-graded algebras. Set

$$X_{(h_1,\ldots,h_t)} = X_{h_1} \otimes \cdots \otimes X_{h_t} \quad \text{for all} \quad h_1 \in H_1, \ldots, h_t \in H_t \tag{10}$$

where, as in Example 5.3, $X_{a_k^i b_k^j} = X_a^i X_b^j$, which gives in this case $X_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X_a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $X_b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X_{ab} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then for any $h \in H$, X_h spans the component of degree h.

Now by Theorem 5.4, any G-grading on $R = M_n(\mathbb{F})$ by an abelian group G can be obtained as $R = C \otimes D$ where $C \cong M_k(\mathbb{F})$ has an "elementary" grading and $D \cong M_l(\mathbb{F})$ has a "fine" grading. Suppose that the G-graded algebra R admits an involution $*$. It was silently assumed in [9] that the subalgebras C and D are stable under $*$ (which does not hold in general), and a canonical form was obtained for the involution under this assumption. A canonical form in the general case was later given in [11]. To state this result, we introduce the following notation. Define a map $\mathrm{sgn} : \langle a \rangle \times \langle b \rangle \to \{\pm 1\}$ by $\mathrm{sgn}(e) = \mathrm{sgn}(a) = \mathrm{sgn}(b) = 1$ and $\mathrm{sgn}(ab) = -1$, and set

$$\mathrm{sgn}(h_1, \ldots, h_t) = \mathrm{sgn}(h_1) \cdots \mathrm{sgn}(h_t) \quad \text{for all} \quad h_1 \in H_1, \ldots, h_t \in H_t.$$

Then we have $(X_h)^T = \mathrm{sgn}(h) X_h$ for all $h \in H$.

Conjugating by a permutation matrix, we can always represent the elementary grading on C by a k-tuple of the form $(g_1^{(q_1)}, \ldots, g_r^{(q_r)})$ where the elements $g_1, \ldots, g_r$ are pairwise distinct and we write $g^{(q)}$ for $\underbrace{g, \ldots, g}_{q \text{ times}}$. Then the identity component R_e is a subalgebra of C isomorphic to $M_{q_1}(\mathbb{F}) \times \cdots \times M_{q_r}(\mathbb{F})$. Clearly, R_e is stable under $*$. Hence $*$ permutes the components of R_e, so we can assume without loss of generality that

$$R_e = M_{q_1}(\mathbb{F}) \times \cdots \times M_{q_m}(\mathbb{F}) \times (M_{q_{m+1}}(\mathbb{F}) \times M_{q_{m+1}}(\mathbb{F})) \times \cdots \times (M_{q_s}(\mathbb{F}) \times M_{q_s}(\mathbb{F}))$$

where the first m components are $*$-stable, and the remaining pairs of components are swapped by $*$.

Theorem 5.11. *Let G be an abelian group. Let $R = M_n(\mathbb{F})$ be equipped with a G-grading by decomposition $R = C \otimes D$ where $C \cong M_k(\mathbb{F})$ has an "elementary" grading and $D \cong M_l(\mathbb{F})$ has a "fine" grading with support $H \subset G$. Suppose that the G-graded algebra R admits an involution $*$. Then $l = 2^t$, $H = H_1 \times \cdots \times H_t$, and for any $h \in H$, the component D_h is spanned by X_h as in (10). Furthermore, there exist integers $s > 0$, $m \geq 0$ and $q_i > 0$ ($i = 1, \ldots, s$) with $q_1 + \cdots + q_m + 2q_{m+1} + \cdots + 2q_s = k$ such that, after conjugating C by a suitable matrix, the G-grading on C is defined by a k-tuple*

$$(g_1^{(q_1)}, \ldots, g_m^{(q_m)}, (g'_{m+1})^{(q_{m+1})}, (g''_{m+1})^{(q_{m+1})}, \ldots, (g'_s)^{(q_s)}, (g''_s)^{(q_s)})$$

where

$$g_1^2 h_1 = \ldots = g_m^2 h_m = g'_{m+1} g''_{m+1} h_{m+1} = \ldots = g'_s g''_s h_s \text{ for some } h_i \in H,$$

and $*$ on R is defined by the following matrix:

$$\Phi = \begin{bmatrix} \Sigma_1 \otimes X_{h_1} & 0 & \dots & 0 & 0 \\ 0 & \Sigma_2 \otimes X_{h_2} & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & 0 & \Sigma_s \otimes X_{h_s} \end{bmatrix}$$

where each Σ_i is one of the matrices I_{q_i} or $\begin{bmatrix} 0 & I_{q_i/2} \\ -I_{q_i/2} & 0 \end{bmatrix}$ for $i = 1, \dots, m$, and one of the matrices $\begin{bmatrix} 0 & I_{q_i} \\ I_{q_i} & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{bmatrix}$ for $i = m+1, \dots, s$, selected according to the following condition: $(\Sigma_i)^T = \mathrm{sgn}(h_i)\Sigma_i$ if $*$ is a transpose involution and $(\Sigma_i)^T = -\mathrm{sgn}(h_i)\Sigma_i$ if $*$ is a symplectic involution.

Conversely, any data $s > 0$, $m \geq 0$, $q_i > 0$, $(g_1, \dots, g_k)$, Σ_i and h_i $(i = 1, \dots, s)$ satisfying the above conditions give rise to a G-grading on $R = C \otimes D$ and an involution on the G-graded algebra R.

Note that the spaces $\mathcal{H}(R, *)$ and $\mathcal{K}(R, *)$ can also be explicitly obtained by computing $Y + Y^*$ and $Y - Y^*$, as Y varies over a spanning set of R:

$$\mathcal{H}(R,*) = \mathrm{Span}\,\{e_i U e_j \otimes X_h + \mathrm{sgn}(h) e_j \Sigma_j U^T \Sigma_i e_i \otimes X_{h_j} X_h X_{h_i} \mid U \in M_{q_i \times q_j}(\mathbb{F}), h \in H\}$$

and

$$\mathcal{K}(R,*) = \mathrm{Span}\,\{e_i U e_j \otimes X_h - \mathrm{sgn}(h) e_j \Sigma_j U^T \Sigma_i e_i \otimes X_{h_j} X_h X_{h_i} \mid U \in M_{q_i \times q_j}(\mathbb{F}), h \in H\}$$

where e_i, $i = 1, \dots, s$, are the idempotent matrices corresponding to the block decomposition of Φ displayed above.

6. Classification results in characteristic zero

Throughout this section, we assume that $\mathbb{F}$ is an algebraically closed field of characteristic zero. Let G be an abelian group.

6.1. **Type $\mathcal{A}_r$, $r \geq 1$.** Any G-grading on the Lie algebra of type $\mathcal{A}_1$ is the restriction to $\mathfrak{sl}_2(\mathbb{F}) \subset M_2(\mathbb{F})$ of either an "elementary" G-grading or the "fine" grading given by (2) for $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the latter can also be obtained as the restriction of an "elementary" grading to $\mathcal{K}(M_3(\mathbb{F}), *)$ where $*$ is the usual transpose. This gives 3 nontrivial gradings up to equivalence (two of which are fine).

Now let $r \geq 2$. We realize the Lie algebra of type $\mathcal{A}_r$ as $L = \mathfrak{sl}_n(\mathbb{F}) \subset M_n(\mathbb{F})$, $n = r + 1$. Then it is not true that any G-grading on L is the restriction of a G-grading on $R = M_n(\mathbb{F})$, because the outer automorphisms of L do not extend to automorphisms of R. However, it is known that any outer automorphism of L is given by $x \mapsto -\varphi(x)$ for some anti-automorphism φ of R — see e.g. [30]. Using this fact, it was shown in [14] that every grading $L = \bigoplus_{g \in G} L_g$ arises from a grading on $R = M_n(F)$ in one of the following two ways:

- I : $L_g = R_g$ for $g \neq e$ and $L_e = R_e \cap L$ where $R = \bigoplus_{g \in G} R_g$ is a G-grading on R;
- II : $L_g = \mathcal{K}(R_g, *) \oplus \mathcal{H}(R_{gh}, *)$ if $g \neq h$ and $L_h = \mathcal{K}(R_h, *) \oplus (\mathcal{H}(R_e, *) \cap L)$ where $R = \bigoplus_{g \in G} R_g$ is a G-grading on R, $*$ is an involution of the G-graded algebra R, and $h \in G$ is an element of order 2.

In view of Theorems 5.4, 5.5 and 5.11, this gives a complete description of gradings on L. Note that for even r we can only have an "elementary" grading in type II, so Lemma 5.9 applies.

6.2. **Type $\mathcal{B}_r$, $r \geq 2$.** We realize the Lie algebra of type $\mathcal{B}_r$ as $L = \mathcal{K}(R, *)$ where $R = M_n(\mathbb{F})$, $n = 2r+1$, and $*$ is a transpose involution. All automorphisms of L are inner and hence extend to R (e.g. [30]), so any G-grading on L is the restriction of a G-grading on R such that $*$ is an involution of G-graded algebras. Now Theorem 5.11 implies that the grading must be "elementary" and hence described by Lemma 5.9.

6.3. **Type C_r, $r \geq 3$.** We realize the Lie algebra of type $\mathcal{C}_r$ as $L = \mathcal{K}(R, *)$ where $R = M_n(\mathbb{F})$, $n = 2r$, and $*$ is a symplectic involution. All automorphisms of L are inner and hence extend to R (e.g. [30]), so any G-grading on L is the restriction of a G-grading on R such that $*$ is an involution of G-graded algebras. Theorem 5.11 gives a complete description of such gradings. In particular, if the 2-torsion subgroup of G is cyclic, then the grading must be "elementary" and hence described by Lemma 5.10.

6.4. **Type $\mathcal{D}_r$, $r > 4$.** We realize the Lie algebra of type $\mathcal{D}_r$ as $L = \mathcal{K}(R, *)$ where $R = M_n(\mathbb{F})$, $n = 2r$, and $*$ is a transpose involution. It is known that any automorphism (inner or outer) of L extends to R (e.g. [30]), so any G-grading on L is the restriction of a G-grading on R such that $*$ is an involution of G-graded algebras. Theorem 5.11 gives a complete description of such gradings. In particular, if the 2-torsion subgroup of G is cyclic, then the grading must be "elementary" and hence described by Lemma 5.9.

The case of $\mathcal{D}_4$ is special, because then L has outer automorphisms that do not extend to R (e.g. the so-called triality automorphism).

6.5. **Type $\mathcal{G}_2$.** We realize the Lie algebra of type $\mathcal{G}_2$ as $L = \mathrm{Der}(\mathbb{O})$ where $\mathbb{O}$ is the algebra of (split) octonions. There is a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of $\mathbb{O}$ such that the multiplication table is as follows:

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	e_1	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	e_1	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	e_1
v_1	v_1	0	e_2	0	0	0	$-u_3$	u_2
v_2	v_2	0	0	e_2	0	u_3	0	$-u_1$
v_3	v_3	0	0	0	e_2	$-u_2$	u_1	0

It will be more convenient to write the grading group G additively. For any $g_1, g_2, g_3 \in G$ with $g_1 + g_2 + g_3 = 0$, there is a grading of $\mathbb{O}$ defined by prescribing e_1, e_2 degree 0, u_i degree g_i, and v_i degree $-g_i$ ($i = 1, 2, 3$). This induces a grading on L as a graded subspace of $\mathrm{Hom}(\mathbb{O}, \mathbb{O})$. Following [10], we will call such gradings (on $\mathbb{O}$ and L) *"elementary"*. These gradings are precisely the toral gradings in the sense of Definition 4.5. Indeed, $M \in \mathrm{SL}_3(\mathbb{F})$ acts as an automorphism of $\mathbb{O}$ by fixing e_1, e_2 and sending $(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3)M$ and $(v_1, v_2, v_3) \mapsto (v_1, v_2, v_3)(M^T)^{-1}$. Hence the diagonal matrices in $\mathrm{SL}_3(\mathbb{F})$ give a maximal torus T in $\mathrm{Aut}(L) = \mathrm{Ad}\,\mathrm{Aut}(\mathbb{O})$. This torus induces gradings on $\mathbb{O}$ and L with universal group $\mathfrak{X}(T) = \langle g_1, g_2, g_3 \,|\, g_1 + g_2 + g_3 = 0\rangle \cong \mathbb{Z}^2$. It is shown in [23] that, except for the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-grading obtained by regarding $\mathbb{O}$ as the result of triple iteration of the Cayley–Dickson doubling process (starting from $\mathbb{F}$), any grading on $\mathbb{O}$ is a coarsening of the $\mathfrak{X}(T)$-grading. It is also shown that the latter has exactly 7 nontrivial proper coarsenings (up to equivalence). In particular, $\mathbb{O}$ admits 2 fine gradings. Since $\mathrm{Aut}\,(L)$ is isomorphic to $\mathrm{Aut}\,(\mathbb{O})$, L also has 2 fine gradings. (The one corresponding to T is, of course, the Cartan grading.) In fact, by Proposition 4.1, the isomorphism (or group-equivalence) classes of G-gradings on L are in one-to-one correspondence with those on $\mathbb{O}$. On the other hand, it is shown in [19] that there are exactly 25 equivalence classes of nontrivial gradings on L: the 2 fine gradings and 23 proper coarsenings of the Cartan grading. The difference with the gradings on $\mathbb{O}$ is caused by the fact that L and $\mathbb{O}$ determine different *saturated* quasitori in their common automorphism group — see Definition 4.2 and Proposition 4.3. Figure 1 displays the partial order of the non-equivalent coarsenings of the Cartan grading[2]. For each grading Γ, its universal group $U(\Gamma)$, the relations one has to impose on $G_0 = \mathfrak{X}(T)$ to obtain $U(\Gamma)$, and the number of Γ according to the list in [19, Theorem 2] are indicated.

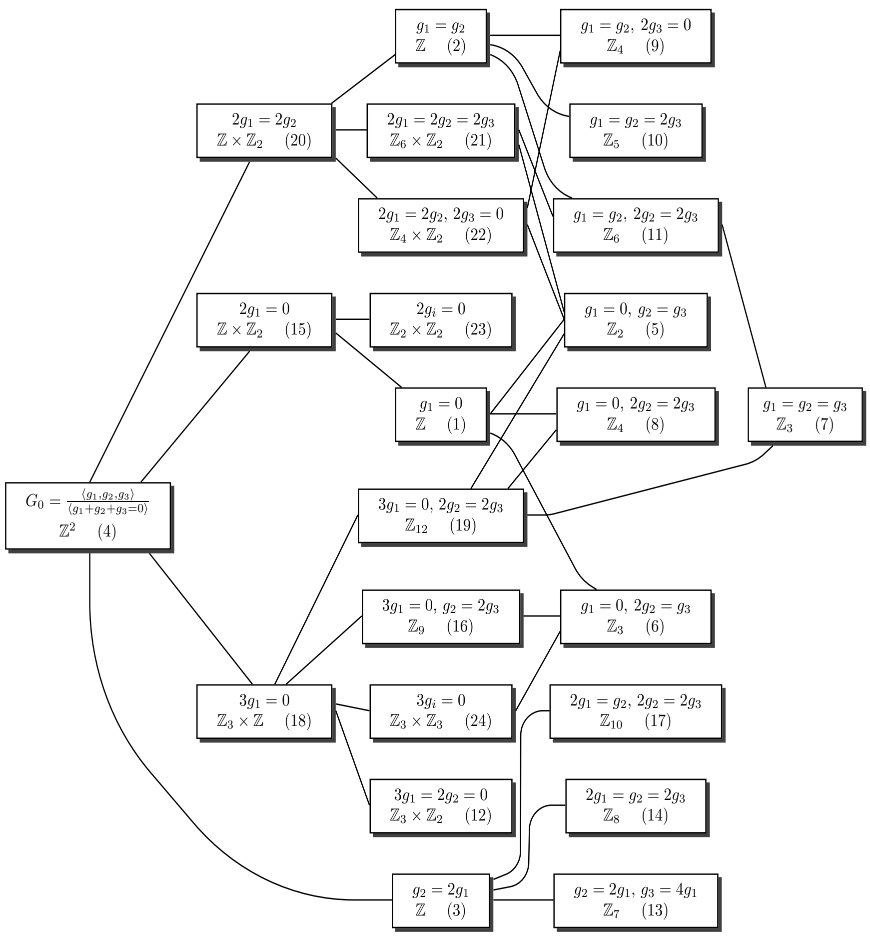

FIGURE 1. Coarsenings of the Cartan grading on the Lie algebra $\mathcal{G}_2$

[2]This diagram was prepared by Andrew Stewart, who worked with me in the framework of NSERC USRA in the summer of 2008.

A nice description of these "elementary" G-gradings on L is given in [10]. Namely, the Lie algebra $\mathfrak{sl}_3(\mathbb{F})$ is imbedded in L by virtue of the action $e_1 \mapsto 0$, $e_2 \mapsto 0$, $(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3)M$ and $(v_1, v_2, v_3) \mapsto -(v_1, v_2, v_3)M^T$ for $M \in \mathfrak{sl}_3(\mathbb{F})$. A complement to $\mathfrak{sl}_3(\mathbb{F})$ in L is spanned by the inner derivations D_{e_1,u_i} and D_{e_2,v_i} $(i = 1, 2, 3)$. (Recall that $D_{x,y} := [L_x, L_y] + [L_x, R_y] + [R_x, R_y]$ where L_x and R_x are the operators of, respectively, left and right multiplication by x on $\mathbb{O}$.) Hence each element $M + \sum_i \alpha_i u_i + \sum_i \beta_i v_i$ of L can be encoded as a "matrix" of the form

$$\left[\begin{array}{ccc|c} & & & \alpha_1 \\ & M & & \alpha_2 \\ & & & \alpha_3 \\ \hline \beta_1 & \beta_2 & \beta_3 & \end{array}\right] \quad \text{where } \ M \in \mathfrak{sl}_3(\mathbb{F}) \text{ and } \alpha_i, \beta_i \in \mathbb{F}.$$

Then the "elementary" G-grading on L defined by the triple (g_1, g_2, g_3) can be visualized as follows:

$$\left[\begin{array}{ccc|c} 0 & g_1 - g_2 & g_1 - g_3 & g_1 \\ g_2 - g_1 & 0 & g_2 - g_3 & g_2 \\ g_3 - g_1 & g_3 - g_2 & 0 & g_3 \\ \hline -g_1 & -g_2 & -g_3 & \end{array}\right].$$

6.6. **Type $\mathcal{F}_4$.** We can realize the Lie algebra of type $\mathcal{F}_4$ as $L = \mathrm{Der}\,(J)$ where J is the Albert algebra: $J = \mathcal{H}(M_3(\mathbb{O}), *)$ with $*$ given by $(a_{ij})^* = (\bar{a}_{ji})$. Clearly, the automorphisms of $\mathbb{O}$ and the automorphisms of $H_3(\mathbb{F}) := \mathcal{H}(M_3(\mathbb{F}), *)$ induce automorphisms of J, so we have $\mathrm{Aut}\,(\mathbb{O}) \times \mathrm{Aut}\,(H_3(\mathbb{F})) \subset \mathrm{Aut}\,(J)$. It follows that we can combine a grading on $\mathbb{O}$ and a grading on $H_3(\mathbb{F})$ to produce a grading on J. Now $\mathbb{O}$ has 2 fine gradings and $H_3(\mathbb{F})$ also has 2 fine gradings (like the Lie algebra $K_3(\mathbb{F})$ of type $\mathcal{A}_1$). It turns out [20] that combining the fine non-toral grading on $\mathbb{O}$ (by $(\mathbb{Z}_2)^3$) with either fine grading on $H_3(\mathbb{F})$ one obtains a fine grading on J. The universal groups are $(\mathbb{Z}_2)^3 \times \mathbb{Z}$ and $(\mathbb{Z}_2)^5$. Using the Tits construction for J, another fine grading on J can be obtained, with universal group $(\mathbb{Z}_3)^3$. Of course, there is also the grading induced by a maximal torus, with universal group $\mathbb{Z}^4$. It is shown in [20, Theorem 3] that the fine non-toral gradings on J admit, up to equivalence, exactly 5 proper coarsenings that are non-toral. The fine toral grading apparently admits a large number of non-equivalent coarsenings.

Since $\mathrm{Aut}\,(L) = \mathrm{Ad}\,\mathrm{Aut}\,(J)$, the Lie algebra L also has 4 fine gradings: the Cartan grading and 3 non-toral ones (with universal groups as above). It is shown in [20, Theorem 5] that the fine non-toral gradings on L also admit, up to equivalence, exactly 5 proper coarsenings that are non-toral. The Cartan grading admits a large number of non-equivalent coarsenings — probably even more than the corresponding grading on J.

6.7. **Type $\mathcal{D}_4$.** Many gradings on the Lie algebra $L = \mathfrak{so}_8(\mathbb{F})$ can be obtained by restricting gradings of the matrix algebra $M_8(\mathbb{F})$ similarly to the case of $\mathcal{D}_r$ with $r > 4$. One may call these "matrix" gradings. However, the gradings related to the triality automorphism of L cannot be obtained in this way. It was announced in [22] without proof that L has 14 fine gradings (including the Cartan grading). In [21] a proof appeared, but it is based on some computer calculations regarding the orbits of the automorphism group of the Cartan grading (i.e., the automorphism group of the root system, which has order 1152) on a certain set of quasitori in $\mathrm{Aut}\,(L)$. A construction involving $\mathbb{O}$ as well as para-Hurwitz and Okubo [36] algebras is given in [21] for the "non-matrix" fine gradings of L. In this regard see also [26].

6.8. **Types $\mathcal{E}_6$, $\mathcal{E}_7$ and $\mathcal{E}_8$.** A complete classification of fine gradings on these algebras has not yet appeared in the literature, but several interesting non-toral gradings have been constructed — see e.g. [26]. Notably, the algebras of types $\mathcal{E}_6$ and $\mathcal{E}_8$ admit gradings corresponding to the so-called *Jordan subgroups* of their automorphism groups [1], which are isomorphic to $(\mathbb{Z}_3)^3$ for $\mathcal{E}_6$ and to $(\mathbb{Z}_2)^5$ or $(\mathbb{Z}_5)^3$ for $\mathcal{E}_8$. (Explicit models of such "Jordan gradings" were constructed for all exceptional simple Lie algebras in [27].)

7. Classification results in prime characteristic

Throughout this section, we assume that $\mathbb{F}$ is an algebraically closed field of positive characteristic $p \neq 2$. Let G be an abelian group. As pointed out in Section 4, if G has no p-torsion, then G-gradings can be studied through automorphisms in the same way as in characteristic zero. For example, one can study such gradings on a classical simple Lie algebra L by looking at the standard matrix realization $L \subset R$ and trying to extend automorphisms from L to R. More generally, let R be a prime associative algebra and let L be the Lie algebra $[R,R]/(Z(R) \cap [R,R])$ where $Z(R)$ is the centre of R. Then any automorphism or the negative of an anti-automorphism of the associative algebra R induces an automorphism of the Lie algebra L. Or let $(R,*)$ be a $*$-prime associative algebra and let $L = \mathcal{K}(R,*)$. Then any $*$-automorphism of R induces an automorphism of L. Can every automorphism of L be obtained in this way? One can also ask the same question for derivations.

I. N. Herstein conjectured in the 1950's that, under certain conditions, the answer to these questions is yes. These conjectures and their generalizations are known as *Herstein's Lie map conjectures.* (They also have versions for Jordan algebras.) All these conjectures were proved, under mild conditions, in [15, 16, 17]. Using duality, we can obtain with a little more work that if G is an abelian group whose p-torsion has period p, then any G-grading on L is induced by a grading on R in a certain way. In order to extend this result to arbitrary abelian groups, one can try to show that, for any cocommutative Hopf algebra K, every K-module structure on the Lie algebra L comes from a K-module structure on the associative algebra R (see Section 4). In [6] the following results were proved.

Theorem 7.1. *Let $\mathbb{F}$ be an algebraically closed field and let $L \subset M_n(\mathbb{F})$ be an algebraic linear Lie algebra such that all derivations of L are inner. Let R be the (unital) associative subalgebra generated by L in $M_n(\mathbb{F})$. Suppose a connected cocommutative bialgebra K acts on L so that L is a K-module algebra. Then the action of K can be uniquely extended to R so that R is a K-module algebra.*

Corollary 7.2. *Let $\mathbb{F}$ be an arbitrary field of characteristic $p \neq 2$. Let $R = M_n(\mathbb{F})$ with $p \nmid n$. Let K be a connected cocommutative bialgebra over $\mathbb{F}$. If the Lie algebra $[R,R]$ is a K-module algebra, then extending the action by $f \cdot 1_R = \varepsilon(f)1_R$ for all $f \in K$ we turn the associative algebra R into a K-module algebra.*

Corollary 7.3. *Let $\mathbb{F}$ be an arbitrary field of characteristic $p \neq 2$. Suppose $R = M_n(F)$ and L is either $\mathfrak{so}_n(F)$ or $\mathfrak{sp}_n(F)$ (n even in the latter case). In the case $L = \mathfrak{so}_n(F)$, assume that $n \neq 4$ and, if $p = 3$, $n \neq 3$. Let K be a connected cocommutative bialgebra. Then any action of K on the Lie algebra L can be uniquely extended to an action of K on the associative algebra R.*

Here are applications to gradings on the Lie algebras of types $\mathcal{B}_r$ ($r \geq 2$), $\mathcal{C}_r$ ($r \geq 3$) and $\mathcal{D}_r$ ($r > 4$):

Theorem 7.4. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \neq 2$. Let L be one of $\mathfrak{so}_n(\mathbb{F})$, $n \geq 5$, $n \neq 8$, and $\mathfrak{sp}_n(\mathbb{F})$, $n \geq 6$, n even. Let G be an abelian group. Then any G-grading on L is the restriction of a G-grading on $M_n(F)$.*

Hence, as in characteristic zero, Theorem 5.11 gives a complete description of gradings on these Lie algebras. In particular, in the case of $\mathcal{B}_r$ all grading on L are induced by "elementary" gradings on R. The same holds for G-gradings in the cases $\mathcal{C}_r$ and $\mathcal{D}_r$ ($r > 4$) if the 2-torsion subgroup of G is cyclic. (Recall that "elementary" gradings are described by Lemmas 5.9 and 5.10.)

For Lie algebras of type $\mathcal{A}_r$, one has to distinguish two cases:

(1) if $p \nmid (r+1)$, then $L = \mathfrak{sl}_{r+1}(\mathbb{F})$ is a simple Lie algebra;
(2) if $p \mid (r+1)$, then $\mathfrak{sl}_{r+1}(\mathbb{F})$ contains the set of scalar matrices $\mathfrak{z}$, so one considers instead the simple Lie algebra $L = \mathfrak{psl}_{r+1}(\mathbb{F}) := \mathfrak{sl}_{r+1}(\mathbb{F})/\mathfrak{z}$.

Theorem 7.1 does not apply in case (2). However, it was shown in [5] that in this case the gradings on L are still obtained in essentially the same way as in characteristic zero. To give a statement that covers both cases, we let $R = M_n(\mathbb{F})$, $Z = Z(R) \cap [R,R]$ and $L = [R,R]/Z$. One can obtain a grading on L in two ways:

I : $L_g = R_g + Z$ for $g \neq e$ and $L_e = R_e \cap [R,R] + Z$ where $R = \bigoplus_{g\in G} R_g$ is a G-grading on R;

II : $L_g = (\mathcal{K}(R_g,*) \oplus \mathcal{H}(R_{gh},*)) + Z$ if $g \neq h$ and $L_h = (\mathcal{K}(R_h,*) \oplus (\mathcal{H}(R_e,*) \cap [R,R])) + Z$ where $R = \bigoplus_{g\in G} R_g$ is a G-grading on R, $*$ is an involution of the G-graded algebra R, and $h \in G$ is an element of order 2.

Theorem 7.5. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \neq 2$. Let $R = M_n(\mathbb{F})$ where $n \neq 3$ if $p = 3$. Let $Z = Z(R) \cap [R,R]$ and $L = [R,R]/Z$. Then any grading of L by an abelian group G is either of type I or of type II above.*

Hence, Theorems 5.4, 5.5 and 5.11 give a complete description of gradings on Lie algebras of type $\mathcal{A}_r$. Note that if the 2-torsion subgroup of G is trivial, then all G-gradings are of type I. If the torsion subgroup of G is a p-group, then all G-grading are of type I and, moreover, induced by "elementary" gradings on R (Corollary 5.6).

The following example shows that the restriction $n \neq 3$ if $p = 3$ in Theorem 7.5 cannot be omitted.

Example 7.6. Let $R = M_3(\mathbb{F})$, $\operatorname{char}\mathbb{F} = 3$, and $L = [R,R]/Z(R)$. Denote by e_{ij} the coset of E_{ij} modulo Z. Then $e_{11} - e_{22} = e_{22} - e_{33}$, e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32} form a basis of L. The following is a grading on L by the cyclic group $\langle a\rangle \cong \mathbb{Z}_3$ that is not induced by a grading on R:

$$\begin{array}{ll} L_e = \operatorname{Span}\{e_{11}-e_{22}, e_{13}, e_{31}\} & \text{and} \\ L_a = \operatorname{Span}\{e_{21}, e_{23}\}, & L_{a^{-1}} = \operatorname{Span}\{e_{12}+e_{23},\ e_{32}-e_{21}\}. \end{array}$$

In the above example, the $\mathbb{Z}_3$-grading on L, though not liftable to R, is conjugate to the grading

$$\begin{array}{ll} L'_e = \operatorname{Span}\{e_{11}-e_{22}, e_{13}, e_{31}\} & \text{and} \\ L'_a = \operatorname{Span}\{e_{21}, e_{23}\}, & L'_{a^{-1}} = \operatorname{Span}\{e_{12}, e_{32}\} \end{array}$$

by an automorphism of L (not liftable to R). The latter grading is obviously induced by the "elementary" grading on R defined by the triple (a, e, a).

All gradings on $\mathfrak{psl}_3(\mathbb{F})$ in the case $\operatorname{char}\mathbb{F} = 3$ can be obtained if one uses, instead of 3×3 matrices, the realization of $\mathfrak{psl}_3(\mathbb{F})$ as the algebra of traceless octonions under commutator (which is a Malcev algebra in general, but turns out to be a Lie algebra in characteristic 3). By [23, Theorem 9], any grading on $\mathfrak{psl}_3(\mathbb{F})$ comes from a gradings on the algebra of octonions, $\mathbb{O}$. Hence any grading on $\mathfrak{psl}_3(\mathbb{F})$ is either isomorphic to a type I grading induced from $M_3(\mathbb{F})$ or group-equivalent to the $(\mathbb{Z}_2)^3$-grading obtained by restricting the fine non-toral grading on $\mathbb{O}$. The latter turns out to be a type II grading induced from $M_3(\mathbb{F})$. Thus any grading on $\mathfrak{psl}_3(\mathbb{F})$ is *isomorphic* to a grading induced from $M_3(\mathbb{F})$.

Coming back to the general setting where $L = [R, R]/(Z(R) \cap [R, R])$ or $L = \mathcal{K}(R, *)$, there is another approach to showing that every G-gradings on L comes from a grading on R, which works in a much more general situation than $R = M_n(\mathbb{F})$. Instead of using duality to translate a G-grading Γ on L into a suitable action, it was proposed in [2] to use the coaction map $\rho = \rho_\Gamma : L \to L \otimes \mathbb{F}G$ defined by (7) to create a *surjective* homomorphism of Lie algebras $\bar{\rho} : L \otimes \mathbb{F}G \to L \otimes \mathbb{F}G$ by setting $\bar{\rho}(x \otimes h) = \rho(x)h$, i.e.,

$$\bar{\rho}(x \otimes h) = x \otimes gh \text{ for all } x \in L_g, g \in G.$$

Then one can apply the theory of *functional identities* [18], which was used to prove Herstein's Lie map conjectures, to show that, under mild conditions on R and G, the Lie homomorphism $\bar{\rho}$ is induced by a homomorphism (possibly combined with an anti-homomorphism) of associative algebras $R \otimes \mathbb{F}G \to R \otimes \mathbb{F}G$ — see [2] for details. In particular, one can use this method to obtain Theorem 7.5 for $n \geq 8$ and Theorem 7.4 for $n \geq 21$. These restrictions on n are the price one has to pay for using functional identities. On the other hand, this method applies to infinite-dimensional simple associative algebras (such as infinite matrices) and thus opens up new possibilities in the study of gradings on infinite-dimensional Lie algebras.

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