GRADED MODULES OVER CLASSICAL SIMPLE LIE ALGEBRAS WITH A GRADING

ALBERTO ELDUQUE* AND MIKHAIL KOCHETOV †

ABSTRACT. Given a grading by an abelian group G on a semisimple Lie algebra \mathcal{L} over an algebraically closed field of characteristic 0, we classify up to isomorphism the simple objects in the category of finite-dimensional G-graded \mathcal{L} -modules. The invariants appearing in this classification are computed in the case when \mathcal{L} is simple classical (except for type D_4 , where a partial result is given). In particular, we obtain criteria to determine when a finite-dimensional simple \mathcal{L} -module admits a G-grading making it a graded \mathcal{L} -module.

1. INTRODUCTION

Let \mathcal{L} be a semisimple finite-dimensional Lie algebra over an algebraically closed field \mathbb{F} of characteristic 0. Suppose $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ is a grading by an abelian group G. We want to study finite-dimensional G-graded modules over \mathcal{L} . Since G is abelian, the universal enveloping algebra $U(\mathcal{L})$ has a unique G-grading such that the canonical imbedding $\mathcal{L} \to U(\mathcal{L})$ is a homomorphism of graded algebras. Thus, a graded \mathcal{L} -module is the same as a graded $U(\mathcal{L})$ -module. The following lemma is a well-known version of Maschke's Theorem in the graded setting:

Lemma 1. Let $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a G-graded associative algebra where G is any group. Let $N \subset M$ be graded \mathcal{A} -modules. If N admits a complement in M as an \mathcal{A} -module then it admits a complement as a graded \mathcal{A} -module.

Applying this to a finite-dimensional graded module W over $\mathcal{A} = U(\mathcal{L})$, we conclude that W is isomorphic to a direct sum of graded-simple \mathcal{A} -modules. Thus we arrive to the problem of classifying graded-simple \mathcal{L} -modules up to isomorphism. This turns out to be closely related to another natural problem: which of the \mathcal{L} -modules admit a G-grading that makes them graded \mathcal{L} -modules?

The first step in solving these problems is a version of Clifford Theory (see Section 3), which describes graded-simple \mathcal{L} -modules in terms of simple (ungraded) \mathcal{L} -modules. In this context, there appears an action of the character group \widehat{G} on dominant integral weights of \mathcal{L} and what we call the graded Schur index of such a weight λ (or of the corresponding simple \mathcal{L} -module V_{λ}). In order to compute the Schur index, we use the graded Brauer group of \mathbb{F} (Section 2). This is a special case of the so-called Brauer-Long group of a commutative ring [7], but since here

²⁰¹⁰ Mathematics Subject Classification. Primary 17B70; Secondary 17B10, 16W50.

Key words and phrases. Graded algebra, Lie algebra, graded module.

^{*}supported by the Spanish Ministerio de Economía y Competitividad—Fondo Europeo de Desarrollo Regional (FEDER) MTM2010-18370-C04-02 and by the Diputación General de Aragón— Fondo Social Europeo (Grupo de Investigación de Álgebra).

[†]supported by a sabbatical research grant of Memorial University and a grant for visiting scientists by Instituto Universitario de Matemáticas y Aplicaciones, University of Zaragoza.

the ring in question is \mathbb{F} , an algebraically closed field, this group has a very simple structure, which allows efficient computation. To each dominant integral weight λ , we assign an element of the Brauer group, which will be called the Brauer invariant of λ (or of V_{λ}) and denoted Br(λ).

The second step, carried out for simple Lie algebras \mathcal{L} (Section 3), is a reduction of $\operatorname{Br}(\lambda)$ for general λ to the case where λ is the sum over a \widehat{G} -orbit of fundamental weights. This orbit always has length ≤ 3 , and length 3 can appear only if \mathcal{L} has type D_4 . Therefore, we concentrate on the case of one fundamental weight or a pair of fundamental weights positioned symmetrically on the Dynkin diagram of \mathcal{L} .

Finally, we compute the Brauer invariants of all simple modules if \mathcal{L} is a simple Lie algebra of type A_r $(r \geq 1, \text{Section 4})$, B_r $(r \geq 2, \text{Section 5})$, C_r $(r \geq 2, \text{Section 6})$ or D_r (r = 3 or r > 4, Section 7), for all *G*-gradings on \mathcal{L} . These *G*-gradings were classified up to isomorphism — i.e., the action of $\text{Aut}(\mathcal{L})$ — in paper [1] (see also the monograph [5] and references therein) in terms of the natural module for \mathcal{L} , which is the simple module of minimal dimension (except for B_2 and D_3). As a by-product, we determine which gradings in series D are "inner" and which are "outer", and classify them up to the action of $\text{Int}(\mathcal{L})$. (For series A, this was already clear in [1].) For type D_4 , we restrict ourselves to "matrix gradings", i.e., those induced from the matrix algebra $M_8(\mathbb{F})$, hence an orbit of length 3 cannot appear.

It turns out that the Brauer invariant of a simple module V_{λ} is in most cases determined by the Brauer invariant of the natural module, which appears as a parameter in the classification of *G*-gradings on the Lie algebra. The exceptions are the following: type A_r with an outer grading and odd $r \geq 3$ when λ is symmetric and involves the central node of the Dynkin diagram, type B_r when λ involves the node corresponding to the spin module, and type D_r when λ involves the nodes corresponding to the half-spin modules. In these cases, we have to invoke other parameters in the classification of *G*-gradings. It is also worth noting that the order of Br(λ) in the graded Brauer group is almost always ≤ 2 , the exceptions being type A_r with an inner grading (then the order is a divisor of r + 1) and type D_r with an inner grading and odd r (then the order is 1, 2 or 4).

As to the exceptional simple Lie algebras, gradings have been classified up to isomorphism for types G_2 and F_4 (see [5] and references therein). The situation is surprisingly simple in the case of G_2 , because the modules corresponding to its fundamental weights are the module of trace zero elements in the Cayley algebra and the adjoint module. Since any grading on G_2 is induced from a grading on the Cayley algebra, it follows that $Br(\lambda)$ is trivial for any dominant integral weight λ , i.e., given any grading on G_2 by an abelian group, any module over G_2 admits a compatible grading. We do not consider the case of F_4 here.

All modules in this paper will be assumed *finite-dimensional* over \mathbb{F} . Unless indicated otherwise, all vector spaces, algebras, tensor products, etc. will be taken over \mathbb{F} .

2. Graded Brauer group

First we briefly recall gradings by abelian groups on matrix algebras — see e.g. [8, 4, 1, 5] and references therein.

Let G be a group and let $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ be a graded-simple finite-dimensional associative algebra over any field \mathbb{F} . Then \mathcal{R} is isomorphic to $\operatorname{End}_{\mathcal{D}}(W)$ where \mathcal{D} is a graded division algebra (i.e., every nonzero homogeneous element of \mathcal{D} is invertible) and W is a finite-dimensional right "vector space" over \mathcal{D} (i.e., a graded right \mathcal{D} -module, which is automatically free). The graded algebra \mathcal{R} has a unique graded-simple module, up to isomorphism and shift of grading. Hence, if G is abelian, the isomorphism class of the graded algebra \mathcal{D} is determined by \mathcal{R} . We will denote this class by $[\mathcal{R}]$. Selecting a homogeneous \mathcal{D} -basis $\{v_1, \ldots, v_k\}$ in W, $v_i \in W_{g_i}$, and setting $\widetilde{W} = \text{span}\{v_1, \ldots, v_k\}$ (over \mathbb{F}), we can write $\mathcal{R} \cong \mathbb{C} \otimes \mathcal{D}$ where $\mathcal{C} = \text{End}(\widetilde{W})$ is a matrix algebra with an *elementary grading*, i.e., a grading induced from its simple module. Explicitly, the elementary grading is determined by (g_1, \ldots, g_k) as follows: deg $E_{ij} = g_i g_j^{-1}$ where E_{ij} are the matrix units associated to the basis $\{v_1, \ldots, v_k\}$.

Assume that \mathbb{F} is algebraically closed. Then \mathcal{D} is a twisted group algebra of its support T (necessarily a subgroup of G), i.e., \mathcal{D} is spanned by elements $X_t, t \in T$, such that $X_s X_t = \sigma(s, t) X_{st}$ where σ is a 2-cocycle of T with values in \mathbb{F}^{\times} . The elements X_t are determined up to a scalar multiple and σ up to a coboundary, but the relation

$X_s X_t = \beta(s, t) X_t X_s$ for all $s, t \in T$

implies that the function $\beta(s,t) = \sigma(s,t)/\sigma(t,s)$ is an alternating bicharacter of G (with values in \mathbb{F}^{\times}) and is uniquely determined by \mathcal{D} . Moreover, \mathcal{R} (or \mathcal{D}) is simple if and only if β is *nondegenerate*, i.e., rad $\beta := \{t \in T \mid \beta(t,s) = 1 \forall s \in T\}$ is trivial. Note that if char $\mathbb{F} = p$ then this condition forces |T| to be coprime with p. Finally, if G is abelian then the isomorphism class of the graded algebra \mathcal{D} is uniquely determined by the subgroup $T \subset G$ and the nondegenerate alternating bicharacter β , and, conversely, each such pair (T, β) gives rise to a graded division algebra [1]. Thus, for matrix algebras \mathcal{R} graded by an abelian group, the classes $[\mathcal{R}]$ are in bijection with the pairs (T, β) .

Fix an abelian group G. Given G-graded matrix algebras $\Re_1 = \operatorname{End}_{\mathcal{D}_1}(W_1)$ and $\Re_2 = \operatorname{End}_{\mathcal{D}_2}(W_2)$ as above, the tensor product $\Re_1 \otimes \Re_2$ is again a G-graded matrix algebra and hence can be written in the form $\operatorname{End}_{\mathcal{D}}(W)$. It is easy to see that \mathcal{D} is determined by \mathcal{D}_1 and \mathcal{D}_2 . Indeed, writing $\Re_i = \mathcal{C}_i \otimes \mathcal{D}_i$ (i = 1, 2) and $\mathcal{D}_1 \otimes \mathcal{D}_2 = \mathcal{C} \otimes \mathcal{D}$, we obtain $\Re_1 \otimes \Re_2 = (\mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}) \otimes \mathcal{D}$ where the first factor has an elementary grading and the second factor has a division grading. Hence we can unambiguously define $[\Re_1][\Re_2]$ as the isomorphism class of \mathcal{D} . This gives an associative commutative multiplication on the set of classes, with the class of \mathbb{F} being the identity element and the class of $\mathcal{D}^{\operatorname{op}}$ being the inverse of the class of \mathcal{D} (since $\mathcal{D} \otimes \mathcal{D}^{\operatorname{op}} \cong \operatorname{End}(\mathcal{D})$ as graded algebras, and $\operatorname{End}(\mathcal{D})$ has an elementary grading induced by the grading of \mathcal{D}). Thus we obtain an abelian group, which will be called the *G*-graded Brauer group of \mathbb{F} .

In order to express \mathcal{D} in terms of \mathcal{D}_1 and \mathcal{D}_2 , it will be convenient to rewrite the gradings in terms of actions. The group of characters \hat{G} acts on any *G*-graded vector space V:

(1)
$$\chi * v = \chi(g)v$$
 for all $v \in V_g, g \in G, \chi \in G$.

If char $\mathbb{F} = 0$ or if char $\mathbb{F} = p$ and G has no p-torsion then the grading can be recovered as the eigenspace decomposition relative to this action. If \mathcal{R} is a matrix algebra with a G-grading then each $\chi \in \widehat{G}$ acts by an automorphism of \mathcal{R} , so there exists an invertible element $u_{\chi} \in \mathcal{R}$ such that $\chi * x = u_{\chi} x u_{\chi}^{-1}$ for all $x \in \mathcal{R}$ (Noether–Skolem Theorem). Explicitly, if $\mathcal{R} = \mathcal{C} \otimes \mathcal{D}$ as above then we can take

$$u_{\chi} = \operatorname{diag}(\chi(g_1), \ldots, \chi(g_k)) \otimes X_t$$

where t is the unique element of T such that $\chi(s) = \beta(t, s)$ for all $s \in T$. Note that $u_{\chi_1}u_{\chi_2} = \beta(t_1, t_2)u_{\chi_2}u_{\chi_1}$. Define an alternating bicharacter $\hat{\beta}$ on \hat{G} (with values in \mathbb{F}^{\times}) by setting $\hat{\beta}(\chi_1, \chi_2) = \beta(t_1, t_2)$. Thus we have

(2) $u_{\chi_1}u_{\chi_2} = \hat{\beta}(\chi_1,\chi_2)u_{\chi_2}u_{\chi_1} \quad \text{for all } \chi_1,\chi_2 \in \widehat{G}.$

This "commutation factor" $\hat{\beta}$ is in fact the obstruction preventing the mapping $\chi \mapsto u_{\chi}$ from being a representation of \hat{G} (cf. [10]). For our purposes, it is important that T and β can be recovered from $\hat{\beta}$ as follows: $T = (\operatorname{rad} \hat{\beta})^{\perp}$ and $\beta(t_1, t_2) = \hat{\beta}(\chi_1, \chi_2)$ where χ_i is any character such that $\hat{\beta}(\psi, \chi_i) = \psi(t_i)$ for all $\psi \in \hat{G}$ (i = 1, 2). It is clear from the characterization (2) that the "commutation factor" of $\mathcal{R}_1 \otimes \mathcal{R}_2$ is the product of those of \mathcal{R}_1 and \mathcal{R}_2 . Hence the class $[\mathcal{R}_1 \otimes \mathcal{R}_2]$ is given by T and β determined by $\hat{\beta} = \hat{\beta}_1 \hat{\beta}_2$ as above.

We can summarize this discussion by saying that the *G*-graded Brauer group of \mathbb{F} is isomorphic to the group of alternating continuous bicharacters of the profinite group \widehat{G}_0 where G_0 is the torsion subgroup of *G* if char $\mathbb{F} = 0$ and the p'-torsion subgroup of *G* if char $\mathbb{F} = p$ (i.e., the set of all elements whose order is finite and coprime with p). The topology of \widehat{G}_0 , which makes it a compact and totally discontinuous topological group, comes from the identification of \widehat{G}_0 with the inverse limit of the finite groups \widehat{H} where *H* ranges over all finite subgroups of G_0 . Equivalently, the topology of \widehat{G}_0 is given by the system of neighborhoods of identity consisting of the subgroups of finite index H^{\perp} for the same *H*. This topology allows us to retain in this setting all the usual properties of duality for finite abelian groups: we just have to restrict our attention to *continuous* characters $\widehat{G}_0 \to \mathbb{F}^{\times}$ (where \mathbb{F} has discrete topology) and *closed* subgroups of \widehat{G}_0 . In particular, any continuous alternating bicharacter $\gamma: \widehat{G}_0 \times \widehat{G}_0 \to \mathbb{F}^{\times}$ has the form $\widehat{\beta}$ for some nondegenerate alternating bicharacter β of the finite subgroup (rad $\gamma)^{\perp} \subset G_0$.

The following property will be useful in the sequel:

Lemma 2. Let \mathcal{R} be a matrix algebra over an algebraically closed field \mathbb{F} . Suppose \mathcal{R} is graded by an abelian group G. If ε is a homogeneous idempotent of \mathcal{R} then $\varepsilon \mathcal{R} \varepsilon$ is a G-graded matrix algebra and $[\varepsilon \mathcal{R} \varepsilon] = [\mathcal{R}]$ in the G-graded Brauer group of \mathbb{F} .

Proof. Write $\mathcal{R} = \operatorname{End}_{\mathcal{D}}(W)$ where \mathcal{D} is a graded division algebra and W is a right vector space over \mathcal{D} . Then $W_0 = \varepsilon W$ is a graded \mathcal{D} -subspace and $\varepsilon \mathcal{R} \varepsilon$ can be identified with $\operatorname{End}_{\mathcal{D}}(W_0)$. The result follows.

Note that the Brauer class $[\mathcal{R}]$ alone does not determine \mathcal{R} as a graded algebra, but it does so in conjunction with another invariant: the *G*-orbit of the multiset (i.e., a set whose elements are assigned multiplicity) $\{g_1T, \ldots, g_kT\}$ in G/T, where *G* acts on G/T by translations [1].

To conclude this section, we make an observation that will be useful later. Any group homomorphism $f: G \to G'$ yields a functor F from the category of G-graded vector spaces (algebras, etc.) to that of G'-graded vector spaces (algebras, etc.). Namely, F sends $V = \bigoplus_{g \in G} V_g$ to $V = \bigoplus_{g' \in G'} V_{g'}$, where $V_{g'} = \bigoplus_{g \in f^{-1}(g')} V_g$, and is identical on morphisms. Since any G-graded algebra can be regarded as a G'-graded algebra in this way, we obtain a homomorphism from the G-graded Brauer group to the G'-graded Brauer group (i.e., the graded Brauer group is a covariant functor in G). In the realization of the graded Brauer group of an algebraically closed field in terms of bicharacters, this homomorphism simply maps $\hat{\beta}$ to $\hat{\gamma} = \hat{\beta} \circ (\hat{f} \times \hat{f})$ where $\hat{f} : \widehat{G'_0} \to \widehat{G_0}$ is induced by the restriction $f : G_0 \to G'_0$. The nondegenerate bicharacter γ corresponding to $\hat{\gamma}$ can be calculated in terms of β using the following fact.

Lemma 3. Let T be a finite abelian group and let β be a symmetric or skewsymmetric bicharacter on T. Assume that β is nondegenerate, so $t \mapsto \beta(t, \cdot)$ is an isomorphism $T \to \hat{T}$. Let $\hat{\beta}$ be the nondegenerate bicharacter on \hat{T} obtained by composing β with the inverse of this isomorphism. Given a subgroup $H \subset T$, consider the restriction $\hat{\gamma}$ of $\hat{\beta}$ to the subgroup $H^{\perp} \subset \hat{T}$. Then $(\operatorname{rad} \hat{\gamma})^{\perp}$ is the subgroup $HH' \subset T$ where

$$H' = \{t \in T \mid \beta(t,h) = 1 \text{ for all } h \in H\}.$$

Moreover, β induces a nondegenerate bicharacter γ on HH'/H, which correponds to $\hat{\gamma}$, as follows: $\gamma(xH, yH) = \beta(x, y)$ for all $x, y \in HH'$.

3. Clifford theory for graded modules

Let \mathcal{L} be a semisimple finite-dimensional Lie algebra over an algebraically closed field \mathbb{F} of characteristic 0. Given a grading $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ by an abelian group G, we want to classify (finite-dimensional) graded-simple \mathcal{L} -modules. When working with a specific graded \mathcal{L} -module W, we may replace G with the subgroup generated by the supports of \mathcal{L} and W. Thus we may assume, without loss of generality, that G is finitely generated.

The classical Clifford theory relates irreducible representations of a group and those of its normal subgroup of finite index. A similar approach can be used to describe simple modules over the smash product $\mathcal{A}\#\mathbb{F}H$ in terms of those of \mathcal{A} , where \mathcal{A} is an associative algebra and H is a finite group acting on \mathcal{A} by automorphisms (see e.g. [9]). To study graded \mathcal{L} -modules, we can take $\mathcal{A} = U(\mathcal{L})$ and $H = \hat{G}$ but note that H is not necessarily finite and hence should be regarded as an *algebraic* group (a quasitorus) in order to have equivalence between graded \mathcal{L} -modules and modules over the smash product $U(\mathcal{L})\#\mathbb{F}\hat{G}$. Therefore, we outline here the relevant version of Clifford theory, interpreting the results in the language of gradings. Some of these ideas already appeared in relation to graded modules in [2], where the representation theory of quantum tori was used instead of Clifford theory.

3.1. Twisting modules by an automorphism. Since \mathcal{L} is *G*-graded, the algebraic group \widehat{G} acts on \mathcal{L} by automorphisms. For $\chi \in \widehat{G}$, we will denote by α_{χ} the corresponding automorphism of \mathcal{L} : $\alpha_{\chi}(x) = \chi * x$ for all $x \in \mathcal{L}$ (see (1) for the definition of *). This automorphism extends uniquely to an automorphism of $U(\mathcal{L})$ (associated with the induced *G*-grading), which we will denote by α_{χ} as well. If W is a graded \mathcal{L} -module then \widehat{G} also acts on W. For $\chi \in \widehat{G}$, we denote by φ_{χ} the corresponding linear transformation of $W: \varphi_{\chi}(w) = \chi * w$ for all $w \in W$. The condition $\mathcal{L}_{q}W_{h} \subset W_{qh}$ translates to the following:

(3)
$$\varphi_{\chi}(xw) = \alpha_{\chi}(x)\varphi_{\chi}(w) \text{ for all } \chi \in G, x \in \mathcal{L}, w \in W.$$

For any \mathcal{L} -module V and $\alpha \in \operatorname{Aut}(\mathcal{L})$, we denote by V^{α} the corresponding twisted \mathcal{L} -module, i.e., the vector space V with a different \mathcal{L} -action: $x \cdot v = \alpha(x)v$ for all $x \in \mathcal{L}$ and $v \in V$. In other words, if V is an \mathcal{L} -module via $\rho: \mathcal{L} \to \mathfrak{gl}(V)$ (or, equivalently, via $\rho: U(\mathcal{L}) \to \operatorname{End}(V)$) then V^{α} is an \mathcal{L} -module via $\rho \circ \alpha$. Clearly, if $\psi: V_1 \to V_2$ is a homomorphism of \mathcal{L} -modules then ψ is also a homomorphism $V_1^{\alpha} \to V_2^{\alpha}$. Hence the group $\operatorname{Aut}(\mathcal{L})$ acts (on the right) on the set of isomorphism classes of \mathcal{L} -modules. It is well known that, for any inner automorphism α and any \mathcal{L} -module V, the twisted module V^{α} is isomorphic to V, so the action of $\operatorname{Aut}(\mathcal{L})$ on the isomorphism classes of \mathcal{L} -modules factors through the quotient group $\operatorname{Out}(\mathcal{L}) := \operatorname{Aut}(\mathcal{L})/\operatorname{Int}(\mathcal{L})$. In particular, the orbits are finite. Moreover, fixing a Cartan subalgebra of \mathcal{L} and a system of simple roots, we can split the quotient map $\operatorname{Aut}(\mathcal{L}) \to \operatorname{Out}(\mathcal{L})$ using the subgroup of diagram automorphisms. Hence the action of $\operatorname{Aut}(\mathcal{L})$ on the isomorphism classes of simple \mathcal{L} -modules can be represented as an action of $\operatorname{Out}(\mathcal{L})$ on dominant integral weights by permuting the vertices of the Dynkin diagram and the corresponding fundamental weights $\omega_1, \ldots, \omega_r$, where $r = \operatorname{rank} \mathcal{L}$. Explicitly, an element $\alpha \in \operatorname{Aut}(\mathcal{L})$ can be uniquely written in the form $\alpha = \alpha_0 \tau$ where $\alpha_0 \in \operatorname{Int}(\mathcal{L})$ and τ is a diagram automorphism, so $V_{\lambda}^{\alpha} \cong V_{\tau^{-1}(\lambda)}$ for any dominant integral weight λ , where V_{λ} is the simple module of highest weight λ .

In particular, any $\chi \in \widehat{G}$ can be used to twist an \mathcal{L} -module V through $\alpha = \alpha_{\chi}$. We will write V^{χ} for $V^{\alpha_{\chi}}$. Now observe that, for a graded \mathcal{L} -module W, equation (3) can be restated by saying that φ_{χ} is an isomorphism $W \to W^{\chi}$. Disregarding the grading of W, we can write W as the direct sum of its isotypic components:

$$W = \bigoplus_{\lambda} W_{\lambda}$$

where λ ranges over a finite set of dominant integral weights of \mathcal{L} and W_{λ} is a sum of copies of V_{λ} . It follows that φ_{χ} maps W_{λ} onto W_{μ} where μ is determined by the condition $V_{\mu} \cong V_{\lambda}^{\chi^{-1}}$, so $\mu = \tau_{\chi}(\lambda)$ where τ_{χ} is the diagram automorphism representing the class of α_{χ} in $Out(\mathcal{L})$. We denote by K_{λ} the *inertia group* of λ :

$$K_{\lambda} = \{ \chi \in \widehat{G} \mid \tau_{\chi}(\lambda) = \lambda \} = \{ \chi \in \widehat{G} \mid V_{\lambda}^{\chi} \cong V_{\lambda} \}.$$

Note that, since $\chi \mapsto \alpha_{\chi}$ is a homomorphism of algebraic groups $\widehat{G} \to \operatorname{Aut}(\mathcal{L})$ and the stabilizer of λ in $\operatorname{Aut}(\mathcal{L})$ is Zariski closed (as it contains the closed subgroup $\operatorname{Int}(\mathcal{L})$ of finite index), the subgroup K_{λ} is Zariski closed in \widehat{G} . Let $H_{\lambda} = K_{\lambda}^{\perp} \subset G$, so $H_{\lambda}^{\perp} = K_{\lambda}$ and $|H_{\lambda}|$ equals the size of the orbit $\widehat{G}\lambda$. Since $\sum_{\mu \in \widehat{G}\lambda} W_{\mu}$ is a \widehat{G} invariant (equivalently, *G*-graded) \mathcal{L} -submodule of *W*, we conclude that all gradedsimple \mathcal{L} -modules have the form $V_{\lambda_1}^k \oplus \cdots \oplus V_{\lambda_s}^k$ where $\{\lambda_1, \ldots, \lambda_s\}$ is a \widehat{G} -orbit and *k* is a positive integer. We will show that, for any orbit, this *k* is uniquely determined.

3.2. Brauer invariant and graded Schur index of a simple module. Let $V = V_{\lambda}$ be the simple module of highest weight λ and consider the corresponding homomorphism $\rho: U(\mathcal{L}) \to \operatorname{End}(V)$. By Density Theorem, this is a surjection. For any $\chi \in K_{\lambda}$, we have $V^{\chi} \cong V$, which means that there exists $u_{\chi} \in \operatorname{GL}(V)$ such that $\rho(\alpha_{\chi}(a)) = u_{\chi}\rho(a)u_{\chi}^{-1}$ for all $a \in U(\mathcal{L})$. Clearly, the inner automorphism $\tilde{\alpha}_{\chi}(x) = u_{\chi}xu_{\chi}^{-1}$ of End(V) is uniquely determined, so the operator u_{χ} is determined up to a scalar multiple. Thus we obtain a representation $K_{\lambda} \to \operatorname{Aut}(\operatorname{End}(V))$ given by $\chi \mapsto \tilde{\alpha}_{\chi}$. In particular, the operators $u_{\chi}, \chi \in K_{\lambda}$, commute up to a scalar. Note that we can select a basis in End(V) consisting of the images of homogeneous elements of $U(\mathcal{L})$ (with respect to the *G*-grading), so the representation $K_{\lambda} \to \operatorname{Aut}(\operatorname{End}(V))$ is a homomorphism of algebraic groups and corresponds to a \overline{G} -grading on End(V) where $\overline{G} = G/H_{\lambda}$. This is the unique \overline{G} -grading on End(V)

such that $\rho: U(\mathcal{L}) \to \operatorname{End}(V)$ is a homomorphism of graded algebras. Here $U(\mathcal{L})$ is considered with the \overline{G} -grading induced by the quotient map $G \to \overline{G}$ (which is a coarsening of the *G*-grading), and the existence of the mentioned \overline{G} -grading on $\operatorname{End}(V)$ can also be seen from the fact that the kernel of ρ is a \overline{G} -graded ideal.

Definition 4. The class $[\operatorname{End}(V_{\lambda})]$ in the (G/H_{λ}) -graded Brauer group will be called the *Brauer invariant* of λ (or of V_{λ}) and denoted by $\operatorname{Br}(\lambda)$. The degree of the graded division algebra \mathcal{D} representing $\operatorname{Br}(\lambda)$ will be called the *(graded) Schur index* of λ (or of V_{λ}).

Recalling the description of the graded Brauer group from the previous section, we see that $Br(\lambda)$ is determined by the "commutation factor" of the operators u_{χ} $(\chi \in K_{\lambda})$, i.e., the alternating bicharacter $\hat{\beta}$ of K_{λ} given by (2).

Proposition 5. Let $V = V_{\lambda}$ and $\overline{G} = G/H_{\lambda}$. The \mathcal{L} -module V^k admits a \overline{G} -grading that makes it a graded-simple \mathcal{L} -module if and only if k equals the Schur index of V. This \overline{G} -grading is unique up to isomorphism and shift.

Proof. If $W = V^k$ admits a \overline{G} -grading making it a graded \mathcal{L} -module then $\operatorname{End}(W)$ has an induced elementary grading and $\rho^{\oplus k} \colon U(\mathcal{L}) \to \operatorname{End}(W)$ is a homomorphism of \overline{G} -graded algebras. Since $\rho^{\oplus k}$ has the same kernel as ρ , we see that $\operatorname{End}(W)$ contains a \overline{G} -graded unital subalgebra \mathcal{R} isomorphic to $\operatorname{End}(V)$. Let \mathcal{C} be the centralizer of \mathcal{R} in $\operatorname{End}(W)$. Then \mathcal{C} is \overline{G} -graded and $\operatorname{End}(W) \cong \mathcal{R} \otimes \mathcal{C}$ as \overline{G} -graded algebras. Hence $[\mathcal{R}] = [\mathcal{C}]^{-1}$ in the \overline{G} -graded Brauer group. If W is graded-simple then \mathcal{C} is a graded division algebra, so $\mathcal{C} \cong \mathcal{D}^{\operatorname{op}}$ where \mathcal{D} is the graded division algebra representing $[\mathcal{R}] = \operatorname{Br}(\lambda)$. Hence $k^2 = \dim \mathcal{C} = \dim \mathcal{D}$, so k equals the Schur index of λ . (If W is \overline{G} -graded but not necessarily graded-simple then k must be a multiple of the Schur index.)

Conversely, suppose k equals the Schur index. Again, write $\operatorname{End}(W) \cong \mathcal{R} \otimes \mathcal{C}$ (as algebras) where \mathcal{R} is the image of $U(\mathcal{L})$ and \mathcal{C} is its centralizer, which is a matrix algebra of degree k and hence can be given a \overline{G} -grading using any isomorphism of algebras $\mathcal{D}^{\operatorname{op}} \to \mathcal{C}$. Then $\operatorname{End}(W)$ also receives a \overline{G} -grading, which is elementary since $[\mathcal{R}][\mathcal{C}]$ is trivial in the graded Brauer group. Therefore, W has a \overline{G} -grading that induces the grading of $\operatorname{End}(W)$. Since \mathcal{C} is a graded division algebra, W is graded-simple as an \mathcal{R} -module. Since \mathcal{R} is a graded-simple algebra, it has a unique graded-simple module, up to isomorphism and shift of grading. \Box

Remark 6. Under the conditions of Proposition 5, let ℓ be the Schur index of V. Fix a \overline{G} -grading on $U = V^{\ell}$ as in Proposition 5 and pick an element $\overline{g} \in \overline{G}$. Since the action of \mathcal{L} factors through $\operatorname{End}(V)$, the shift $U^{[\overline{g}]}$ is isomorphic to U as a graded \mathcal{L} -module if and only if \overline{g} belongs to the support of the graded division algebra representing $[\operatorname{End}(V)]$.

3.3. Induced graded space. To classify graded-simple \mathcal{L} -modules up to isomorphism, we need another ingredient: the construction of a graded space induced from a quotient group. Let H be a finite subgroup of G and let $U = \bigoplus_{\overline{g} \in \overline{G}} U_{\overline{g}}$ be a \overline{G} -graded vector space where $\overline{G} = G/H$. Let $K = H^{\perp} \subset \widehat{G}$. Then U is a K-module, so we can consider $W = \operatorname{Ind}_{K}^{\widehat{G}} U := \mathbb{F}\widehat{G} \otimes_{\mathbb{F}K} U$, which is a \widehat{G} -module and thus a G-graded space. Note that, as a K-module or \overline{G} -graded space, W is just the sum of s copies of U, where s = |H|. The G-grading on W can be explicitly described as follows. Take $\widehat{H} = \{\chi_1, \ldots, \chi_s\}$ and extend each χ_i to a character of

G in some way. We will denote the extensions by χ_i as well, and we may assume that χ_1 is the trivial character. Then $\{\chi_1, \ldots, \chi_s\}$ is a transversal for the subgroup K in \hat{G} and hence

$$\mathrm{Ind}_K^G U = \chi_1 \otimes U \oplus \cdots \oplus \chi_s \otimes U$$

One verifies that, for any $g \in G$, the homogeneous component W_q is given by

$$W_g = \Big\{ \sum_{j=1}^s \chi_j \otimes \chi_j(g)^{-1} u \mid u \in U_{\overline{g}} \Big\}.$$

We will denote the G-graded space $W = \bigoplus_{q \in G} W_g$ by $\operatorname{Ind}_{G/H}^G U$.

If U is a \overline{G} -graded \mathcal{L} -module then \mathcal{L} acts on W as follows: $x \cdot (\chi \otimes u) = \chi \otimes \alpha_{\chi^{-1}}(x)u$ for all $x \in \mathcal{L}, \chi \in \widehat{G}$ and $u \in U$. One verifies that this action is well defined and turns W into a G-graded \mathcal{L} -module. Moreover, if U is simple as a \overline{G} -graded \mathcal{L} -module and K is its inertia group then W is simple as a G-graded \mathcal{L} -module. Indeed, let $W_0 \subset W$ be a nonzero G-graded \mathcal{L} -submodule. Since $\chi_i \otimes U \cong U \chi_i^{-1}$ are simple as \overline{G} -graded \mathcal{L} -modules and pairwise non-isomorphic, we conclude that there exists i such that $\chi_i \otimes U \subset W_0$. But $\chi_i \otimes U$ generates W as a \widehat{G} -module, hence $W_0 = W$.

3.4. Classification of graded-simple modules. Let λ be a dominant integral weight. If $\mu \in \widehat{G}\lambda$ then $H_{\lambda} = H_{\mu}$ (since \widehat{G} is abelian) and the (G/H_{λ}) -graded algebras $\operatorname{End}(V_{\lambda})$ and $\operatorname{End}(V_{\mu})$ are isomorphic (with an isomorphism induced by $\alpha_{\chi} : U(\mathcal{L}) \to U(\mathcal{L})$ for some $\chi \in \widehat{G}$). Hence λ and μ have the same graded Schur index and Brauer invariant.

Definition 7. For each \widehat{G} -orbit 0 in the set of dominant integral weights, we select a representative λ and equip $U = V_{\lambda}^{\ell}$ with a (G/H_{λ}) -grading as in Proposition 5, where ℓ is the graded Schur index of V_{λ} . Then $W(0) := \operatorname{Ind}_{G/H_{\lambda}}^{G}U$ is a gradedsimple \mathcal{L} -module.

Theorem 8. Let \mathcal{L} be a semisimple finite-dimensional Lie algebra over an algebraically closed field of characteristic 0. Suppose \mathcal{L} is graded by an abelian group G. Then, for any graded-simple finite-dimensional \mathcal{L} -module W, there exist a \widehat{G} -orbit \mathbb{O} of dominant integral weights and an element $g \in G$ such that W is isomorphic to $W(\mathbb{O})^{[g]}$, with $W(\mathbb{O})$ as in Definition 7. Moreover, two such graded modules, $W(\mathbb{O})^{[g]}$ and $W(\mathbb{O}')^{[g']}$, are isomorphic if and only if $\mathbb{O}' = \mathbb{O}$ and $g'G_{\lambda} = gG_{\lambda}$ where G_{λ} is the pre-image of the support of the Brauer invariant $[\operatorname{End}(V_{\lambda})]$ under the quotient map $G \to G/H_{\lambda}$, with λ being a representative of \mathbb{O} .

Proof. We already know that, as an ungraded \mathcal{L} -module, W can be written as $W_1 \oplus \cdots \oplus W_s$ where $W_i \cong V_{\lambda_i}^k$ and $\mathcal{O} = \{\lambda_1, \ldots, \lambda_s\}$ is a \widehat{G} -orbit. Let $\lambda = \lambda_1$ be the representative selected for \mathcal{O} in Definition 7 and let $K_{\lambda} = H_{\lambda}^{\perp} \subset \widehat{G}$. Then the isotypic component W_1 is invariant under all $\varphi_{\chi}, \chi \in K_{\lambda}$, so W_1 is a \overline{G} -graded \mathcal{L} -module for $\overline{G} = G/H_{\lambda}$. Let $\{\chi_1, \ldots, \chi_s\}$ be a transversal for K_{λ} in \widehat{G} where χ_1 is the trivial character. We order the χ_i in such a way that $\lambda_i = \tau_{\chi_i}(\lambda)$ and hence φ_{χ_i} maps W_1 onto W_i . For any \overline{G} -graded \mathcal{L} -submodule $W_0 \subset W_1$, the direct sum $\sum_i \varphi_{\chi_i}(W_0)$ is a G-graded \mathcal{L} -submodule of W. It follows that W_1 is graded-simple and hence $k = \ell$ by Proposition 5. Moreover, there exists $g \in G$ such that $U^{[\overline{g}]}$ is isomorphic to W_1 as a \overline{G} -graded \mathcal{L} -module. We fix an

isomorphism $\varphi_1 \colon U^{[\overline{g}]} \to W_1$ and set $\varphi_i = \varphi_{\chi_i} \circ \varphi_1$. Then $\varphi_i \colon U^{[\overline{g}]} \to W_i^{\chi_i}$ is an isomorphism of \overline{G} -graded \mathcal{L} -modules. Define a linear isomorphism $\varphi \colon W(\mathcal{O})^{[g]} \to W$ by setting $\varphi(\chi_i \otimes u) = \chi_i(g)^{-1}\varphi_i(u)$. We claim that φ is an isomorphism of G-graded \mathcal{L} -modules. It follows from the definition of \mathcal{L} -action on $W(\mathcal{O})$ that φ is an isomorphism of \mathcal{L} -modules. It remains to show that φ is \widehat{G} -equivariant.

Recall that $\chi * w = \varphi_{\chi}(w)$ for all $w \in W$ and $\chi \in \widehat{G}$. Now fix *i* and χ . Then there exist unique *j* and $\chi_0 \in K_{\lambda}$ such that $\chi\chi_i = \chi_j\chi_0$. Since χ_0 sends H_{λ} to 1, we regard it as a character of \overline{G} with $\chi_0(g) = \chi_0(\overline{g})$. By definition of φ_1 , we have

(4)
$$\varphi_1(\chi_0(g)(\chi_0 * u)) = \chi_0 * \varphi_1(u) = \varphi_{\chi_0}(\varphi_1(u))$$

for any $u \in U$. By definition of induced module, we have

$$\chi*(\chi_i\otimes u)=\chi\chi_i\otimes u=\chi_j\chi_0\otimes u=\chi_j\otimes\chi_0*u.$$

Hence we compute, using the definition of φ and equation (4):

$$\varphi(\chi(g)\chi*(\chi_i\otimes u)) = \chi_j(g)^{-1}\chi(g)\varphi_j(\chi_0*u)$$

$$=\chi_i(g)^{-1}\varphi_j(\chi_0(g)(\chi_0*u))$$

$$=\chi_i(g)^{-1}\varphi_{\chi_j}(\varphi_1(\chi_0(g)(\chi_0*u)))$$

$$=\chi_i(g)^{-1}\varphi_{\chi_j}(\varphi_{\chi_0}(\varphi_1(u)))$$

$$=\chi_i(g)^{-1}\varphi_\chi(\varphi_{\chi_i}(\varphi_1(u)))$$

$$=\chi_i(g)^{-1}\varphi_\chi(\varphi_i(u)) = \chi*\varphi(\chi_i\otimes u),$$

which is exactly what is required for $\varphi \colon W(\mathcal{O})^{[g]} \to W$ to be \widehat{G} -equivariant.

If $g \in G_{\lambda}$ then \overline{g} is in the support of the graded division algebra representing $[\operatorname{End}(V_{\lambda})]$ and hence, by Remark 6, the \overline{G} -graded \mathcal{L} -module $U^{[\overline{g}]}$ is isomorphic to U. Then the above argument allows us to extend an isomorphism $\varphi_1 \colon U^{[\overline{g}]} \to U$ to an isomorphism $\varphi \colon W(\mathfrak{O})^{[g]} \to W(\mathfrak{O})$. Conversely, suppose that $\varphi \colon W(\mathfrak{O})^{[g]} \to W(\mathfrak{O})$ is an isomorphism of G-graded \mathcal{L} -modules. Looking at the isotypic components as \mathcal{L} -modules (disregarding the grading), we see that φ must map $\chi_1 \otimes U$ onto $\chi_1 \otimes U$, so it restricts to an isomorphism $(\chi_1 \otimes U)^{[\overline{g}]} \to \chi_1 \otimes U$ of \overline{G} -graded \mathcal{L} -modules. By Remark 6, \overline{g} must be in the support of the graded division algebra representing $[\operatorname{End}(V_{\lambda})]$, so $g \in G_{\lambda}$.

Corollary 9. An \mathcal{L} -module V admits a G-grading that would make it a graded \mathcal{L} -module if and only if, for any dominant integral weight λ , the multiplicities of $V_{\lambda_1}, \ldots, V_{\lambda_s}$ in V are equal to each other and divisible by the graded Schur index of λ , where $\{\lambda_1, \ldots, \lambda_s\}$ is the \widehat{G} -orbit of λ .

3.5. Comparison of graded modules for two isomorphic gradings on \mathcal{L} . Gradings by abelian groups have been classified up to isomorphism [1, 5] for the classical simple Lie algebras except D_4 . Let us compare the theories of graded modules for two isomorphic *G*-gradings on a semisimple Lie algebra \mathcal{L} . So let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ and $\Gamma' : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}'_g$ be two such gradings and let α be an automorphism of \mathcal{L} such that $\alpha(\mathcal{L}_g) = \mathcal{L}'_g$ for all $g \in G$. We will use prime to distinguish the objects associated to Γ' from those associated to Γ . For any $\chi \in \widehat{G}$, we have $\alpha'_{\chi} = \alpha \alpha_{\chi} \alpha^{-1}$. Suppose $V_{\lambda} \cong V^{\alpha}_{\mu}$. Then

$$K_{\lambda} = \{ \chi \in \widehat{G} \mid V_{\lambda}^{\alpha_{\chi}} \cong V_{\lambda} \} = \{ \chi \in \widehat{G} \mid V_{\mu}^{\alpha\alpha_{\chi}} \cong V_{\mu}^{\alpha} \} = \{ \chi \in \widehat{G} \mid V_{\mu}^{\alpha'_{\chi}} \cong V_{\mu} \} = K'_{\mu}.$$

Since α maps the kernel of $\rho_{\lambda} \colon U(\mathcal{L}) \to \operatorname{End}(V_{\lambda})$ onto the kernel of $\rho_{\mu} \colon U(\mathcal{L}) \to \operatorname{End}(V_{\mu})$, we obtain a commutative diagram

$$\begin{array}{c} U(\mathcal{L}) & \stackrel{\alpha}{\longrightarrow} & U(\mathcal{L}) \\ \rho_{\lambda} & \downarrow & \downarrow^{\rho_{\mu}} \\ \text{End}(V_{\lambda}) & \stackrel{\widetilde{\alpha}}{\longrightarrow} & \text{End}(V_{\mu}) \end{array}$$

where the vertical arrows are surjective and $\tilde{\alpha}$ is an isomorphism of algebras. Clearly, $\tilde{\alpha}$ maps the \overline{G} -grading of $\operatorname{End}(V_{\lambda})$ induced by Γ to the \overline{G} -grading of $\operatorname{End}(V_{\mu})$ induced by Γ' , where $\overline{G} = G/H_{\lambda} = G/H'_{\mu}$. Hence $\operatorname{Br}(\lambda) = \operatorname{Br}'(\mu)$. We conclude that the theory of graded modules for Γ' can be obtained from that for Γ as long as an isomorphism α is known. In particular, if α is inner then we have $H'_{\lambda} = H_{\lambda}$ and $\operatorname{Br}'(\lambda) = \operatorname{Br}(\lambda)$ for all λ .

For the classical simple Lie algebras of type A_1 and of series B and C, all automorphisms are inner so the H_{λ} are trivial, and it suffices to calculate $\operatorname{Br}(\lambda)$ for one representative of each isomorphism class of gradings. For series A (except A_1) and D (except D_4), the index of $\operatorname{Int}(\mathcal{L})$ in $\operatorname{Aut}(\mathcal{L})$ is 2 so the H_{λ} have order at most 2, and each isomorphism class of gradings consists of at most 2 orbits under the action of $\operatorname{Int}(\mathcal{L})$. Clearly, the isomorphism class of a grading Γ is one $\operatorname{Int}(\mathcal{L})$ orbit if and only if $\operatorname{Stab}(\Gamma)$ (i.e., the group of automorphisms of \mathcal{L} as a G-graded algebra) contains an outer automorphism. This is definitely the case if Γ is an "outer grading" (i.e., the image of \overline{G} in $\operatorname{Aut}(\mathcal{L})$ contains an outer automorphism). For series A, the gradings are already classified in [1] as inner (Type I) and outer (Type II), and it is clear when two inner gradings belong to one $\operatorname{Int}(\mathcal{L})$ -orbit. For series D, this will be done in Section 7. It should be noted that type D_4 is no different from other members of the series if we restrict our attention to "matrix gradings", i.e., those that can be obtained by restricting a grading of $M_8(\mathbb{F})$ (or, equivalently, such that \widehat{G} fixes the isomorphism class of the natural module).

3.6. Calculation of Brauer invariants. We conclude this section by showing how, for simple Lie algebras, the calculation of Brauer invariants of all dominant integral weights can be reduced to the fundamental weights $\omega_1, \ldots, \omega_r$.

Proposition 10. Let λ_1 and λ_2 be dominant integral weights of a semisimple Lie algebra \mathcal{L} and let $\mu = \lambda_1 + \lambda_2$. Suppose \mathcal{L} is equipped with a grading by an abelian group G such that $H_{\lambda_1} \subset H_{\mu}$ (equivalently, $H_{\lambda_2} \subset H_{\mu}$). Then $\operatorname{Br}(\mu) =$ $\operatorname{Br}(\lambda_1)\operatorname{Br}(\lambda_2)$ in the (G/H_{μ}) -graded Brauer group.

Proof. Denote $\Re_i = \operatorname{End}(V_{\lambda_i})$ (i = 1, 2) and $\overline{G} = G/H_{\mu}$. Let $V = V_{\lambda_1} \otimes V_{\lambda_2}$ and $\Re = \operatorname{End}(V)$. By definition of H_{λ_i} , the algebra \Re_i is (G/H_{λ_i}) -graded in such a way that the surjection $\rho_i \colon U(\mathcal{L}) \to \Re_i$ is a homomorphism of (G/H_{λ_i}) algebras, hence a fortiori a homomorphism of \overline{G} -graded algebras. Let $\rho \colon U(\mathcal{L}) \to \Re$ be the tensor product representation, i.e., the representation of \mathcal{L} on V given by $\rho(x) = \rho_1(x) \otimes 1 + 1 \otimes \rho_2(x)$ for all $x \in \mathcal{L}$. If we give \Re a \overline{G} -grading by identifying \Re and $\Re_1 \otimes \Re_2$ in the natural way then $\rho \colon U(\mathcal{L}) \to \Re$ is a homomorphism of \overline{G} graded algebras. This homomorphism is not surjective, in general, so let \mathcal{A} be its image, which is a graded unital subalgebra of \Re . It is known that the simple module V_{μ} occurs in V with multiplicity 1. Let $\varepsilon \in \Re$ be the projection of Vonto V_{μ} associated with the decomposition of V into isotypic components (as an \mathcal{L} -module). Then ε is the identity element in the block \mathcal{A}_0 of the semisimple algebra \mathcal{A} associated to the \mathcal{A} -module V_{μ} . By definition of H_{μ} , the kernel I of the surjection $\rho: U(\mathcal{L}) \to \operatorname{End}(V_{\mu})$ is a \overline{G} -graded ideal of $U(\mathcal{L})$, hence $\rho(I)$ is a graded ideal of \mathcal{A} . But \mathcal{A}_0 is the annihilator of $\rho(I)$ in \mathcal{A} , so \mathcal{A}_0 is a graded subalgebra of \mathcal{R} . Moreover, $\mathcal{A}_0 \cong \operatorname{End}(V_{\mu})$ as \overline{G} -graded algebras. Therefore,

$$Br(\mu) = [End(V_{\mu})] = [\mathcal{A}_0] = [\mathcal{E}\mathcal{R}\mathcal{E}] = [\mathcal{R}] = [\mathcal{R}_1][\mathcal{R}_2] = Br(\lambda_1)Br(\lambda_2),$$

where we have used Lemma 2.

If \mathcal{L} is simple then $\operatorname{Out}(\mathcal{L})$ has order at most 2 in all cases except D_4 . For D_4 , we have $\operatorname{Out}(\mathcal{L}) \cong S_3$ so the image of the abelian group \widehat{G} in $\operatorname{Out}(\mathcal{L})$ has order at most 3. It follows that, in all cases, the \widehat{G} -orbits in the set of dominant integral weights have length at most 3, and \widehat{G} acts cyclically on each orbit. Consider $\lambda = \sum_{i=1}^{s} m_i \omega_{k_i}$ where the $\{\omega_{k_1}, \ldots, \omega_{k_s}\}$ is an orbit and m_i are nonnegative integers. Since \widehat{G} cyclically permutes the ω_{k_i} , we have one of the following two possibilities: either all m_i are equal or $H_{\lambda} = H_{\omega_{k_i}}$ (for any *i*). It follows that, if we know the Brauer invariant of $\sum_{i=1}^{s} \omega_{k_i}$ for each orbit, then we can compute it for any dominant integral weight using Proposition 10.

Proposition 11. Let λ_1 and λ_2 be dominant integral weights of a semisimple Lie algebra \mathcal{L} and let $\mu = \lambda_1 + \lambda_2$. Suppose \mathcal{L} is equipped with a grading by an abelian group G such that $\{\lambda_1, \lambda_2\}$ is a \widehat{G} -orbit, so $H_{\mu} = \{e\}$ and $H_{\lambda_i} = \langle h \rangle$, i = 1, 2, where h has order 2, and the algebra $\mathcal{R} = \operatorname{End}(V_{\lambda_1} \otimes V_{\lambda_2})$ has a \overline{G} -grading by identification with $\operatorname{End}(V_{\lambda_1}) \otimes \operatorname{End}(V_{\lambda_2})$, where $\overline{G} = G/\langle h \rangle$. Fix $\chi \in \widehat{G}$ such that $\chi(h) = -1$ and pick isomorphisms $u' \colon V_{\lambda_1} \to V_{\lambda_2}^{\chi}$ and $u'' \colon V_{\lambda_2} \to V_{\lambda_1}^{\chi}$. Then $\operatorname{Br}(\mu) = [\mathcal{R}]$ in the G-grading as follows:

 $\mathfrak{R}_g = \{ x \in \mathfrak{R}_{\overline{g}} \mid uxu^{-1} = \chi(g)x \} \quad \textit{for all } g \in G,$

where $u = (u'' \otimes u') \circ \tau$ and τ is the flip $v' \otimes v'' \mapsto v'' \otimes v'$ for all $v' \in V_{\lambda_1}$ and $v'' \in V_{\lambda_2}$ (so $u \in \mathbb{R}$).

Proof. Let $\mathcal{R}_i = \text{End}(V_{\lambda_i})$. Then $\rho_i \colon U(\mathcal{L}) \to \mathcal{R}_i$ are homomorphisms of \overline{G} -graded algebras and the following diagram commutes:



where $\alpha'(a) = u'a(u')^{-1}$ for all $a \in \mathcal{R}_1$ and $\alpha''(b) = u''b(u'')^{-1}$ for all $b \in \mathcal{R}_2$. Since α_{χ} is an isomorphism of \overline{G} -graded algebras, so are α' and α'' . Define an automorphism $\widetilde{\alpha} : \mathcal{R} \to \mathcal{R}$ (as a \overline{G} -graded algebra) by setting

$$\widetilde{\alpha}(a \otimes b) = \alpha''(b) \otimes \alpha'(a) \quad \text{for all } a \in \mathfrak{R}_1, b \in \mathfrak{R}_2$$

Then $\tilde{\alpha}^2(a \otimes b) = \alpha''\alpha'(a) \otimes \alpha'\alpha''(b) = (\chi^2 * a) \otimes (\chi^2 * b)$ where, as usual, * denotes the actions of $K = K_{\lambda_1} = K_{\lambda_2}$ associated to the \overline{G} -gradings. Hence $\tilde{\alpha}^2$ acts as the scalar operator $\chi^2(\overline{g})$ on the homogeneous component $\mathcal{R}_{\overline{g}}$ (where we regard χ^2 as a character of \overline{G}), and we obtain a G-grading on \mathcal{R} by setting

$$\mathcal{R}_g = \{ x \in \mathcal{R}_{\overline{g}} \mid \widetilde{\alpha}(x) = \chi(g)x \} \text{ for all } g \in G.$$

For $\rho = \rho_1 \otimes \rho_2$, one checks that $\rho(\alpha_{\chi}(x)) = \tilde{\alpha}(\rho(x))$ for all $x \in \mathcal{L}$. It follows that $\rho: U(\mathcal{L}) \to \mathcal{R}$ is a homomorphism of *G*-graded algebras. By the same argument as in the proof of Proposition 10, we obtain $\operatorname{Br}(\mu) = [\mathcal{R}]$ in the *G*-graded Brauer group (H_{μ} is trivial in our case). It remains to observe that

$$\widetilde{\alpha}(a \otimes b) = u'' b(u'')^{-1} \otimes u' a(u')^{-1} = (u'' \otimes u') \tau(a \otimes b) \tau(u'' \otimes u')^{-1} = u(a \otimes b) u^{-1}$$

for all $a \in \mathcal{R}_1$ and $b \in \mathcal{R}_2$.

The result is especially simple in the case when V_{λ_1} and V_{λ_2} are dual to each other.

Proposition 12. Let λ_1 and λ_2 be dominant integral weights of a semisimple Lie algebra \mathcal{L} such that $V_{\lambda_2} \cong V^*_{\lambda_1}$, and let $\mu = \lambda_1 + \lambda_2$. Suppose \mathcal{L} is equipped with a grading by an abelian group G such that $\{\lambda_1, \lambda_2\}$ is a \widehat{G} -orbit. Then $Br(\mu)$ is trivial.

Proof. We will use the notation of Proposition 11 and the abbreviations $V = V_{\lambda_1}$ and $V^* = V_{\lambda_2}$. Then we have the following commutative diagram:



where adj denotes the map sending an operator $a \in \text{End}(V)$ to its adjoint $a^* \in \text{End}(V^*)$, which is determined by

$$\langle x, ay \rangle = \langle a^*x, y \rangle$$
 for all $x \in V^*, y \in V$,

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between V^* and V. It follows that $[\operatorname{End}(V^*)]$ is the inverse of $[\operatorname{End}(V)]$ in the \overline{G} -graded Brauer group. Hence the \overline{G} -grading on $\mathcal{R} = \operatorname{End}(V \otimes V^*)$ is elementary. We claim that this elementary grading on \mathcal{R} is induced from the \overline{G} -grading on the vector space $V \otimes V^*$ coming from the natural isomorphism $V \otimes V^* \cong \operatorname{End}(V)$. Indeed, for any $\psi \in K := K_{\lambda_1} = K_{\lambda_2}$, we have $\rho_1(\alpha_{\psi}(x)) = u_{\psi}\rho_1(x)u_{\psi}^{-1}$ for all $x \in \mathcal{L}$, hence $\rho_2(\alpha_{\psi}(x)) = (u_{\psi}^*)^{-1}\rho_2(x)u_{\psi}^*$ for all $x \in \mathcal{L}$. It follows that the \overline{G} -grading on the algebra $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ is associated to the K-action $\psi * x = (u_{\psi} \otimes (u_{\psi}^*)^{-1})x(u_{\psi} \otimes (u_{\psi}^*)^{-1})^{-1}$. Under the natural isomorphism $V \otimes V^* \cong \operatorname{End}(V)$, the operator $u_{\psi} \otimes (u_{\psi}^*)^{-1}$ on $V \otimes V^*$ corresponds to the inner automorphism $\operatorname{Ad}(u_{\psi})$ of $\operatorname{End}(V)$, and these inner automorphisms determine the \overline{G} -grading of $\operatorname{End}(V)$.

Now consider $\chi \in \widehat{G} \setminus K$, i.e., $\chi \in \widehat{G}$ with $\chi(h) = -1$. We may choose the associated isomorphisms $u': V \to (V^*)^{\chi}$ and $u'': V^* \to V^{\chi}$ so that $u'' = ((u')^*)^{-1}$. Define a nondegenerate bilinear form on V as follows:

$$(x,y) = \langle u'x, y \rangle$$
 for all $x, y \in V$,

and then define a linear map $\varphi \colon \operatorname{End}(V) \to \operatorname{End}(V)$ by setting

$$(ax, y) = (x, \varphi(a)y)$$
 for all $x, y \in V, a \in \text{End}(V)$,

i.e., $\varphi(a)$ is the adjoint of a with respect to the bilinear form (\cdot, \cdot) . Clearly, φ is an anti-automorphism. One checks that $\varphi(a)^* = u'a(u')^{-1}$ for all $a \in \text{End}(V)$.

Therefore, we have the following commutative diagram:



Since $-\alpha_{\chi}$ is a homomorphism of \overline{G} -graded algebras, so is φ . Moreover, φ^2 acts as the scalar operator $\chi^2(\overline{g})$ on the homogeneous component $\operatorname{End}(V)_{\overline{g}}$. It follows that we obtain a G-grading on the vector space $\operatorname{End}(V)$ by setting

(5)
$$\operatorname{End}(V)_g = \{a \in \operatorname{End}(V)_{\overline{g}} \mid \varphi(a) = -\chi(g)a\} \text{ for all } g \in G.$$

Since $-\varphi$ is a Lie homomorphism, this is actually a *G*-grading on the Lie algebra $\operatorname{End}(V)^{(-)}$. By construction, $\rho_1: \mathcal{L} \to \operatorname{End}(V)^{(-)}$ is a homomorphism of *G*-graded algebras. Finally, we claim that the operator $u = (u'' \otimes u')\tau$ on $V \otimes V^*$ corresponds to φ on $\operatorname{End}(V)$ under the natural isomorphism $V \otimes V^* \cong \operatorname{End}(V)$. Indeed, for any $z \in V$ and $f \in V^*$, the element $z \otimes f$ corresponds to $a(\cdot) = f(\cdot)z \in \operatorname{End}(V)$. For any $x, y \in V$, we compute:

$$\begin{aligned} ((z\otimes f)x,y) &= (f(x)z,y) = \langle f,x\rangle \langle u'z,y\rangle = \langle u'x,u''f\rangle \langle u'z,y\rangle \\ &= (x,(u'z)(y)(u''f)) = (x,u(z\otimes f)(y)), \end{aligned}$$

proving the claim. Therefore, in our case, the *G*-grading on $\mathcal{R} = \text{End}(\text{End}(V))$ in Proposition 11 is induced from the above *G*-grading on the vector space End(V). Thus $\text{Br}(\mu) = [\mathcal{R}]$ is trivial.

4. Series A

Consider the simple Lie algebra $\mathcal{L} = \mathfrak{sl}_{r+1}(\mathbb{F})$ of type A_r , where \mathbb{F} is an algebraically closed field of characteristic 0. Given a *G*-grading on \mathcal{L} , we are going to find the Brauer invariant of every simple \mathcal{L} -module. The parameters used in [1] (see also [5]) to determine the *G*-grading on \mathcal{L} (up to isomorphism) include what are, in our present terminology, the inertia group and the Brauer invariant of the natural module of \mathcal{L} .

4.1. **Preliminaries.** Recall that the calculation of Brauer invariants reduces to modules of the form V_{λ} where λ is the sum of fundamental weights forming a \widehat{G} -orbit. The simple \mathcal{L} -module of highest weight ω_i $(i = 1, \ldots, r)$ can be realized as $\wedge^i V$ where V is the natural module of \mathcal{L} , which has highest weight ω_1 and dimension n = r + 1.

Proposition 13. For the simple Lie algebra of type A_r graded by an abelian group G and for any i = 1, ..., r, we have $Br(\omega_i) = Br(\omega_1)^i$ in the (G/H_{ω_1}) -graded Brauer group.

Proof. Denote $\overline{G} = G/H_{\omega_1}$, $\mathfrak{R} = \operatorname{End}(V)$ and $\rho \colon \mathcal{L} \to \mathfrak{R}$ the natural representation. Then the algebra $\widetilde{\mathfrak{R}} = \operatorname{End}(V^{\otimes i})$ is \overline{G} -graded by identification with $\mathfrak{R}^{\otimes i}$, and $\rho^{\otimes i} \colon U(\mathcal{L}) \to \widetilde{\mathfrak{R}}$ is a homomorphism of \overline{G} -graded algebras. We identify $\wedge^i V$ with the space of skew-symmetric tensors in $V^{\otimes i}$. Let $\varepsilon \in \widetilde{\mathfrak{R}}$ be the standard projection $V^{\otimes i} \to \wedge^i V$, i.e.,

$$\varepsilon(v_1 \otimes \cdots \otimes v_i) = \frac{1}{i!} \sum_{\pi \in S_i} (-1)^{\pi} v_{\pi(1)} \otimes \cdots \otimes v_{\pi(i)} \quad \text{for all } v_1, \dots, v_i \in V.$$

Then $\operatorname{End}(\wedge^i V)$ can be identified with $\varepsilon \widetilde{\mathcal{R}} \varepsilon$. With this identification, we have $\rho^{\wedge i}(a) = \varepsilon \rho^{\otimes i}(a)\varepsilon$ for all $a \in U(\mathcal{L})$.

Now let K be the group of characters of \overline{G} . If the \overline{G} -grading on \mathfrak{R} is associated to the action $\chi * x = u_{\chi} x u_{\chi}^{-1}, \chi \in K, x \in \mathfrak{R}$, then the grading on $\widetilde{\mathfrak{R}}$ is associated to the action $\chi * x = \widetilde{u}_{\chi} x \widetilde{u}_{\chi}^{-1}, \chi \in K, x \in \widetilde{\mathfrak{R}}$, where

$$\tilde{u}_{\chi}(v_1 \otimes \cdots \otimes v_i) = u_{\chi}(v_1) \otimes \cdots \otimes u_{\chi}(v_i) \text{ for all } v_1, \dots, v_i \in V.$$

Clearly, $\tilde{u}_{\chi} \varepsilon \tilde{u}_{\chi}^{-1} = \varepsilon$ for all $\chi \in K$, so ε is a homogeneous idempotent of $\widetilde{\mathfrak{R}}$. Therefore,

$$\operatorname{Br}(\omega_i) = [\operatorname{End}(\wedge^i V)] = [\varepsilon \widehat{\mathcal{R}} \varepsilon] = [\widehat{\mathcal{R}}]^i = \operatorname{Br}(\omega_1)^i,$$

where we have used Lemma 2 (cf. the proof of Proposition 10).

4.2. Inner gradings on $\mathfrak{sl}_{r+1}(\mathbb{F})$. Gradings of Type I, i.e., such that the image of \widehat{G} in $\operatorname{Aut}(\mathcal{L})$ consists of inner automorphisms, are classified by the corresponding gradings on $\mathcal{R} = \operatorname{End}(V)$, and the latter by the graded division algebra \mathcal{D} representing the class $[\mathcal{R}]$ in the *G*-graded Brauer group and the multiset Ξ in G/T (determined up to a shift) representing the grading on a right vector space W over \mathcal{D} such that $\mathcal{R} = \operatorname{End}_{\mathcal{D}}(W)$, where $T \subset G$ is the support of \mathcal{D} . Explicitly, we can select a homogeneous \mathcal{D} -basis $\{v_1, \ldots, v_k\}$ of W and identify V with $W \otimes_{\mathcal{D}} N = \widetilde{W} \otimes N$ where $\widetilde{W} = \operatorname{span}\{v_1, \ldots, v_k\}$ (over \mathbb{F}) and N is the natural left (ungraded) module for the matrix algebra \mathcal{D} , so dim $N = \ell$ is the graded Schur index of V and $n = k\ell$.

Remark 14. The G-grading on V^{ℓ} in Proposition 5, where $\lambda = \omega_1$ and H_{λ} is trivial, is obtained by identifying V^{ℓ} with $V \otimes M = \widetilde{W} \otimes \mathcal{D} = W$ where M is the natural right module for the matrix algebra \mathcal{D} .

The *G*-grading on \mathcal{R} then comes from the identification of \mathcal{R} with $\mathcal{C} \otimes \mathcal{D}$ where $\mathcal{C} = \operatorname{End}(\widetilde{W})$. The multiset Ξ is $\{g_1T, \ldots, g_kT\}$ where $g_i = \deg v_i$. In [1, 5], the \mathcal{D} -basis was chosen in such a way that $g_i = g_j$ whenever $g_iT = g_jT$; here it will be convenient not to impose this condition. Finally, the *G*-grading on \mathcal{L} comes from the identification of \mathcal{L} with the graded subspace $[\mathcal{R}, \mathcal{R}]$ of \mathcal{R} . Two Type I gradings, Γ and Γ' , belong to one $\operatorname{Int}(\mathcal{L})$ -orbit if and only if T' = T, $\beta' = \beta$ and $\Xi' = g\Xi$ for some $g \in G$ (see [1] or [5, Theorem 3.53]).

Theorem 15. Let \mathcal{L} be the simple Lie algebra of type A_r over an algebraically closed field \mathbb{F} of characteristic 0. Suppose \mathcal{L} is graded by an abelian group G such that the image of \hat{G} in Aut(\mathcal{L}) consists of inner automorphisms (Type I grading). Then, for any dominant integral weight $\lambda = \sum_{i=1}^{r} m_i \omega_i$, we have $H_{\lambda} = \{e\}$ and $\operatorname{Br}(\lambda) = \hat{\beta} \sum_{i=1}^{r} im_i$, where $\hat{\beta} \colon \hat{G} \times \hat{G} \to \mathbb{F}^{\times}$ is the commutation factor associated to the parameters (T, β) of the grading on \mathcal{L} .

Proof. Since H_{ω_i} is trivial for all *i*, we can apply Proposition 10:

$$\operatorname{Br}(\lambda) = \prod_{i=1}^{r} \operatorname{Br}(\omega_i)^{m_i}$$

But by Proposition 13, we have $Br(\omega_i) = Br(\omega_1)^i = \hat{\beta}^i$. The result follows. \Box

Corollary 16. The simple \mathcal{L} -module V_{λ} admits a *G*-grading making it a graded \mathcal{L} -module if and only if the number $\sum_{i=1}^{r} im_i$ is divisible by the exponent of the group *T*.

14

Example 17. Consider $\mathcal{L} = \mathfrak{sl}_2(\mathbb{F})$. Then either $T = \{e\}$ or $T = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_2^2$. In the first case, any simple \mathcal{L} -module has trivial Brauer invariant and hence admits a G-grading, as expected because the grading on \mathcal{L} is a coarsening of Cartan grading. In the second case (so-called Pauli grading on \mathcal{L}), T has a unique nondegenerate alternating bicharacter β (defined on the generators by $\beta(a, a) = \beta(b, b) = 1$ and $\beta(a, b) = \beta(b, a) = -1$), for which $\beta^2 = 1$. Hence $\operatorname{Br}(m\omega_1)$ is trivial for even m and equals $\hat{\beta}$ for odd m. In particular, the simple \mathcal{L} -modules of even highest weight admit a G-grading but those of odd highest weight do not: they have graded Schur index 2.

4.3. Outer gradings on $\mathfrak{sl}_{r+1}(\mathbb{F})$. If r > 1 then there exist gradings of Type II, i.e., such that the image of \widehat{G} in $\operatorname{Aut}(\mathcal{L})$ contains an outer automorphism. Every such grading has a distinguished element $h \in G$ of order 2, which is characterized by the property that the corresponding \overline{G} -grading is of Type I, where $\overline{G} = G/\langle h \rangle$. (Using our present notation, we have $H_{\omega_1} = \langle h \rangle$.) Gradings of Type II with a fixed distinguished element h are classified by the corresponding \overline{G} -graded algebras \mathcal{R} equipped with an anti-automorphism φ representing (the negative of) the action of a fixed $\chi \in \widehat{G}$ with $\chi(h) = -1$. Namely, the Type II grading on \mathcal{L} corresponding to (\mathcal{R}, φ) comes from the identification of \mathcal{L} with the graded subspace $[\mathcal{R}, \mathcal{R}]$ of the Lie algebra $\mathcal{R}^{(-)}$ equipped with the G-grading given by (5).

The existence of φ forces $\overline{T} \subset \overline{G}$ to be an elementary 2-group, where \overline{T} is the support of the graded division algebra \mathcal{D} representing $[\mathcal{R}]$ in the \overline{G} -graded Brauer group. Moreover, for any $x \in \mathcal{R}$, $\varphi(x)$ is the adjoint of x with respect to a suitable nondegenerate \mathbb{F} -bilinear form $B: W \times W \to \mathcal{D}$ which is \mathcal{D} -sesquilinear relative to an involution of the graded algebra \mathcal{D} (see [4] or [5]). To write φ explicitly, we can fix a "standard realization" of \mathcal{D} as a matrix algebra in the following way. To simplify notation, we temporarily omit bars and write T instead of \overline{T} . Select a symplectic basis $\{a_j\} \cup \{b_j\}$ of T (as a vector space over the field of two elements) with respect to the nondegenerate alternating bicharacter $\beta: T \times T \to \mathbb{F}^{\times}$, i.e., $\beta(a_j, b_j) = -1$ and all other values of β on basis elements are equal to 1. Then we can take $\mathcal{D} = M_2(\mathbb{F}) \otimes \cdots \otimes M_2(\mathbb{F})$ with the following T-grading:

$$X_{a_j} = 1 \otimes \cdots \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes 1$$
 and $X_{b_j} = 1 \otimes \cdots \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes 1$,

where the indicated matrix appears in the j-th factor, and

$$X_{(\prod a_j^{\xi_j})(\prod b_j^{\eta_j})} = \big(\prod X_{a_j}^{\xi_j}\big)\big(\prod X_{b_j}^{\eta_j}\big).$$

Note that we have ${}^{t}X_{a_{j}} = X_{a_{j}}^{-1} = X_{a_{j}}$ and ${}^{t}X_{b_{j}} = X_{b_{j}}^{-1} = X_{b_{j}}$ for all j, and hence (6) ${}^{t}X_{s} = X_{s}^{-1} = \beta(s)X_{s}$ for all $s \in T$, where $\beta \colon T \to \{\pm 1\}$ is a quadratic form on T (as a vector space over the field of

where $\beta: T \to \{\pm 1\}$ is a quadratic form on T (as a vector space over the field of two elements) whose polar bilinear form is $\beta(\cdot, \cdot)$, i.e., $\beta(s,t) = \beta(st)\beta(s)\beta(t)$ for all $s, t \in T$.

Remark 18. This matrix realization of \mathcal{D} can be presented in a concise way if we write $T = A \times B$, where $A = \langle a_j \rangle$ and $B = \langle b_j \rangle$, and take for N the vector space $\mathbb{F}B = \text{span} \{e_b \mid b \in B\}$ with the following action of \mathcal{D} : $X_{(a,b)}e_{b'} = \beta(a,bb')e_{bb'}$ for all $a \in A$ and $b, b' \in B$.

Lemma 19. If $|T| \neq 4$ then $\det(X_t) = 1$ for all $t \in T$. If |T| = 4 then $\det(X_t) = \beta(c,t)$ for all $t \in T$, where c = ab.

Fix a standard matrix realization of \mathcal{D} . Adjusting the sesquilinear form B, we may assume that the involution of \mathcal{D} is the matrix transpose. Selecting a suitable homogeneous \mathcal{D} -basis $\{v_1, \ldots, v_k\}$ in W, deg $v_i = \overline{g}_i$, we may assume that

(7)
$$\overline{g}_1^2 \overline{t}_1 = \dots = \overline{g}_q^2 \overline{t}_q = \overline{g}_{q+1} \overline{g}_{q+2} = \dots = \overline{g}_{q+2s-1} \overline{g}_{q+2s} = \overline{g}_0^{-1}$$

where q + 2s = k, $\overline{g}_0 \in \overline{G}$, $\overline{t}_i \in \overline{T}$, and the sesquilinear form B is represented by the following block-diagonal matrix:

(8)
$$\Phi = \operatorname{diag}\left(X_{t_1}, \dots, X_{t_q}, \begin{pmatrix} 0 & I\\ \mu_1 I & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I\\ \mu_s I & 0 \end{pmatrix}\right)$$

where $\mu_i \in \mathbb{F}^{\times}$ and $I = X_e \in \mathcal{D}$ (see [4] or [5, Theorem 3.31]). The element \overline{g}_0 has the meaning of the degree of B as a linear map $W \otimes W \to \mathcal{D}$. The scalars μ_i can be expressed in terms of a single scalar μ_0 satisfying $\mu_0^2 = \chi^2(\overline{g}_0^{-1})$, namely, $\mu_i = \mu_0 \chi^2(\overline{g}_{q+2i-1}^{-1})$. The gradings of Type II with distinguished element h are classified by the graded division algebra \mathcal{D} , the multiset $\Xi = \{\overline{g}_1 \overline{T}, \ldots, \overline{g}_k \overline{T}\}$, and the elements $\mu_0 \in \mathbb{F}^{\times}$ and $\overline{g}_0 \in \overline{G}$ (see [1] or [5, Theorem 3.53]). Identifying \mathcal{R} with $M_k(\mathcal{D})$ through the \mathcal{D} -basis $\{v_1, \ldots, v_k\}$ and using the fixed matrix realization of \mathcal{D} , we can write the anti-automorphism φ in matrix form as follows:

$$\varphi(X) = \Phi^{-1}({}^{t}X)\Phi \quad \text{for all } X \in M_n(\mathbb{F}),$$

where ${}^{t}X$ is the transpose of X. Note that $\varphi(X)$ is the adjoint of X with respect to the nondegenerate bilinear form $(v', v'')_{\Phi} = {}^{t}v'\Phi v''$ on V, which can also be expressed, using the identification $V = W \otimes_{\mathcal{D}} N = \widetilde{W} \otimes N$, as follows:

(9)
$$(w' \otimes x, w'' \otimes y)_{\Phi} = {}^{t}xB(w', w'')y \text{ for all } w', w'' \in W, x, y \in N,$$

where the expression in the right-hand side is matrix product (using the fixed matrix realization of \mathcal{D}).

Recall that the action of a character ψ of \overline{G} on the \overline{G} -graded algebra \mathcal{R} is given by $\psi * x = u_{\psi} x u_{\psi}^{-1}$ ($x \in \mathcal{R}$) where

(10)
$$u_{\psi} = \operatorname{diag}(\psi(\overline{g}_1), \dots, \psi(\overline{g}_k)) \otimes X_{\overline{t}}$$

and $\overline{t} \in \overline{T}$ is determined by the condition $\beta(\overline{t}, \overline{s}) = \psi(\overline{s})$ for all $\overline{s} \in \overline{T}$.

Lemma 20. For any $\psi \in \widehat{G}$ with $\psi(h) = 1$, we have $(u_{\psi}v', u_{\psi}v'')_{\Phi} = \psi(\overline{g}_0^{-1})(v', v'')_{\Phi}$ for all $v', v'' \in V$.

Proof. It is sufficient to verify ${}^{t}u_{\psi}\Phi u_{\psi} = \psi(\overline{g}_{0}^{-1})\Phi$, which follows from orthogonality of $X_{\overline{t}}$ and the equations

$$\psi(\overline{g}_i)^2 X_{\overline{t}}^{-1} X_{\overline{t}_i} X_{\overline{t}} = \psi(\overline{g}_i)^2 \beta(\overline{t}, \overline{t}_i) X_{\overline{t}_i} = \psi(\overline{g}_i)^2 \psi(\overline{t}_i) X_{\overline{t}_i} = \psi(\overline{g}_0^{-1}) X_{\overline{t}_i},$$

for each $i = 1, \ldots, q$, as well as the equations $\psi(\overline{g}_{q+2j-1})\psi(\overline{g}_{q+2j}) = \psi(\overline{g}_0^{-1})$, for each $j = 1, \ldots, s$.

We will need the following notation to state our main result on Type II gradings. Let ℓ be the \overline{G} -graded Schur index of V, so ℓ is a power of 2, $|\overline{T}| = \ell^2$ and dim $V = n = k\ell$. If n is even, then set

(11)
$$\bar{h}' = \begin{cases} \overline{g}_0^{n/2} (\overline{g}_1 \cdots \overline{g}_k)^\ell & \text{if } \ell \neq 2, \\ (\overline{c}\overline{g}_0)^{n/2} (\overline{g}_1 \cdots \overline{g}_k)^\ell & \text{if } \ell = 2, \end{cases}$$

where, for $\ell = 2$, \bar{c} is the element of \overline{T} as in Lemma 19 ($\bar{c} = \bar{a}\bar{b}$ where $\{\bar{a}, \bar{b}\}$ is the selected basis of \overline{T}). Note that relations (7) imply that \bar{h}' has order at most 2 and depends only on $\bar{g}_1, \ldots, \bar{g}_q$.

16

Theorem 21. Let \mathcal{L} be the simple Lie algebra of type A_r (r > 1) over an algebraically closed field \mathbb{F} of characteristic 0. Suppose \mathcal{L} is graded by an abelian group G such that the image of \widehat{G} in Aut (\mathcal{L}) contains outer automorphisms and hence $H_{\omega_1} = \langle h \rangle$ for some $h \in G$ of order 2 (Type II grading with distinguished element h). Let $K = K_{\omega_1} = \langle h \rangle^{\perp}$ and fix $\chi \in \widehat{G} \setminus K$ (i.e., $\chi(h) = -1$). Then, for a dominant integral weight $\lambda = \sum_{i=1}^{r} m_i \omega_i$, we have the following possibilities:

- 1) If $m_i \neq m_{r+1-i}$ for some *i*, then $H_{\lambda} = \langle h \rangle$, $K_{\lambda} = K$, and $\operatorname{Br}(\lambda) = \hat{\beta} \sum_{i=1}^{r} im_i = \hat{\beta} \sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}$, where $\hat{\beta} \colon K \times K \to \mathbb{F}^{\times}$ is the commutation factor associated to the parameters (\overline{T}, β) of the grading on \mathcal{L} .
- 2a) If r is even and $m_i = m_{r+1-i}$ for all i, then $H_{\lambda} = \{e\}$ and $Br(\lambda) = 1$.
- 2b) If r is odd and $m_i = m_{r+1-i}$ for all i, then $H_{\lambda} = \{e\}$ and $\operatorname{Br}(\lambda) = \hat{\gamma}^{m_{(r+1)/2}}$, where $\hat{\gamma} \colon \widehat{G} \times \widehat{G} \to \mathbb{F}^{\times}$ is the extension of the alternating bicharacter $\hat{\beta}^{(r+1)/2} \colon K \times K \to \mathbb{F}^{\times}$ given by $\hat{\gamma}(\chi, \psi) = \psi(h')$ for all $\psi \in \widehat{G}$, where $\hat{\beta}$ is the commutation factor associated to the parameters (\overline{T}, β) of the grading on \mathcal{L} and $h' \in G$ is the unique element satisfying $\chi(h') = 1$ in the coset $\overline{h'}$ defined by (11). The order of h' is at most 2.

Proof. Since χ acts by an outer automorphism, we have $V_{r+1-i} \cong V_i^{\chi}$. The \hat{G} -orbits in the set of fundamental weights are the pairs $\{\omega_i, \omega_{r+1-i}\}, i = 1, \ldots, \lfloor r/2 \rfloor$, and, if r is odd, also the singleton $\{\omega_{(r+1)/2}\}$.

1) In this case, we actually deal with a Type I grading by $\overline{G}=G/\langle h\rangle,$ so Theorem 15 applies.

2a) Here $\lambda = \sum_{i=1}^{r/2} m_i (\omega_i + \omega_{r+1-i})$. Since $H_{\omega_i + \omega_{r+1-i}}$ is trivial for all *i*, we can apply Proposition 10:

$$\operatorname{Br}(\lambda) = \prod_{i=1}^{r/2} \operatorname{Br}(\omega_i + \omega_{r+1-i})^{m_i}.$$

Since V_{ω_i} and $V_{\omega_{r+1-i}}$ are dual to each other, Proposition 12 gives the result. 2b) Now $\lambda = m_{(r+1)/2} \omega_{(r+1)/2} + \sum_{i=1}^{(r-1)/2} m_i(\omega_i + \omega_{r+1-i})$ and hence

Br(
$$\lambda$$
) = Br($\omega_{(r+1)/2}$) ^{$m_{(r+1)/2}$} $\prod_{i=1}^{(r-1)/2}$ Br($\omega_i + \omega_{r+1-i}$) ^{m_i} = Br($\omega_{(r+1)/2}$) ^{$m_{(r+1)/2}$}

by Propositions 10 and 12. It remains to consider the module $V_{\omega_p} = \wedge^p V$ for $p = \frac{r+1}{2}$ and compute the associated commutation factor $\hat{\gamma} : \hat{G} \times \hat{G} \to \mathbb{F}^{\times}$. The restriction of $\hat{\gamma}$ to $K \times K$ is equal to $\hat{\beta}^p$ according to Proposition 13. Since $\hat{\gamma}(\chi, \chi) = 1$, it suffices to show that $\hat{\gamma}(\chi, \psi) = \psi(\bar{h}')$ for all $\psi \in K$. By definition of the commutation factor, we have

(12)
$$\tilde{u}_{\chi}\tilde{u}_{\psi} = \hat{\gamma}(\chi,\psi)\tilde{u}_{\psi}\tilde{u}_{\chi},$$

where $\tilde{u}_{\chi}: (\wedge^p V)^{\chi^{-1}} \to \wedge^p V$ and $\tilde{u}_{\psi}: (\wedge^p V)^{\psi^{-1}} \to \wedge^p V$ are isomorphisms of \mathcal{L} -modules. We can take for \tilde{u}_{ψ} the mapping $x_1 \wedge \cdots \wedge x_p \mapsto u_{\psi} x_1 \wedge \cdots \wedge u_{\psi} x_p$ $(x_i \in V)$ where u_{ψ} is given by (10). We will now describe \tilde{u}_{χ} .

Let $u': V^{\chi^{-1}} \to V^*$ be an isomorphism of \mathcal{L} -modules. Then, for any $a \in \mathcal{R}$, $\varphi(a)$ is the adjoint of a with respect to the bilinear form $(x, y) = \langle u'x, y \rangle$ on V— see the proof of Proposition 12 (with $\lambda_1 = \omega_1$). Multiplying u' by a scalar if necessary, we may assume that $(x, y)_{\Phi} = \langle u'x, y \rangle$ for all $x, y \in V$. Let \tilde{u} be the induced isomorphism $(\wedge^p V)^{\chi^{-1}} \to \wedge^p (V^*)$, which is given by $\tilde{u}(x_1 \wedge \cdots \wedge x_p) = u'x_1 \wedge \cdots \wedge u'x_p$ for all $x_i \in V$. The \mathcal{L} -module $\wedge^p (V^*)$ can be identified with $(\wedge^p V)^*$ through the canonical pairing

$$\langle f_1 \wedge \dots \wedge f_q, y_1 \wedge \dots \wedge y_q \rangle = \sum_{\pi \in S_p} (-1)^{\pi} \langle f_1, y_{\pi(1)} \rangle \dots \langle f_p, y_{\pi(p)} \rangle \text{ for all } y_i \in V, \ f_i \in V^*.$$

Hence $\langle \tilde{u}x, y \rangle = (x, y)_{\Phi}$ for all $x, y \in \wedge^p V$, where the bilinear form $(\cdot, \cdot)_{\Phi}$ is defined on $\wedge^p V$ as follows:

$$(x_1 \wedge \dots \wedge x_q, y_1 \wedge \dots \wedge y_q)_{\Phi} = \sum_{\pi \in S_p} (-1)^{\pi} (x_1, y_{\pi(1)})_{\Phi} \cdots (x_p, y_{\pi(p)})_{\Phi} \text{ for all } x_i, y_i \in V.$$

Note that, by Lemma 20, we have

(13)
$$(\tilde{u}_{\psi}x, \tilde{u}_{\psi}y)_{\Phi} = \psi(\overline{g}_0^{-p})(x, y)_{\Phi} \quad \text{for all } x, y \in \wedge^p V.$$

Since $2p = \dim V$, the \mathcal{L} -module $\wedge^p V$ is isomorphic to its dual $(\wedge^p V)^*$ via the bilinear form on $\wedge^p V$ given by $(x, y) = x \wedge y$. Namely, an isomorphism $\theta \colon \wedge^p V \to (\wedge^p V)^*$ is defined by the rule $\langle \theta(x), y \rangle = (x, y)$ for all $x, y \in \wedge^p V$. Hence $\theta^{-1}\tilde{u}$ is an isomorphism $(\wedge^p V)^{\chi^{-1}} \to \wedge^p V$, so we can take $\tilde{u}_{\chi} = \theta^{-1}\tilde{u}$. By construction, we have

$$(\tilde{u}_{\chi}x, y) = \langle \tilde{u}x, y \rangle = (x, y)_{\Phi}$$
 for all $x, y \in \wedge^p V$.

Finally, we can calculate $\hat{\gamma}(\chi, \psi)$ using equation (12). For any $x, y \in \wedge^p V$, on the one hand, we obtain

$$(\tilde{u}_{\chi}\tilde{u}_{\psi}x,y) = (\tilde{u}_{\psi}x,y)_{\Phi} = \psi(\overline{g}_0^{-p})(x,\tilde{u}_{\psi}^{-1}y)_{\Phi},$$

where we have used (13). On the other hand, we obtain

$$(\tilde{u}_{\psi}\tilde{u}_{\chi}x,y) = \det(u_{\psi})(\tilde{u}_{\chi}x,\tilde{u}_{\psi}^{-1}y) = \det(u_{\psi})(x,\tilde{u}_{\psi}^{-1}y)_{\Phi}.$$

It follows that

$$\hat{\gamma}(\chi,\psi)^{-1} = \psi(\overline{g}_0^p) \det(u_\psi).$$

Looking at (10), we see that $\det(u_{\psi}) = \left(\prod_{i=1}^{k} \psi(\overline{g}_{i})^{\ell}\right) \det(X_{\overline{t}})^{k}$. Taking into account Lemma 19 and recalling that $\beta(\overline{c},\overline{t}) = \psi(\overline{c})$ by definition of u_{ψ} , we obtain

$$\hat{\gamma}(\chi,\psi)^{-1} = \begin{cases} \psi(\overline{g}_0^p)\psi\big(\prod_{i=1}^k \overline{g}_i^\ell\big) & \text{if } \ell \neq 2, \\ \psi(\overline{g}_0^p)\psi\big(\prod_{i=1}^k \overline{g}_i^\ell\big)\psi(\overline{c})^k & \text{if } \ell = 2, \end{cases}$$

so $\hat{\gamma}(\chi,\psi) = \psi(\bar{h}')^{-1} = \psi(\bar{h}')$ since \bar{h}' has order at most 2. Substituting $\psi = \chi^2$, we get $1 = \hat{\gamma}(\chi,\chi^2) = \chi^2(\bar{h}')$. Let h' be one of the two elements in the coset \bar{h}' . Since $\chi(h')^2 = \chi^2(\bar{h}') = 1$, we see that $(h')^2 \neq h$ and hence $(h')^2 = e$. Replacing h' with h'h if necessary, we obtain $\chi(h') = 1$. Then $\hat{\gamma}(\chi,\psi) = \psi(h')$ for all $\psi \in \hat{G}$. \Box

Remark 22. If $r \equiv 3 \pmod{4}$ then $\hat{\beta}^{(r+1)/2} = 1$ and the support T of the graded division algebra representing $\operatorname{Br}(\omega_{(r+1)/2})$ is $\{e\}$ if h' = e and $\langle h, h' \rangle \cong \mathbb{Z}_2^2$ if $h' \neq e$.

Proof. The property $\hat{\gamma}(\chi, \psi) = \psi(h')$ implies that $\operatorname{rad} \hat{\gamma} = \widehat{G}$ if h' = e and $\operatorname{rad} \hat{\gamma} = K \cap \langle h' \rangle^{\perp}$ if $h' \neq e$. Since $T = (\operatorname{rad} \hat{\gamma})^{\perp}$, we get $T = \{e\}$ if h' = e and $T = \langle h, h' \rangle$ if $h' \neq e$.

Remark 23. If $r \equiv 1 \pmod{4}$ then $\hat{\beta}^{(r+1)/2} = \hat{\beta}$ and the support T of the graded division algebra representing $\operatorname{Br}(\omega_{(r+1)/2})$ is \overline{T} if $\overline{h}' \in \overline{T}$ and $\overline{T} \times \langle h, h' \rangle \cong \overline{T} \times \mathbb{Z}_2^2$ (orthogonal sum relative to γ) if $\overline{h}' \notin \overline{T}$, where \overline{T} is regarded as a subgroup of G by identifying $\overline{t} \in \overline{T}$ with the unique element t in the coset \overline{t} satisfying $\chi(t) = \beta(\overline{h}', \overline{t})$ in the case $\overline{h}' \in \overline{T}$ and $\chi(t) = 1$ in the case $\overline{h}' \notin \overline{T}$.

Proof. The property $\hat{\gamma}(\chi, \psi) = \psi(h')$ implies that $\operatorname{rad} \hat{\gamma} = \operatorname{rad} \hat{\beta} \cap \langle h' \rangle^{\perp}$ if $\operatorname{rad} \hat{\beta} \not\subset \langle h' \rangle^{\perp}$. The condition $\operatorname{rad} \hat{\beta} \subset \langle h' \rangle^{\perp}$ is equivalent to $\bar{h}' \in \overline{T}$. If this is the case, then $\operatorname{rad} \hat{\gamma} = \langle \operatorname{rad} \hat{\beta}, \tilde{\chi} \rangle$ where $\tilde{\chi} = \chi \psi_0$ and ψ_0 is any element of K satisfying $\psi_0|_{\overline{T}} = \beta(\bar{h}', \cdot)$. Let H be the pre-image of \overline{T} under the quotient map $G \to \overline{G}$, so $H^{\perp} = \operatorname{rad} \hat{\beta}$.

If $\bar{h}' \in \overline{T}$ then $T = (\operatorname{rad} \hat{\gamma})^{\perp} = H \cap \langle \tilde{\chi} \rangle^{\perp}$ is a subgroup of index 2 in H (since $\tilde{\chi} \notin \operatorname{rad} \hat{\beta}$ and $\tilde{\chi}^2 \in \operatorname{rad} \hat{\beta}$), so T is precisely the isomorphic copy of \overline{T} in G obtained as indicated.

If $\overline{h}' \notin \overline{T}$ then $T = \langle H, h' \rangle$. The definition of $\hat{\gamma}(\cdot, \cdot)$ and the fact that $\hat{\gamma}(\chi, \psi) = \psi(h')$ for all $\psi \in \widehat{G}$ imply that $\gamma(t, h') = \chi(t)$ for all $t \in T$. In particular, $\gamma(h, h') = -1$ and hence the restriction of $\gamma(\cdot, \cdot)$ to $\langle h, h' \rangle$ is nondegenerate. Also, $\psi(h) = \hat{\gamma}(\chi', \psi)$ for all $\psi \in \widehat{G}$, where $\chi' \in \operatorname{rad} \hat{\beta}$. It follows that $T = \overline{T} \times \langle h, h' \rangle$, where \overline{T} is a subgroup of G obtained as indicated. \Box

Corollary 24. The simple \mathcal{L} -module V_{λ} admits a *G*-grading making it a graded \mathcal{L} -module if and only if 1) for all $i, m_i = m_{r+1-i}$ and 2) either r is even or r is odd and one of the following conditions is satisfied: (i) $m_{(r+1)/2}$ is even or (ii) $r \equiv 3 \pmod{4}$ and $\overline{h}' = \overline{e}$ in \overline{G} or (iii) $r \equiv 1 \pmod{4}$, $\overline{T} = \{\overline{e}\}$ and $\overline{h}' = \overline{e}$ in \overline{G} .

5. Series B

Consider the simple Lie algebra $\mathcal{L} = \mathfrak{so}_{2r+1}(\mathbb{F})$ of type B_r , $r \geq 2$, where \mathbb{F} is an algebraically closed field of characteristic 0. In this case all automorphisms of \mathcal{L} are inner. For a given *G*-grading on \mathcal{L} , we are going to find the Brauer invariants of all simple \mathcal{L} -modules.

In [1], the G-gradings on \mathcal{L} are classified in terms of the corresponding G-graded algebras $\mathcal{R} = \operatorname{End}(V)$ equipped with an orthogonal involution φ where V is the natural module of \mathcal{L} , which has highest weight ω_1 and dimension n = 2r + 1. Namely, the grading on \mathcal{L} corresponding to (\mathcal{R}, φ) comes from the identification of \mathcal{L} with the graded subspace $\mathcal{K}(\mathcal{R}, \varphi) = \{x \in \mathcal{R} \mid \varphi(x) = -x\}.$

The existence of the involution φ forces $T \subset G$ to be an elementary 2-group, where T is the support of the graded division algebra \mathcal{D} representing $[\mathcal{R}]$ in the Ggraded Brauer group. But |T| is a divisor of the odd number n^2 , hence $T = \{e\}$, i.e., the Brauer invariant of the natural module is trivial. In other words, the G-grading on $\mathcal{R} = \operatorname{End}(V)$ is induced by some G-grading $V = \bigoplus_{g \in G} V_g$, which is determined up to a shift. Since, for $i = 1, \ldots, r - 1$, the simple \mathcal{L} -module of highest weight ω_i can be realized as $\wedge^i V$, which has a natural G-grading induced from V, we see that the Brauer invariants of all fundamental weights, except possibly ω_r , are trivial. The simple \mathcal{L} -module V_{ω_r} is called the *spin module* and can be constructed in the following way, using the Clifford algebra $\mathfrak{Cl}(V, Q)$ where Q is the quadratic form associated to the orthogonal involution φ .

For any $x \in \mathcal{R}$, $\varphi(x)$ is the adjoint of x with respect to a nondegenerate symmetric F-bilinear form $B: V \times V \to \mathbb{F}$. Selecting a suitable homogeneous basis $\{v_1, \ldots, v_n\}$ in V, $\deg v_i = g_i$, we may assume that

(14)
$$g_1^2 = \ldots = g_q^2 = g_{q+1}g_{q+2} = \ldots = g_{q+2s-1}g_{q+2s} = g_0^{-1},$$

where q + 2s = n, $g_0 \in G$, and the bilinear form B is represented by the following block-diagonal matrix:

(15)
$$\Phi = \operatorname{diag}\left(1, \dots, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

where the number of 1's is q (see [4] or [5, Theorem 3.42]). The element g_0 has the meaning of the degree of B as a linear map $V \otimes V \to \mathbb{F}$. Since q is odd, we have $q \geq 1$ and hence g_0 is a square in G, so we may, and will, shift the G-grading on V so that g_0 becomes e. The G-gradings on \mathcal{L} are classified by the multiset $\Xi = \{g_1, \ldots, g_n\}$ (see [1] or [5, Theorem 3.65]).

Remark 25. Since \mathbb{F} is algebraically closed, we may take, at the expense of adding blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the elements g_1, \ldots, g_q to be distinct.

The quadratic form Q is given by $Q(v) = \frac{1}{2}B(v, v)$ so that B is the polar form of Q, i.e., B(u, v) = Q(u + v) - Q(u) - Q(v). In particular $Q(v_i) = \frac{1}{2}$ for $i = 1, \ldots, q$ and $Q(v_i) = 0$ for $i = q + 1, \ldots, q + 2s$.

Recall that the Clifford algebra $\mathfrak{Cl}(V) = \mathfrak{Cl}(V,Q)$ of a quadratic space (V,Q)is the quotient of the tensor algebra T(V) by the ideal generated by the elements $v \otimes v - Q(v)1$, for all $v \in V$. We will denote by $x \cdot y$ the product of elements in $\mathfrak{Cl}(V)$. The algebra $\mathfrak{Cl}(V)$ is naturally graded by \mathbb{Z}_2 : $\mathfrak{Cl}(V) = \mathfrak{Cl}_{\bar{0}}(V) \oplus \mathfrak{Cl}_{\bar{1}}(V)$, with V contained in the odd part.

The Lie algebra $\mathfrak{so}(V, Q)$ imbeds in $\mathfrak{Cl}(V, Q)^{(-)}$ (actually, in its even part) as the subspace $[V, V]^{\cdot} = \operatorname{span} \{ u \cdot v - v \cdot u \mid u, v \in V \}$. Indeed, for $u, v, w \in V$,

(16)

$$ad_{[u,v]}(w) = [[u,v]',w]'$$

$$= u \cdot v \cdot w - v \cdot u \cdot w - w \cdot u \cdot v + w \cdot v \cdot u$$

$$= (B(v,w)u - u \cdot w \cdot v) - (B(u,w)v - v \cdot w \cdot u)$$

$$- (B(u,w)v - u \cdot w \cdot v) + (B(v,w)u - v \cdot w \cdot u)$$

$$= -2(B(u,w)v - B(v,w)u),$$

and the linear maps

(

(17)
$$\sigma_{u,v}: w \mapsto B(u,w)v - B(v,w)u$$

span $\mathfrak{so}(V,Q)$. Hence we have an imbedding $\rho \colon \mathfrak{so}(V,Q) \to \mathfrak{Cl}_{\bar{0}}(V,Q)$ defined by the property $\rho(\sigma_{u,v}) = -\frac{1}{2}[u,v]^{\cdot}$.

In the case at hand, dim V is odd, so the even Clifford algebra $\mathfrak{Cl}_{\bar{0}}(V)$ is simple, and its unique (up to isomorphism) irreducible module is the spin module of \mathcal{L} .

The *G*-grading on *V* induces a *G*-grading on T(V), and the elements $v \otimes v - Q(v)1$ are homogeneous because *B* is a homogeneous map $V \otimes V \to \mathbb{F}$ of degree *e*. Hence $\mathfrak{Cl}(V)$ inherits a *G*-grading from T(V), and *V* imbeds in $\mathfrak{Cl}(V)$ as a graded subspace. The *G*-grading and the \mathbb{Z}_2 -grading on $\mathfrak{Cl}(V)$ are compatible in the sense that the homogeneous components of one are graded subspaces with respect to the other. Since the *G*-gradings on \mathcal{L} and on $\mathfrak{Cl}_{\bar{0}}(V)$ are both induced from *V*, the imbedding $\rho: \mathcal{L} \to \mathfrak{Cl}_{\bar{0}}(V)^{(-)}$ is a homomorphism of *G*-graded algebras.

Recall that, for $\chi \in \widehat{G}$, the action of χ on V is given by the matrix:

$$u_{\chi} = \operatorname{diag}(\chi(g_1), \dots, \chi(g_{q+2s}))$$

and the action of χ on \Re and \mathcal{L} is the conjugation by u_{χ} . Let $\lambda_i = \chi(g_i)$. Note that equation (14), with $g_0 = e$, implies that $\lambda_i^2 = 1$ for $i = 1, \ldots, q$ and $\lambda_{q+2j-1}\lambda_{q+2j} =$ 1 for $j = 1, \ldots, s$. Thus u_{χ} is in the orthogonal group O(V, Q). Note that the square of the element $g_1 \cdots g_q$ is e, so we may shift the grading on V by this element and still have $g_0 = e$. Therefore, we may assume without loss of generality that $g_1 \cdots g_q = e$. Then we have $\lambda_1 \cdots \lambda_q = 1$ and hence $\det(u_{\chi}) = \lambda_1 \cdots \lambda_{q+2s} = 1$, i.e., u_{χ} is actually in the special orthogonal group SO(V, Q).

For any $\chi \in \widehat{G}$, the orthogonal transformation u_{χ} induces an automorphism $\mathfrak{Cl}(u_{\chi})$ of $\mathfrak{Cl}(V,Q)$ whose restriction to V is u_{χ} . This automorphism gives the action of χ on the G-graded algebra $\mathfrak{Cl}(V,Q)$ and on its graded subalgebra $\mathfrak{Cl}_{\bar{0}}(V,Q)$.

Since $u_{\chi} \in \mathrm{SO}(V, Q)$, there is an element $s_{\chi} \in \mathrm{Spin}(V, Q)$, unique up to sign, such that $u_{\chi}(v) = s_{\chi} \cdot v \cdot s_{\chi}^{-1}$ for all $v \in V$. Recall that

$$\operatorname{Spin}(V,Q) = \{ x \in \mathfrak{Cl}_{\bar{0}}(V,Q) \, | \, x \cdot V \cdot x^{-1} = V \text{ and } x \cdot \tau(x) = 1 \}_{2}$$

where τ is the *canonical involution* of $\mathfrak{Cl}(V, Q)$, i.e., the unique involution whose restriction to V is the identity.

Given isotropic elements $u, v \in V$ with B(u, v) = 1, and a nonzero scalar $\alpha \in \mathbb{F}$, the element

$$s = (\alpha u + \alpha^{-1}v) \cdot (u + v) = \alpha u \cdot v + \alpha^{-1}v \cdot u \in \operatorname{Spin}(V, Q)$$

commutes with all the vectors in V orthogonal to u and v, and satisfies

$$s \cdot u = (\alpha u \cdot v + \alpha^{-1} v \cdot u) \cdot u = \alpha u \cdot v \cdot u = \alpha u,$$
$$u \cdot s = u \cdot (\alpha u \cdot v + \alpha^{-1} v \cdot u) = \alpha^{-1} u \cdot v \cdot u = \alpha^{-1} u,$$

so $s \cdot u \cdot s^{-1} = \alpha^2 u$ and, in the same vein, $s \cdot v \cdot s^{-1} = \alpha^{-2} v$.

Hence, if we choose square roots $\lambda_{q+i}^{1/2}$ for $i = 1, \ldots, 2s$ in such a way that $\lambda_{q+2j-1}^{1/2} \lambda_{q+2j}^{1/2} = 1$ for all $j = 1, \ldots, s$, then we may take

$$(18) \ s_{\chi} = \Big(\prod_{\substack{1 \le i \le q \\ \lambda_i = -1}} 2^{1/2} v_i\Big) \cdot \Big(\prod_{1 \le j \le s} \left(\lambda_{q+2j-1}^{1/2} v_{q+2j-1} \cdot v_{q+2j} + \lambda_{q+2j}^{1/2} v_{q+2j} \cdot v_{q+2j-1}\right)\Big).$$

Note that there is an even number of indices i with $\lambda_i = -1$ $(1 \le i \le q)$. But for u, v as above and $0 \ne \alpha, \beta \in \mathbb{F}$, we have

$$(\alpha u \cdot v + \alpha^{-1}v \cdot u) \cdot (\beta u \cdot v + \beta^{-1}v \cdot u) = \alpha \beta u \cdot v \cdot u \cdot v + \alpha^{-1}\beta^{-1}v \cdot u \cdot v \cdot u$$
$$= \alpha \beta u \cdot v + (\alpha \beta)^{-1}v \cdot u.$$

Also, if u_1, \ldots, u_m are orthogonal elements and I, J are two subsets in $\{1, \ldots, m\}$ of even size, then $u_I \cdot u_J = (-1)^{|I \cap J|} u_J \cdot u_I$ where $u_I = \prod_{i \in I} u_i$ and $u_J = \prod_{i \in J} u_i$. It follows that, for any $\chi_1, \chi_2 \in \widehat{G}$,

$$s_{\chi_1} \cdot s_{\chi_2} = \hat{\gamma}(\chi_1, \chi_2) s_{\chi_2} \cdot s_{\chi_1},$$

where

(19)
$$\hat{\gamma}(\chi_1,\chi_2) = (-1)^{|\{1 \le i \le q \mid \chi_1(g_i) = \chi_2(g_i) = -1\}|}$$

By construction, the conjugation by s_{χ} is the automorphism $\mathfrak{Cl}(u_{\chi})$ of $\mathfrak{Cl}(V)$, for all $\chi \in \widehat{G}$, so the commutation factor $\widehat{\gamma}$ is precisely $\operatorname{Br}(\omega_r) = [\mathfrak{Cl}_{\overline{0}}(V)]$.

The following result, which will be used in Section 7, can be obtained by arguments similar to the above computation of $\hat{\gamma}$.

Lemma 26. Let (V,Q) be a quadratic space of even dimension over a field of characteristic different from 2. Let $u_1, u_2 \in O(V,Q)$ be commuting semisimple elements. Since $\mathfrak{Cl}(V,Q)$ is central simple, there exist invertible elements $s_1, s_2 \in \mathfrak{Cl}(V,Q)$, unique up to scalar, such that $\mathfrak{Cl}(u_i)(x) = s_i \cdot x \cdot s_i^{-1}$ for all $x \in \mathfrak{Cl}(V,Q)$, i = 1, 2. Then the group commutator $[s_1, s_2] := s_1 \cdot s_2 \cdot s_1^{-1} \cdot s_2^{-1}$ is given by $[s_1, s_2] = (-1)^{p_1 p_2 + d}$ where $p_i = 0$ if $u_i \in \mathrm{SO}(V,Q)$ and $p_i = 1$ if $u_i \notin \mathrm{SO}(V,Q)$, i = 1, 2, and d is the dimension of the common (-1, -1)-eigenspace of u_1, u_2 . \Box

It is convenient to introduce some notation before we state our main result for Series B. As mentioned above, a *G*-grading on \mathcal{L} is determined by a multiset $\Xi = \{g_1, \ldots, g_{2r+1}\}$ satisfying (14) with $g_0 = e$. For $i = 1, \ldots, q$, set

$$\tilde{g}_i = g_1 \cdots g_{i-1} g_{i+1} \cdots g_q$$

Then $\tilde{g}_i^2 = e$ and $\tilde{g}_1 \cdots \tilde{g}_q = e$. Consider the group homomorphism

(20)
$$f_{\Xi}: \widehat{G} \to \mathbb{Z}_2^q, \chi \mapsto (x_1, \dots, x_q) \text{ where } \chi(\widetilde{g}_i) = (-1)^{x_i}$$

Note that the image of f_{Ξ} is contained in the hyperplane $(\mathbb{Z}_2^q)_0$ determined by the equation $x_1 + \cdots + x_q = 0$. Define $x \bullet y = \sum_{i=1}^q x_i y_i$ for all $x, y \in \mathbb{Z}_2^q$. Note that this is a symmetric bilinear form whose restriction to $(\mathbb{Z}_2^q)_0$ is alternating and nondegenerate.

Theorem 27. Let \mathcal{L} be the simple Lie algebra of type B_r $(r \geq 2)$ over an algebraically closed field \mathbb{F} of characteristic 0. Suppose \mathcal{L} is graded by an abelian group G. Then, for any dominant integral weight $\lambda = \sum_{i=1}^{r} m_i \omega_i$, we have $H_{\lambda} = \{e\}$ and $\operatorname{Br}(\lambda) = \hat{\gamma}^{m_r}$, where $\hat{\gamma} \colon \widehat{G} \times \widehat{G} \to \mathbb{F}^{\times}$ is given by $\hat{\gamma}(\chi_1, \chi_2) = (-1)^{f \equiv (\chi_1) \bullet f \equiv (\chi_2)}$ with f_{Ξ} as in (20) associated to the parameter Ξ of the grading on \mathcal{L} .

Proof. Since H_{ω_i} is trivial for all *i*, we can apply Proposition 10:

$$\operatorname{Br}(\lambda) = \prod_{i=1}^{r} \operatorname{Br}(\omega_i)^{m_i}$$

But $\operatorname{Br}(\omega_i) = 1$ for all i < r and $\operatorname{Br}(\omega_r) = \hat{\gamma}$ according to (19) in which we substitute $\tilde{g}_i = g_i(g_1 \dots g_q)$ for g_i .

Remark 28. The support of the graded division algebra representing $[Br(\omega_r)]$ is the subgroup of $\langle \tilde{g}_1, \ldots, \tilde{g}_q \rangle$ given by

$$\begin{aligned} &\{\tilde{g}_1^{x_1}\cdots\tilde{g}_q^{x_q} \mid x\in\mathbb{Z}_2^q \text{ such that } x \bullet y = 0 \text{ for all } y\in\mathbb{Z}_2^q \text{ satisfying } \tilde{g}_1^{y_1}\cdots\tilde{g}_q^{y_q} = e \\ &= \{\tilde{g}_1^{x_1}\cdots\tilde{g}_q^{x_q} \mid x\in f_{\Xi}(\widehat{G}) \}. \end{aligned}$$

Thus, it is an elementary 2-group of (even) rank $\leq q - 1$.

Proof. Let $T = \mathbb{Z}_2^q$. Define a homomorphism $\alpha \colon T \to G$ by $\alpha(e_i) = \tilde{g}_i$ where $\{e_1, \ldots, e_q\}$ is the standard basis of \mathbb{Z}_2^q . Let $H = \ker \alpha$. We can identify \hat{T} with T using the standard basis. Then $H^{\perp} \subset \hat{T}$ is precisely the image of f_{Ξ} . The result now follows from Lemma 3 using $\beta(x, y) = \hat{\beta}(x, y) = (-1)^{x \bullet y}$ and regarding $\hat{\gamma}$ of Theorem 27 as a bicharacter on the image of f_{Ξ} .

Alternatively, let $Q = \langle \tilde{g}_1, \ldots, \tilde{g}_q \rangle$, so $Q^{\perp} = \ker f_{\Xi}$. For any $x \in \mathbb{Z}_2^q$, set $\tilde{g}^x := \tilde{g}_1^{x_1} \cdots \tilde{g}_q^{x_q}$. Then, for any $\chi \in \widehat{G}$ and $x \in \mathbb{Z}_2^q$,

$$\chi(\tilde{g}^x) = (-1)^{x \bullet f_\Xi(\chi)}$$

and, in particular,

(21)
$$\tilde{g}^x = e \text{ if and only if } x \in f_{\Xi}(\widehat{G})',$$

where prime denotes the orthogonal subspace relative to the nondegenerate symmetric bilinear form • on \mathbb{Z}_2^q . Then, since $Q = (\ker f_{\Xi})^{\perp} \subset (\operatorname{rad} \hat{\gamma})^{\perp}$, we have

$$(\operatorname{rad} \hat{\gamma})^{\perp} = \{g \in Q \mid \chi(g) = 1 \ \forall \chi \in \widehat{G} \text{ such that } \hat{\gamma}(\chi, \widehat{G}) = 1\} \\ = \{g \in Q \mid \chi(g) = 1 \ \forall \chi \in \widehat{G} \text{ such that } f_{\Xi}(\chi) \bullet f_{\Xi}(\widehat{G}) = 0\} \\ = \{\widetilde{g}^x \mid x \bullet f_{\Xi}(\chi) = 0 \ \forall \chi \in \widehat{G} \text{ such that } f_{\Xi}(\chi) \bullet f_{\Xi}(\widehat{G}) = 0\} \\ = \{\widetilde{g}^x \mid x \bullet f_{\Xi}(\chi) = 0 \ \forall \chi \in \widehat{G} \text{ such that } f_{\Xi}(\chi) \in f_{\Xi}(\widehat{G})'\} \\ = \{\widetilde{g}^x \mid x \in f_{\Xi}(\widehat{G}) \cap f_{\Xi}(\widehat{G})'\} \\ = \{\widetilde{g}^x \mid x \in f_{\Xi}(\widehat{G}) + f_{\Xi}(\widehat{G})'\} \\ = \{\widetilde{g}^x \mid x \in f_{\Xi}(\widehat{G})\},$$

where in the last equation we used (21).

Corollary 29. The simple \mathcal{L} -module V_{λ} admits a *G*-grading making it a graded \mathcal{L} -module if and only if m_r is even or the elements $\tilde{g}_1, \ldots, \tilde{g}_q$ have the following property: for any $x \in f_{\Xi}(\widehat{G})$, $\tilde{g}_1^{x_1} \cdots \tilde{g}_q^{x_q} = e$.

6. Series C

Consider the simple Lie algebra $\mathcal{L} = \mathfrak{sp}_{2r}(\mathbb{F})$ of type $C_r, r \geq 2$, where \mathbb{F} is an algebraically closed field of characteristic 0. In this case all automorphisms of \mathcal{L} are inner. For a given *G*-grading on \mathcal{L} , we are going to find the Brauer invariants of all simple \mathcal{L} -modules.

In [1], the G-gradings on \mathcal{L} are classified in terms of the corresponding G-graded algebras $\mathcal{R} = \operatorname{End}(V)$ equipped with a symplectic involution φ where V is the natural module of \mathcal{L} , which has highest weight ω_1 and dimension n = 2r. Namely, the grading on \mathcal{L} corresponding to (\mathcal{R}, φ) comes from the identification of \mathcal{L} with the graded subspace $\mathcal{K}(\mathcal{R}, \varphi) = \{x \in \mathcal{R} \mid \varphi(x) = -x\}.$

The existence of the involution φ forces $T \subset G$ to be an elementary 2-group, where T is the support of the graded division algebra \mathcal{D} representing $[\mathcal{R}]$ in the G-graded Brauer group. Although the isomorphism class of a grading on \mathcal{L} is not determined by the parameters (T, β) of \mathcal{D} (see [1] or [5, Theorem 3.69]), it turns out that these are all we need to obtain the Brauer invariants of all simple \mathcal{L} -modules. The reason is that, for any $i = 1, \ldots, r$, the simple \mathcal{L} -module of highest weight ω_i is contained with multiplicity 1 in $\wedge^i V$ (see e.g. [3, VIII, Section 13], where it is shown that $\wedge^i V \cong V_{\omega_i} \oplus \wedge^{i-2} V$), so we can express its Brauer invariant in terms of that of V.

Proposition 30. For the simple Lie algebra of type C_r graded by an abelian group G and for any i = 1, ..., r, we have $Br(\omega_i) = Br(\omega_1)^i$ in the G-graded Brauer group.

Proof. Denote $\mathfrak{R} = \operatorname{End}(V)$ and $\rho: \mathcal{L} \to \mathfrak{R}$ the natural representation. As in the proof of Proposition 13, the algebra $\widetilde{\mathfrak{R}} = \operatorname{End}(V^{\otimes i})$ is *G*-graded by identification with $\mathfrak{R}^{\otimes i}$, and $\rho^{\otimes i}: U(\mathcal{L}) \to \widetilde{\mathfrak{R}}$ is a homomorphism of *G*-graded algebras. Moreover, identifying $\operatorname{End}(\wedge^{i}V)$ with $\varepsilon \widetilde{\mathfrak{R}} \varepsilon$ where $\varepsilon: V^{\otimes i} \to \wedge^{i}V$ is the standard projection,

we have $\rho^{\wedge i}(a) = \varepsilon \rho^{\otimes i}(a)\varepsilon$ for all $a \in U(\mathcal{L})$. Since ε is a homogeneous idempotent, $\rho^{\wedge i} \colon U(\mathcal{L}) \to \varepsilon \widetilde{\mathcal{R}} \varepsilon$ is a homomorphism of *G*-graded algebras. Let $\varepsilon_0 \in \varepsilon \mathcal{R} \varepsilon$ be the projection of $\wedge^i V$ onto V_{ω_i} associated to the decomposition of $\wedge^i V$ into isotypic components (as an \mathcal{L} -module). As in the proof of Proposition 10, we see that ε_0 is homogeneous and $\varepsilon_0 \widetilde{\mathcal{R}} \varepsilon_0 \cong \operatorname{End}(V_{\omega_i})$ as *G*-graded algebras. Therefore,

$$\operatorname{Br}(\omega_i) = [\operatorname{End}(V_{\omega_i})] = [\varepsilon_0 \widetilde{\mathfrak{R}} \varepsilon_0] = [\widetilde{\mathfrak{R}}]^i = \operatorname{Br}(\omega_1)^i,$$

where we have used Lemma 2.

Theorem 31. Let \mathcal{L} be the simple Lie algebra of type C_r $(r \geq 2)$ over an algebraically closed field \mathbb{F} of characteristic 0. Suppose \mathcal{L} is graded by an abelian group G. Then, for any dominant integral weight $\lambda = \sum_{i=1}^{r} m_i \omega_i$, we have $H_{\lambda} = \{e\}$ and $\operatorname{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}}$, where $\hat{\beta} \colon \widehat{G} \times \widehat{G} \to \mathbb{F}^{\times}$ is the commutation factor associated to the parameters (T, β) of the grading on \mathcal{L} .

Proof. Since H_{ω_i} is trivial for all *i*, we can apply Proposition 10:

$$\operatorname{Br}(\lambda) = \prod_{i=1}^{\prime} \operatorname{Br}(\omega_i)^{m_i}.$$

But by Proposition 30, we have $Br(\omega_i) = Br(\omega_1)^i = \hat{\beta}^i$. Also, $\hat{\beta}^2 = 1$. The result follows.

Corollary 32. The simple \mathcal{L} -module V_{λ} admits a *G*-grading making it a graded \mathcal{L} -module if and only if $T = \{e\}$ or the number $\sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}$ is even.

7. Series D

Consider the simple Lie algebra $\mathcal{L} = \mathfrak{so}_{2r}(\mathbb{F})$ of type D_r , $r \geq 3$, where \mathbb{F} is an algebraically closed field of characteristic 0. The natural module V of \mathcal{L} has highest weight ω_1 and dimension n = 2r. Suppose \mathcal{L} is given a G-grading. In the case r = 4, assume that this is a matrix grading, i.e., the action of \widehat{G} fixes ω_1 . Our goal is to find the Brauer invariants of all simple \mathcal{L} -modules.

7.1. **Preliminaries.** In [1], the *G*-gradings on \mathcal{L} are classified in terms of the corresponding *G*-graded algebras $\mathcal{R} = \operatorname{End}(V)$ equipped with an orthogonal involution φ . Namely, the grading on \mathcal{L} corresponding to (\mathcal{R}, φ) comes from the identification of \mathcal{L} with the graded subspace $\mathcal{K}(\mathcal{R}, \varphi) = \{x \in \mathcal{R} \mid \varphi(x) = -x\}.$

The existence of the involution φ forces $T \subset G$ to be an elementary 2-group, where T is the support of the graded division algebra \mathcal{D} representing $[\mathcal{R}]$ in the Ggraded Brauer group. Moreover, for any $x \in \mathcal{R}$, $\varphi(x)$ is the adjoint of x with respect to a suitable nondegenerate \mathbb{F} -bilinear form $B \colon W \times W \to \mathcal{D}$ which is \mathcal{D} -sesquilinear relative to an involution of the graded algebra \mathcal{D} . Here, as in Subsection 4.2, W is a right vector space over \mathcal{D} such that $\mathcal{R} = \operatorname{End}_{\mathcal{D}}(W)$ as a G-graded algebra. Recall that V can be identified with $W \otimes_{\mathcal{D}} N$ where N is the natural left (ungraded) module for \mathcal{D} , $\ell = \dim N$ is the graded Schur index of V and $|T| = \ell^2$. Fix a standard matrix realization of \mathcal{D} as described at the beginning of Subsection 4.3. Adjusting the sesquilinear form B, we may assume that the involution of \mathcal{D} is the matrix transpose. Selecting a suitable homogeneous \mathcal{D} -basis $\{v_1, \ldots, v_k\}$ in W, deg $v_i = g_i$, we may assume that

(22)
$$g_1^2 t_1 = \ldots = g_q^2 t_q = g_{q+1}g_{q+2} = \ldots = g_{q+2s-1}g_{q+2s} = g_0^{-1},$$

24

where q + 2s = k, $g_0 \in G$, $t_i \in T$, and the sesquilinear form B is represented by the following block-diagonal matrix:

(23)
$$\Phi = \operatorname{diag}\left(X_{t_1}, \dots, X_{t_q}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right),$$

where $I = X_e \in \mathcal{D}$ and all X_{t_i} are symmetric (see [4] or [5, Theorem 3.42]). The element g_0 has the meaning of the degree of B as a linear map $W \otimes W \to \mathcal{D}$. For $r \neq 4$, the G-gradings on \mathcal{L} are classified by the graded division algebra \mathcal{D} , the multiset $\Xi = \{g_1T, \ldots, g_kT\}$, and the element $g_0 \in G$ (see [1] or [5, Theorem 3.74]). The same proof works for matrix gradings in the case r = 4. Note, however, that this classification is up to isomorphism (matrix isomorphism in the case r = 4), so one set of parameters may correspond to one or two $\operatorname{Int}(\mathcal{L})$ -orbits in the set of G-gradings on \mathcal{L} .

For $i = 1, \ldots, r-2$, the simple \mathcal{L} -module of highest weight ω_i can be realized as $\wedge^i V$, whose Brauer invariant can be computed in the same way as in Proposition 13. Thus, for i < r-1, we have $H_{\omega_i} = H_{\omega_1} = \{e\}$ and $\operatorname{Br}(\omega_i) = \operatorname{Br}(\omega_1)^i$ in the *G*-graded Brauer group. As to the simple \mathcal{L} -modules $V_{\omega_{r-1}}$ and V_{ω_r} , which are called the *half-spin modules*, they are fixed by the automorphisms of \mathcal{L} given by $x \mapsto uxu^{-1}$ with $u \in \operatorname{SO}(V,Q)$ and permuted by such automorphisms with $u \in \operatorname{O}(V,Q) \setminus \operatorname{SO}(V,Q)$. Here Q is the quadratic form associated to the involution φ , namely, $Q(v) = \frac{1}{2}(v, v)_{\Phi}$ where the bilinear form $(\cdot, \cdot)_{\Phi}$ on $V = W \otimes_{\mathcal{D}} N$ is given by (9). To calculate $H_{\omega_{r-1}} = H_{\omega_r}$, it will be convenient to work with similitudes of Q. Recall that $f \in \operatorname{GL}(V)$ is a *similitude of multiplier* μ if $Q(f(v)) = \mu Q(v)$ for all $v \in V$; f is proper if $\det(f) = \mu^r$ and *improper* if $\det(f) = -\mu^r$. Then $x \mapsto fxf^{-1}$ is an automorphism of \mathcal{L} ; it fixes the half-spin modules if f is proper and swaps them if f is improper. For any $\chi \in \hat{G}$, the action of χ on \mathcal{R} is the conjugation by the matrix

(24)
$$u_{\chi} = \operatorname{diag}\left(\chi(g_1), \dots, \chi(g_k)\right) \otimes X_t,$$

where the element $t \in T$ is such that $\chi(s) = \beta(t, s)$ for any $s \in T$. By Lemma 20 (with G instead of \overline{G}), u_{χ} is a similitude of multiplier $\chi(g_0)^{-1}$. On the other hand,

$$\det(u_{\chi}) = \left(\chi(g_1)\cdots\chi(g_k)\right)^{\ell} \det(X_t)^k$$

and $k\ell = n = 2r$.

Lemma 33. For the similitude u_{χ} given by (24), we have:

- If |T| > 4, then u_{χ} is proper for any $\chi \in \widehat{G}$.
- If |T| = 4, so $T = \{e, a, b, c = ab\}$, with X_a and X_b symmetric, then u_{χ} is proper if and only if $\chi(h) = 1$, where

$$h = \begin{cases} t_1 \cdots t_q & \text{if } r \text{ is even,} \\ ct_1 \cdots t_q & \text{if } r \text{ is odd.} \end{cases}$$

 If |T| = 1, then u_χ is proper if and only if χ(h) = 1, where h = g^r₀g₁ ··· g_k. (Note that relations (22) imply that h has order at most 2 and depends only on g₁,..., g_q.)

Proof. If |T| > 4, then det $(X_t) = 1$ for any t by Lemma 19, and

$$(\chi(g_1)\cdots\chi(g_k))^2 = \chi(g_1^2)\cdots\chi(g_q^2)\chi(g_{q+1}g_{q+2})^2\cdots\chi(g_{q+2s-1}g_{q+2s})^2 = \chi(g_0)^{-(q+2s)}\chi(t_1\cdots t_q) = \chi(g_0)^{-k}\chi(t_1\cdots t_k),$$

hence

$$\det(u_{\chi}) = \left(\chi(g_1)\cdots\chi(g_k)\right)^{2\frac{\ell}{2}} = \chi(g_0)^{-k\frac{\ell}{2}} = \chi(g_0)^{-r}.$$

This proves the first part.

If |T| = 4, then k = r, and the computations above give

 $\det(u_{\chi}) = \chi(g_0)^{-k} \chi(t_1 \cdots t_q) \det(X_t)^k,$

so for even k, u_{χ} is proper if and only if $\chi(t_1 \cdots t_q) = 1$, while for odd k, u_{χ} is proper if and only if $\chi(t_1 \cdots t_q) = \det(X_t)$, and, by Lemma 19, $\det(X_t) = \beta(c, t) = \chi(c)$, whence the result.

If |T| = 1, then u_{χ} is proper if and only if $\chi(g_1 \cdots g_k) = \chi(g_0)^{-r}$, which is equivalent to $\chi(g_0^r g_1 \cdots g_k) = 1$.

Thus, if |T| > 4, then the grading on \mathcal{L} is inner and $H_{\omega_{r-1}} = H_{\omega_r} = \{e\}$. If |T| = 1 or 4, then the grading is inner if and only if h = e where $h \in G$ is the element defined in Lemma 33, also $H_{\omega_{r-1}} = H_{\omega_r} = \langle h \rangle$. We state for future reference:

Definition 34. The *distinguished element* of a *G*-grading on $\mathfrak{so}_{2r}(\mathbb{F})$ is the element $h \in G$ of order at most 2 defined by

$$h = \begin{cases} g_0^r g_1 \cdots g_k & \text{if } |T| = 1, \\ t_1 \cdots t_q & \text{if } |T| = 4 \text{ and } r \text{ is even} \\ ct_1 \cdots t_q & \text{if } |T| = 4 \text{ and } r \text{ is odd}, \\ e & \text{if } |T| > 4. \end{cases}$$

The half-spin modules $V_{\omega_{r-1}}$ and V_{ω_r} can be constructed using the Clifford algebra $\mathfrak{Cl}(V) = \mathfrak{Cl}(V,Q)$ in the following way. The algebra $\mathfrak{Cl}(V)$ is simple but its even part $\mathfrak{Cl}_{\bar{0}}(V)$ is a direct sum of two simple ideals. The center of $\mathfrak{Cl}_{\bar{0}}(V)$ is

(25)
$$Z(\mathfrak{Cl}_{\bar{0}}(V,Q)) = \mathbb{F}1 \oplus \mathbb{F}2,$$

where z is the element, unique up to sign, in Spin(V, Q) that satisfies $z \cdot v = -v \cdot z$ for any $v \in V$. If $\{v_1, \ldots, v_{2r}\}$ is an orthogonal basis with $Q(v_i) = 1$ for all *i*, then we may take $z = \pm v_1 \cdots v_{2r}$. Thus, conjugation by z gives the automorphism Υ of $\mathfrak{Cl}(V)$ associated to the \mathbb{Z}_2 -grading, i.e., Υ is id on $\mathfrak{Cl}_{\bar{l}}(V)$ and -id on $\mathfrak{Cl}_{\bar{l}}(V)$. Note that $z^2 = (-1)^r 1$. Moreover, the center of Spin(V, Q) is

$$Z(\operatorname{Spin}(V,Q)) = \{\pm 1, \pm z\} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } r \text{ is even} \\ \mathbb{Z}_4 & \text{if } r \text{ is odd.} \end{cases}$$

For even r, the elements $\varepsilon_{\pm} := \frac{1\pm z}{2}$ are central idempotents of $\mathfrak{Cl}_{\bar{0}}(V)$, so $\mathfrak{Cl}_{\bar{0}}(V) = \mathfrak{Cl}_{\bar{0}}(V)\varepsilon_{+} \oplus \mathfrak{Cl}_{\bar{0}}(V)\varepsilon_{-}$ is the decomposition of the even Clifford algebra as the direct sum of simple ideals. For odd r, the elements $\varepsilon_{\pm} = \frac{1\pm \sqrt{-1}z}{2}$ are central idempotents of $\mathfrak{Cl}_{\bar{0}}(V)$.

Let S^{\pm} be the unique (up to isomorphism) irreducible module for $\mathfrak{Cl}_{\bar{0}}(V)\varepsilon_{\pm}$. Denote by $\pi_{\pm}: \mathfrak{Cl}_{\bar{0}}(V) \to \mathfrak{Cl}_{\bar{0}}(V)\varepsilon_{\pm} \cong \operatorname{End}(S^{\pm}), x \mapsto x\varepsilon_{\pm}$, the projections on each of the simple ideals. Hence,

(26)
$$\pi_{\pm}(z) = \begin{cases} \pm \mathrm{id} & \mathrm{if} \ r \ \mathrm{is \ even}, \\ \pm \sqrt{-1} \mathrm{id} & \mathrm{if} \ r \ \mathrm{is \ odd}. \end{cases}$$

Recall that the Lie algebra $\mathcal{L} = \mathfrak{so}(V, Q)$ imbeds in $\mathfrak{Cl}_{\bar{0}}(V, Q)$ as the subspace $[V, V]^{\cdot}$ (see (16) and (17) where we have to substitute the bilinear form $(\cdot, \cdot)_{\Phi}$ for B). This

26

gives a Lie algebra homomorphism $\rho: \mathcal{L} \to \mathfrak{Cl}(V)^{(-)}$. Composing ρ with π_{\pm} , we get the two half-spin representations ρ_{\pm} , and we may identify S^+ with $V_{\omega_{r-1}}$ and S^- with V_{ω_r} or vice versa, depending on the choice of the sign for z.

7.2. *G*-grading on the even Clifford algebra. We want to construct a *G*-grading on $\mathfrak{Cl}_{\bar{0}}(V)$ such that ρ is a homomorphism of *G*-graded algebras. Since S^+ and S^- are non-isomorphic irreducible \mathcal{L} -modules, the image of the extension $\rho: U(\mathcal{L}) \to \mathfrak{Cl}(V)$ is the entire $\mathfrak{Cl}_{\bar{0}}(V)$, so the *G*-grading on $\mathfrak{Cl}_{\bar{0}}(V)$ will be uniquely determined by this property.

Given a similitude f of multiplier μ on a quadratic space (V,Q) and a square root $\mu^{1/2}$, we have $\mu^{-1/2}f \in O(V,Q)$, and hence there is an automorphism $\mathfrak{Cl}(\mu^{-1/2}f)$ of $\mathfrak{Cl}(V,Q)$ that maps v to $\mu^{-1/2}f(v)$ for all $v \in V$. Note that the restriction of $\mathfrak{Cl}(\mu^{-1/2}f)$ to the even part $\mathfrak{Cl}_{\bar{0}}(V,Q)$ does not depend on the choice of the square root. Thus we obtain a homomorphism from the group of similitudes $\mathrm{GO}(V,Q)$ to $\mathrm{Aut}(\mathfrak{Cl}_{\bar{0}}(V,Q))$. In fact, this is a homomorphism of algebraic groups because it can be defined without using square roots in the following way. First, $\mathfrak{Cl}_{\bar{0}}(V,Q)$ can be defined as the quotient $T(V \otimes V)/I_1(Q) + I_2(Q)$, where $I_1(Q)$ is the ideal generated by the elements $v \otimes v = Q(v)1$ for $v \in V$, and $I_2(Q)$ is the ideal generated by the elements $u \otimes v \otimes w - Q(u)u \otimes w$, for $u, v, w \in V$ (see e.g. [6, Lemma 8.1]). Then, for a similitude f of multiplier μ , the restriction of the above mapping $\mathfrak{Cl}(\mu^{-1/2}f)$ to $V \otimes V$ is given by $v \otimes w \mapsto \mu^{-1}f(v) \otimes f(w)$ for all $v, w \in V$. The extension of this latter to $T(V \otimes V)$ leaves $I_1(Q)$ and $I_2(Q)$ invariant and hence induces an automorphism of $\mathfrak{Cl}_{\bar{0}}(V,Q)$, which coincides with $\mathfrak{Cl}(\mu^{-1/2}f)|_{\mathfrak{Cl}_{\bar{0}}(V,Q)}$.

For any $\chi \in \widehat{G}$, u_{χ} is a similitude of multiplier $\chi(g_0)^{-1}$ (Lemma 20), so we fix a square root of $\chi(g_0)$ and consider the automorphism $\tilde{\alpha}_{\chi} = \mathfrak{Cl}(\chi(g_0)^{1/2}u_{\chi})$ of $\mathfrak{Cl}(V)$. Recall that the imbedding $\rho \colon \mathcal{L} \to \mathfrak{Cl}(V)$ is defined by $\rho(\sigma_{u,v}) = -\frac{1}{2}[u,v]$ for all $u, v \in V$. As usual, let α_{χ} be the action of χ on \mathcal{L} .

Lemma 35. For any $\chi \in \widehat{G}$, we have $\rho \alpha_{\chi} = \widetilde{\alpha}_{\chi} \rho$.

Proof. For any $u, v, w \in V$, we compute:

$$\begin{aligned} u_{\chi}\sigma_{u,v}(w) &= (u,w)_{\Phi} \, u_{\chi}(v) - (v,w)_{\Phi} \, u_{\chi}(u) \\ &= \chi(g_0)(u_{\chi}(u), u_{\chi}(w))_{\Phi} \, u_{\chi}(v) - \chi(g_0)(u_{\chi}(v), u_{\chi}(w))_{\Phi} \, u_{\chi}(u) \\ &= \chi(g_0)\sigma_{u_{\chi}(u), u_{\chi}(v)}(u_{\chi}(w)), \end{aligned}$$

so we get $u_{\chi}\sigma_{u,v}u_{\chi}^{-1} = \chi(g_0)\sigma_{u_{\chi}(u),u_{\chi}(v)}$. Hence

$$\begin{split} \rho \alpha_{\chi}(\sigma_{u,v}) &= \chi(g_0) \rho \left(\sigma_{u_{\chi}(u), u_{\chi}(v)} \right) = -\frac{1}{2} \chi(g_0) [u_{\chi}(u), u_{\chi}(v)]^{\cdot} \\ &= -\frac{1}{2} [\tilde{\alpha}_{\chi}(u), \tilde{\alpha}_{\chi}(v)]^{\cdot} = \tilde{\alpha}_{\chi} \left(-\frac{1}{2} [u, v]^{\cdot} \right) \\ &= \tilde{\alpha}_{\chi} \rho(\sigma_{u,v}), \end{split}$$

which proves the result.

For any $\chi_1, \chi_2 \in \widehat{G}$, we have $u_{\chi_1}u_{\chi_2} = \hat{\beta}(\chi_1, \chi_2)u_{\chi_2}u_{\chi_1}$, where $\hat{\beta} = \operatorname{Br}(\omega_1)$ is the commutation factor of the natural module $V = V_{\omega_1}$, which takes values in $\{\pm 1\}$. Thus the restriction to V of $\tilde{\alpha}_{\chi_1}\tilde{\alpha}_{\chi_2}$ equals the restriction to V of $\hat{\beta}(\chi_1, \chi_2)\tilde{\alpha}_{\chi_2}\tilde{\alpha}_{\chi_1}$.

Therefore, the commutator of $\tilde{\alpha}_{\chi_1}$ and $\tilde{\alpha}_{\chi_2}$ in the group $\operatorname{Aut}(\mathfrak{Cl}(V))$ is given by

(27)
$$[\tilde{\alpha}_{\chi_1}, \tilde{\alpha}_{\chi_2}] = \begin{cases} \text{id} & \text{if } \hat{\beta}(\chi_1, \chi_2) = 1, \\ \Upsilon & \text{if } \hat{\beta}(\chi_1, \chi_2) = -1, \end{cases}$$

where Υ is the automorphism of $\mathfrak{Cl}(V)$ associated to the \mathbb{Z}_2 -grading. Note that, in any case, the restrictions $\tilde{\alpha}_{\chi_1}|_{\mathfrak{Cl}_{\bar{0}}(V)}$ and $\tilde{\alpha}_{\chi_2}|_{\mathfrak{Cl}_{\bar{0}}(V)}$ commute. We claim that the mapping $\widehat{G} \to \operatorname{Aut}(\mathfrak{Cl}_{\bar{0}}(V))$ sending χ to $\tilde{\alpha}_{\chi}|_{\mathfrak{Cl}_{\bar{0}}(V)}$ is a homomorphism of algebraic groups. Indeed, this mapping is the composition of two homomorphisms: $\widehat{G} \to \operatorname{PGO}(V,Q)$ given by $\chi \mapsto \alpha_{\chi}$ and $\operatorname{PGO}(V,Q) \to \operatorname{Aut}(\mathfrak{Cl}_{\bar{0}}(V))$ induced by the homomorphism $\operatorname{GO}(V,Q) \to \operatorname{Aut}(\mathfrak{Cl}_{\bar{0}}(V))$ defined above. Here we identified $\operatorname{PGO}(V,Q) := \operatorname{GO}(V,Q)/\mathbb{F}^{\times}$ with $\operatorname{Aut}(\mathcal{L})$ (a subgroup of $\operatorname{Aut}(\mathcal{L})$ in the case r = 4).

Corollary 36. The homomorphism of algebraic groups $\widehat{G} \to \operatorname{Aut}(\mathfrak{Cl}_{\overline{0}}(V))$ given by

$$\chi \mapsto \tilde{\alpha}_{\chi}|_{\mathfrak{Cl}_{\bar{0}}(V)}$$

endows $\mathfrak{Cl}_{\bar{0}}(V)$ with a G-grading such that the imbedding $\rho: \mathcal{L} \to \mathfrak{Cl}_{\bar{0}}(V)^{(-)}$ is a homomorphism of G-graded algebras.

Remark 37. If |T| = 1, then V itself is G-graded. For $u \in V_a$ and $v \in V_b$, the element $u \cdot v \in \mathfrak{Cl}_{\bar{0}}(V)$ is homogeneous of degree abg_0 . Note, however, that $\mathfrak{Cl}(V)$ is not G-graded.

We can now obtain relations between the Brauer invariants of the half-spin modules and that of the natural module. Let $H = H_{\omega_{r-1}} = H_{\omega_r}$. Then the *G*-grading on $\mathfrak{Cl}_{\bar{0}}(V) \cong \operatorname{End}(S^+) \times \operatorname{End}(S^-)$ induces (G/H)-gradings on $\operatorname{End}(S^+)$ and $\operatorname{End}(S^-)$ such that $\rho_{\pm} \colon U(\mathcal{L}) \to \operatorname{End}(S^{\pm})$ are (surjective) homomorphisms of (G/H)-graded algebras. By definition, $\operatorname{Br}(S^{\pm}) = [\operatorname{End}(S^{\pm})]$ in the (G/H)-graded Brauer group.

For any $u \in O(V,Q)$, there exists an invertible element $s \in \mathfrak{Cl}(V,Q)$, unique up to scalar, such that $\mathfrak{Cl}(u)(x) = s \cdot x \cdot s^{-1}$ for all $x \in \mathfrak{Cl}(V,Q)$. It follows from $s \cdot V \cdot s^{-1} = V$ that the element *s* must be even or odd; moreover, it can be normalized so that $s \cdot \tau(s) = 1$ where τ is the canonical involution of $\mathfrak{Cl}(V,Q)$. If $u \in \mathrm{SO}(V,Q)$ then *s* is even, so it be taken in $\mathrm{Spin}(V,Q)$. Also, the conjugation by the element $\pi_{\pm}(s) = s\varepsilon_{\pm}$ is the restriction of $\mathfrak{Cl}(u)$ to the simple ideal $\mathfrak{Cl}_{\bar{0}}(V,Q)\varepsilon_{\pm} \cong \mathrm{End}(S^{\pm})$. If $u \notin \mathrm{SO}(V,Q)$ then *s* is odd and $\mathfrak{Cl}(u)$ swaps the simple ideals $\mathfrak{Cl}_{\bar{0}}(V,Q)\varepsilon_{\pm}$.

Lemma 38. For any $\chi \in \widehat{G}$, fix an invertible element $s_{\chi} \in \mathfrak{Cl}(V)$ such that $\widetilde{\alpha}_{\chi}$ is the conjugation by s_{χ} . Then the group commutator $[s_{\chi_1}, s_{\chi_2}] := s_{\chi_1} \cdot s_{\chi_2} \cdot s_{\chi_1}^{-1} \cdot s_{\chi_2}^{-1}$ belongs to $\{\pm 1\}$ if $\widehat{\beta}(\chi_1, \chi_2) = 1$ and to $\{\pm z\}$ if $\widehat{\beta}(\chi_1, \chi_2) = -1$, with z as in (25).

Proof. The element $[s_{\chi_1}, s_{\chi_2}]$ is even and independent of the choice of s_{χ} . It follows that $[s_{\chi_1}, s_{\chi_2}] \in \text{Spin}(V, Q)$. By (27), the conjugation by $[s_{\chi_1}, s_{\chi_2}]$ is id if $\hat{\beta}(\chi_1, \chi_2) = 1$ and Υ if $\hat{\beta}(\chi_1, \chi_2) = -1$. The result follows.

The following proposition is reminiscent of Theorem 9.12 in [6], which deals with the ordinary Brauer group of the field \mathbb{F} (which is there of arbitrary characteristic and, of course, not assumed algebraically closed).

Proposition 39. Let the simple Lie algebra of type D_r be graded by an abelian group G. If r = 4, assume that the highest weight ω_1 of the natural module is fixed by \hat{G} . Let $H = H_{\omega_{r-1}} = H_{\omega_r}$. Then we have the following relations in the (G/H)-graded Brauer group:

GRADED MODULES OVER CLASSICAL SIMPLE LIE ALGEBRAS WITH A GRADING 29

- If r is odd then Br(ω_{r-1})Br(ω_r) = 1 and Br(ω_{r-1})² = Br(ω_r)² = Br(ω₁).
 If r is even then Br(ω_{r-1})Br(ω_r) = Br(ω₁) and Br(ω_{r-1})² = Br(ω_r)² = 1.

Proof. Let $K = H^{\perp} \subset \widehat{G}$ and let $\widehat{\gamma}_{\pm} \colon K \times K \to \mathbb{F}^{\times}$ be the commutation factors of the half-spin modules S^{\pm} . For any $\chi \in K$, the element s_{χ} is even, so the conjugation by $\pi_{\pm}(s_{\chi}) = s_{\chi}\varepsilon_{\pm}$ is the restriction of $\tilde{\alpha}_{\chi}$ to the simple ideal $\mathfrak{Cl}_{\bar{0}}(V)\varepsilon_{\pm} \cong \operatorname{End}(S^{\pm})$. For $\chi_1, \chi_2 \in K$, we have $\hat{\gamma}_{\pm}(\chi_1, \chi_2) = [\pi_{\pm}(s_{\chi_1}), \pi_{\pm}(s_{\chi_2})].$

• If $\hat{\beta}(\chi_1, \chi_2) = 1$, then Lemma 38 tells us that $[s_{\chi_1}, s_{\chi_2}]$ is either 1 or -1, hence

$$\hat{\gamma}_+(\chi_1,\chi_2) = \hat{\gamma}_-(\chi_1,\chi_2) \in \{\pm 1\}$$

- If $\hat{\beta}(\chi_1, \chi_2) = -1$, then Lemma 38 tells us that $[s_{\chi_1}, s_{\chi_2}]$ is either z or -z. Taking into account (26), we see:
 - If r is even, then z acts as id on S^+ and -id on S^- , hence

$$\hat{\gamma}_+(\chi_1,\chi_2) = -\hat{\gamma}_-(\chi_1,\chi_2) \in \{\pm 1\}.$$

- If r is odd, then z acts as $\sqrt{-1}$ id on S⁺ and as $-\sqrt{-1}$ id on S⁻, hence

$$\hat{\gamma}_+(\chi_1,\chi_2) = -\hat{\gamma}_-(\chi_1,\chi_2) \in \{\pm\sqrt{-1}\}.$$

The result follows.

Corollary 40. Br $(\omega_{r-1}) = Br(\omega_r)$ if and only if $Br(\omega_1) = 1$.

7.3. Inner gradings on $\mathfrak{so}_{2r}(\mathbb{F})$. Assume that the given grading on $\mathcal{L} = \mathfrak{so}_{2r}(\mathbb{F})$ is inner, i.e., the similitudes u_{χ} are proper for all $\chi \in \widehat{G}$. Recall that this happens if and only if h = e where $h \in G$ is the distinguished element (see Definition 34).

We will now determine the commutation factors $\gamma_{\pm} : \widehat{G} \times \widehat{G} \to \mathbb{F}^{\times}$ of the halfspin modules S^{\pm} of \mathcal{L} . First we calculate $\hat{\gamma}_{\pm}(\chi_1, \chi_2)$ in a special case. Recall the quadratic form $\beta: T \to \{\pm 1\}$ defined by equation (6).

Lemma 41. For any $\chi \in \widehat{G}$, fix a square root $\chi(g_0)^{1/2}$ and an invertible element $s_{\chi} \in \mathfrak{Cl}(V)$ such that $\widetilde{\alpha}_{\chi} = \mathfrak{Cl}(\chi(g_0)^{1/2}u_{\chi})$ is the conjugation by s_{χ} . Consider $\chi, \psi \in$ \widehat{G} such that the similitudes u_{χ} and u_{ψ} are proper and $\beta(t') = \beta(t'') = \beta(t', t'') = 1$ where $\chi|_T = \beta(t', \cdot)$ and $\psi|_T = \beta(t'', \cdot)$. Then the group commutator $[s_{\chi}, s_{\psi}]$, which does not depend on the choice of the square roots, can be calculated as follows:

• If |T| > 16, then $[s_{\chi}, s_{\psi}] = 1$.

• If
$$|T| > 16$$
, then $[s_{\chi}, s_{\psi}] = 1$.
• If $|T| = 16$, then $[s_{\chi}, s_{\psi}] = \begin{cases} 1 & \text{if } \chi|_{T} = 1 & \text{or } \psi|_{T} = 1 & \text{or } \chi|_{T} = \psi|_{T}, \\ (-1)^{|\{1 \le i \le q \mid \chi(t_{i}) = \psi(t_{i}) = 1\}|} & \text{otherwise.} \end{cases}$

• If
$$|T| = 4$$
, then

$$[s_{\chi}, s_{\psi}] = \begin{cases} 1 & \text{if } \chi|_{T} = 1 \text{ and } \psi|_{T} = 1, \\ (-1)^{|\{1 \le i \le q \mid \chi(t_{i}) = 1, \mu_{i} = -1\}|} & \text{if } \chi|_{T} \neq 1 \text{ and } \psi|_{T} = 1, \\ (-1)^{|\{1 \le i \le q \mid \psi(t_{i}) = 1, \lambda_{i} = -1\}|} & \text{if } \chi|_{T} = 1 \text{ and } \psi|_{T} \neq 1, \\ (-1)^{|\{1 \le i \le q \mid \chi(t_{i}) = \psi(t_{i}) = 1, \lambda_{i} = \mu_{i}\}|} & \text{if } \chi|_{T} \neq 1 \text{ and } \psi|_{T} \neq 1, \end{cases}$$

where $\lambda_i = \chi(g_0)^{1/2} \chi(g_i)$ and $\mu_i = \psi(g_0)^{1/2} \psi(g_i)$ for i = 1, ..., q. • If |T| = 1, then $[s_{\chi}, s_{\psi}] = (-1)^{|\{1 \le i \le q \mid \lambda_i = \mu_i = -1\}|}$, with λ_i and μ_i as above.

Proof. First of all, changing the square root $\chi(g_0)^{1/2}$ is tantamaunt to replacing s_{χ} by zs_{χ} , but $[zs_{\chi}, s_{\psi}] = [s_{\chi}, s_{\psi}]$ since s_{ψ} is even. Recall that

$$u_{\chi} = \operatorname{diag}\left(\chi(g_1), \dots, \chi(g_k)\right) \otimes X_{t'} \quad \text{and} \quad u_{\psi} = \operatorname{diag}\left(\psi(g_1), \dots, \psi(g_k)\right) \otimes X_{t''}.$$

Since $\langle t', t'' \rangle$ is a totally isotropic subspace of T with respect to the quadratic form $\beta(\cdot)$, Witt Extension Theorem implies that we may choose totally isotropic subspaces \widetilde{A} and \widetilde{B} such that $T = \widetilde{A} \times \widetilde{B}$ and $t', t'' \in \widetilde{A}$. Then we can pick a symplectic basis $\{\widetilde{a}_j\} \cup \{\widetilde{b}_j\}$ of T with $\widetilde{A} = \langle \widetilde{a}_j \rangle$ and $\widetilde{B} = \langle \widetilde{b}_j \rangle$. We use tildes to distinguish from the symplectic basis associated to the standard matrix realization of \mathcal{D} leading to equations (22) and (23), which was fixed earlier. Since $X_{a_j}^2 = 1$, $X_{b_j}^2 = 1$ and $X_{a_i}X_{b_j} = (-1)^{\delta_{ij}}X_{b_j}X_{a_j}$ (Kronecker delta) are defining relations for \mathcal{D} , the mapping $X_{a_j} \mapsto X_{\tilde{a}_j}, X_{b_j} \mapsto X_{\tilde{b}_j}$ extends to an automorphism of \mathcal{D} and thus leads to a new matrix realization of \mathcal{D} , in which the elements $X_{t'}$ and $X_{t''}$ are diagonal. Namely, using the notation $N = \mathbb{F}\widetilde{B}$ as in Remark 18, we have $X_{t'}\widetilde{e}_b = \beta(t', b)\widetilde{e}_b$ and $X_{t''}\widetilde{e}_b = \beta(t'', b)\widetilde{e}_b$ for all $b \in \widetilde{B}$. Therefore, the basis $v_{i,b} := v_i \otimes \widetilde{e}_b$ $(i = 1, \ldots, k, b \in \widetilde{B})$ of $V = W \otimes_{\mathcal{D}} N$ consists of common eigenvectors of u_{χ} and u_{ψ} . Setting $\lambda_i = \chi(g_0)^{1/2}\chi(g_i)$ and $\mu_i = \psi(g_0)^{1/2}\psi(g_i)$ for $i = 1, \ldots, k$, we obtain:

$$\chi(g_0)^{1/2} u_{\chi}(v_{i,b}) = \lambda_{i,b} v_{i,b} \text{ where } \lambda_{i,b} = \lambda_i \beta(t',b) = \lambda_i \chi(b),$$

$$\psi(g_0)^{1/2} u_{\psi}(v_{i,b}) = \mu_{i,b} v_{i,b} \text{ where } \mu_{i,b} = \mu_i \beta(t'',b) = \mu_i \psi(b).$$

Relations (22) imply that $\lambda_i^2 = \chi(t_i)$ for $i = 1, \ldots, q$ and $\lambda_{q+2j-1}\lambda_{q+2j} = 1$ for $j = 1, \ldots, s$, and similarly for the μ_i . It follows that

$$\begin{split} \lambda_{i,b}^2 &= \chi(t_i), \, \mu_{i,b}^2 = \psi(t_i) \text{ for } 1 \leq i \leq q, \, b \in \widehat{B}; \\ \lambda_{q+2j-1,b} \lambda_{q+2j,b} &= \mu_{q+2j-1,b} \mu_{q+2j,b} = 1 \text{ for } 1 \leq j \leq s, \, b \in \widetilde{B} \end{split}$$

By Lemma 26, we have $[s_{\chi}, s_{\psi}] = (-1)^d$ where d is the dimension of the common (-1, -1)-eigenspace V_{-1} of the proper isometries $\chi(g_0)^{1/2}u_{\chi}$ and $\psi(g_0)^{1/2}u_{\psi}$. Note that $v_{q+2j-1,b} \in V_{-1}$ if and only if $v_{q+2j,b} \in V_{-1}$, so the vectors $v_{i,b}$ with i > q do not affect the parity of d. Also, if $\chi(t_i) = -1$ or $\psi(t_i) = -1$ for some $i \leq q$ then $v_{i,b} \notin V_{-1}$ for all $b \in B$. Therefore, we can restrict our attention to $1 \leq i \leq q$ such that $\chi(t_i) = \psi(t_i) = 1$. For any such i, we have $v_{i,b} \in V_{-1}$ if and only if $\chi(b) = -\lambda_i$ and $\psi(b) = -\mu_i$.

If |T| = 1, then $\tilde{B} = \{e\}$, so $v_{i,e} \in V_{-1}$ if and only if $\lambda_i = \mu_i = -1$. Otherwise $\ell = |\tilde{B}|$ is a nontrivial power of 2. Note that, for given $\xi, \eta \in \{\pm 1\}$, the set $\tilde{B}(\xi,\eta) := \{b \in \tilde{B} \mid \chi(b) = \xi, \psi(b) = \eta\}$ has size 0 or $\ell/2$ or ℓ unless the restrictions $\chi|_{\tilde{B}}$ and $\psi|_{\tilde{B}}$ are distinct and nontrivial. In this latter case (which is impossible for $\ell = 2$), the set $\tilde{B}(\xi,\eta)$ has size $\ell/4$. If $\ell > 4$ then all of these numbers are even, hence so is d. If $\ell = 4$ then we get size 1 if $\chi|_{\tilde{B}}$ and $\psi|_{\tilde{B}}$ are distinct and nontrivial, and even size otherwise. Finally, consider $\ell = 2$. We get even size unless at least one of $\chi|_{\tilde{B}}$ and $\psi|_{\tilde{B}}$ is nontrivial. If, say, $\chi|_{\tilde{B}} \neq 1$ then $|\tilde{B}(\xi,\eta)| = 1$ if and only if $\psi|_{\tilde{B}} = 1, \eta = 1$ or $\psi|_{\tilde{B}} = \chi|_{\tilde{B}}, \eta = \xi$. It remains to observe that, since $t', t'' \in \tilde{A}$, we have $\chi(\tilde{A}) = 1$ and $\psi(\tilde{A}) = 1$, so we may replace the restrictions of $\chi|_{\tilde{B}}$ and $\psi|_{\tilde{B}}$ by $\chi|_T$ and $\psi|_T$, respectively, in the above conditions.

• Consider the case |T| = 1.

The G-grading on \mathcal{L} is induced by a G-grading on the natural module $V = V_{\omega_1}$ and is determined by $g_0 \in G$ and a multiset $\Xi = \{g_1, \ldots, g_{2r}\}$ satisfying (14). Note that q is even. Let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}_2^q$ and consider the map

(28) $f_{\Xi,g_0}: \widehat{G} \to \mathbb{Z}_2^q/\langle \mathbf{1} \rangle, \ \chi \mapsto (x_1, \dots, x_q) + \langle \mathbf{1} \rangle \text{ where } \chi(g_0)^{1/2} \chi(g_i) = (-1)^{x_i}.$

Note that this map does not depend on the choice of the square roots and is a homomorphism of groups. Since $h = g_0^{q/2}g_1 \cdots g_q = e$, the image of f_{Ξ,g_0} is contained in $(\mathbb{Z}_2^q)_0/\langle \mathbf{1} \rangle$ where $(\mathbb{Z}_2^q)_0$ is the hyperplane determined by the equation $x_1 + \cdots + x_q = 0$. Define $x \bullet y = \sum_{i=1}^q x_i y_i$ for all $x, y \in \mathbb{Z}_2^q$. Note that this is a symmetric bilinear form whose restriction to $(\mathbb{Z}_2^q)_0$ is alternating and degenerate. The radical of this restriction is precisely $\langle \mathbf{1} \rangle$, so we obtain an alternating nondegenerate form on $(\mathbb{Z}_2^q)_0/\langle \mathbf{1} \rangle$, which we still denote by $x \bullet y$. By Lemma 41, we obtain:

(29)
$$\hat{\gamma}_{+}(\chi_{1},\chi_{2}) = \hat{\gamma}_{-}(\chi_{1},\chi_{2}) = (-1)^{f_{\Xi,g_{0}}(\chi_{1})\bullet f_{\Xi,g_{0}}(\chi_{2})}.$$

The following is analogous to Remark 28:

Remark 42. If q = 0 then $\operatorname{Br}(\omega_{r-1}) = \operatorname{Br}(\omega_r) = 1$. Otherwise g_0 is a square in G, so we may shift the grading on V and assume $g_0 = e$. Then we can define $f_{\Xi} \colon \widehat{G} \to \mathbb{Z}_2^q, \chi \mapsto (x_1, \ldots, x_q)$, where $\chi(g_i) = (-1)^{x_i}$, and the support of the graded division algebra representing $[\operatorname{Br}(\omega_{r-1})] = [\operatorname{Br}(\omega_r)]$ is the subgroup of $\langle g_1, \ldots, g_q \rangle$ given by

$$\{g_1^{x_1}\cdots g_q^{x_q} \mid x \in \mathbb{Z}_2^q \text{ such that } x \bullet y = 0 \text{ for all } y \in \mathbb{Z}_2^q \text{ satisfying } g_1^{y_1}\cdots g_q^{y_q} = e\}$$
$$= \{g_1^{x_1}\cdots g_q^{x_q} \mid x \in f_{\Xi}(\widehat{G})\}.$$

Thus, it is an elementary 2-group of (even) rank $\leq q - 2$.

• Consider the case |T| = 4.

Here $T = \{e, a, b, c\}$, $\beta(a) = \beta(b) = 1$ (i.e., X_a and X_b are symmetric). The G-grading on \mathcal{L} is determined by $g_0 \in G$ and a multiset $\Xi = \{g_1, \ldots, g_r\}$ satisfying (22) with k = r. Note that q and r have the same parity. Also note that $t_i \in \{e, a, b\}$ for all $i = 1, \ldots, q$.

Fix $\chi_a, \chi_b \in \widehat{G}$ such that $\chi_a(t) = \beta(a, t)$ and $\chi_b(t) = \beta(b, t)$ for all $t \in T$. Set $\chi_e = 1$ and $\chi_c = \chi_a \chi_b$. Then $\{\chi_e, \chi_a, \chi_b, \chi_c\}$ is a transversal for the subgroup T^{\perp} of \widehat{G} .

Since $\hat{\beta}(\chi_a, \chi_b) = \beta(a, b) = -1$, Lemma 38 tells us that the group commutator of the corresponding invertible elements s_{χ_a} and s_{χ_b} of $\mathfrak{Cl}_{\bar{0}}(V)$ is a nonscalar central element of the spin group, so we may set

 $z := [s_{\chi_a}, s_{\chi_b}].$

Then we have, by (26), that $\hat{\gamma}_{\pm}(\chi_a, \chi_b) = [\pi_{\pm}(s_{\chi_a}), \pi_{\pm}(s_{\chi_b})] = \pm 1$ if r is even and $\pm \mathbf{i}$ if r is odd, where $\mathbf{i} = \sqrt{-1}$.

For any $\psi \in T^{\perp}$, we have $\hat{\beta}(\chi_a, \psi) = \hat{\beta}(\chi_b, \psi) = 1$, so Lemma 41 yields

$$\hat{\gamma}_{\pm}(\chi_{a},\psi) = [\pi_{\pm}(s_{\chi_{a}}),\pi_{\pm}(s_{\psi})] = (-1)^{|\{1 \le i \le q \mid t_{i} \in \{e,a\},\,\mu_{i}=-1\}|},\\ \hat{\gamma}_{\pm}(\chi_{b},\psi) = [\pi_{\pm}(s_{\chi_{b}}),\pi_{\pm}(s_{\psi})] = (-1)^{|\{1 \le i \le q \mid t_{i} \in \{e,b\},\,\mu_{i}=-1\}|},$$

where $\mu_i = \psi(g_0)^{1/2} \psi(g_i)$. For any $t \in T$, define

$$I_t = \{ 1 \le i \le q \mid t_i = t \}.$$

Then $I_c = \emptyset$ and the sets I_e , I_a , I_b form a partition of $\{1, \ldots, q\}$. Now, the distinguished element h is given by

$$h = \begin{cases} a^{|I_a|}b^{|I_b|} & \text{if } r \text{ is even,} \\ a^{|I_a|+1}b^{|I_b|+1} & \text{if } r \text{ is odd.} \end{cases}$$

Since h = e, we conclude that the sizes of I_a and I_b have the same parity as r. It follows that the same is true of I_e . Set

$$g_a = g_0^{(|I_e| + |I_a|)/2} \prod_{i \in I_e \cup I_a} g_i$$
 and $g_b = g_0^{(|I_e| + |I_b|)/2} \prod_{i \in I_e \cup I_b} g_i$

Note that $g_a^2 = e$, $g_b^2 = e$ if r is even and $g_a^2 = a$, $g_b^2 = b$ if r is odd. Set $g_e = e$ and $g_c = g_a g_b$. Then, for any $\psi \in T^{\perp}$, we have $\hat{\gamma}(\chi_e, \psi) = \psi(g_e)$ and we can restate the above formulas as

$$\hat{\gamma}_{\pm}(\chi_a, \psi) = \psi(g_a) \text{ and } \hat{\gamma}_{\pm}(\chi_b, \psi) = \psi(g_b).$$

It follows that

$$\hat{\gamma}_{\pm}(\chi_c,\psi) = \hat{\gamma}_{\pm}(\chi_a,\psi)\hat{\gamma}_{\pm}(\chi_b,\psi) = \psi(g_a)\psi(g_b) = \psi(g_c)$$

Also note that, for any $\psi', \psi'' \in T^{\perp}$, we have $\hat{\gamma}_{\pm}(\psi', \psi'') = 1$ by Lemma 41.

• If r is even, then we have $\hat{\gamma}_+(\chi_a, \chi_b) = 1$ and hence also $\hat{\gamma}_+(\chi_a, \chi_c) = 1$ and $\hat{\gamma}_+(\chi_c, \chi_b) = 1$. It follows that

(30)
$$\hat{\gamma}_{+}(\psi'\chi_{t'},\psi''\chi_{t''})=\psi'(g_{t''})\psi''(g_{t'}),$$

for all $t', t'' \in T$ and $\psi', \psi'' \in T^{\perp}$. Since every element of \widehat{G} can be written uniquely in the form $\psi\chi_t$ ($\psi \in T^{\perp}, t \in T$), the above formula determines $\hat{\gamma}_+(\chi_1, \chi_2)$ for all $\chi_1, \chi_2 \in \widehat{G}$. Note that $\hat{\gamma}_- = \hat{\beta}\hat{\gamma}_+$ by Proposition 39.

• If r is odd, then we have $\hat{\gamma}_+(\chi_a, \chi_b) = \mathbf{i}$ and hence also $\hat{\gamma}_+(\chi_a, \chi_c) = \mathbf{i}$ and $\hat{\gamma}_+(\chi_c, \chi_b) = \mathbf{i}$. It follows that

(31)
$$\hat{\gamma}_{+}(\psi'\chi_{t'},\psi''\chi_{t''}) = \beta^{1/2}(t',t'')\psi'(g_{t''})\psi''(g_{t'}),$$

for all $t', t'' \in T$ and $\psi', \psi'' \in T^{\perp}$, where

$$\beta^{1/2}(t',t'') := \begin{cases} 1 & \text{if } \beta(t',t'') = 1, \\ \mathbf{i} & \text{if } \beta(t',t'') = -1 \text{ and } t' < t'', \\ -\mathbf{i} & \text{if } \beta(t',t'') = -1 \text{ and } t' > t'', \end{cases}$$

and the nontrivial elements of T are formally ordered a < c < b. This determines $\hat{\gamma}_+(\chi_1, \chi_2)$ for all $\chi_1, \chi_2 \in \widehat{G}$, and $\hat{\gamma}_- = \hat{\gamma}_+^{-1}$ by Proposition 39.

Remark 43. Let $Q = \langle T, g_a, g_b \rangle$. If r is odd then $\psi \chi_t$ is in rad $\hat{\gamma}_+$ if and only if t = e and $\psi(g_a) = \psi(g_b) = 1$, hence Q is the support of the graded division algebra representing $\hat{\gamma}_+$ and $\hat{\gamma}_-$, $Q \cong \mathbb{Z}_4^2$. If r is even, then $\psi \chi_t$ is in rad $\hat{\gamma}_+$ if and only if $g_t \in T$ and $\psi(g_a) = \psi(g_b) = 1$, hence the support of the graded division algebra representing $\hat{\gamma}_+$ is the subgroup of Q consisting of all elements $x \in Q$ satisfying $\chi_a(x) = 1$ if $g_a \in T$, $\chi_b(x) = 1$ if $g_b \in T$ and $\chi_a(x) = \chi_b(x)$ if $g_a g_b \in T$; this is an elementary 2-group of rank 0, 2 or 4.

• Consider the case |T| = 16 (hence r is even).

Here $T = \langle a_1, a_2 \rangle \times \langle b_1, b_2 \rangle$, $\beta(a_j) = \beta(b_j) = 1$ for j = 1, 2. The *G*-grading on \mathcal{L} is determined by $g_0 \in G$ and a multiset $\Xi = \{g_1, \ldots, g_k\}$ satisfying (22) with k = r/2, where $\beta(t_i) = 1$ for all $i = 1, \ldots, q$. Note that q and r/2 have the same parity.

Pick elements χ_{a_j}, χ_{b_j} of $\widehat{G}, j = 1, 2$, such that $\chi_{a_j}(t) = \beta(a_j, t)$ and $\chi_{b_j}(t) = \beta(b_j, t)$ for all $t \in T$. It follows from Lemma 41 that T^{\perp} lies in the radicals of $\hat{\gamma}_+$ and $\hat{\gamma}_-$. Indeed, if $\psi', \psi'' \in T^{\perp}$ then $\hat{\gamma}_{\pm}(\psi', \psi'') = [\pi_{\pm}(s_{\psi'}), \pi_{\pm}(s_{\psi''})] = 1$. Also,

 $\hat{\gamma}_{\pm}(\chi_{a_i},\psi)=1$ and $\hat{\gamma}_{\pm}(\chi_{b_i},\psi)=1$ for any $\psi\in T^{\perp}$. Since T^{\perp} together with χ_{a_i} and χ_{b_i} , j = 1, 2, generate the group \widehat{G} , we conclude that ψ is in rad $\widehat{\gamma}_{\pm}$, as claimed. Define a homomorphism

 $f: \widehat{G} \to \mathbb{Z}_2^4, \chi \mapsto (x_1, x_2, y_1, y_2)$ where $\chi(a_j) = (-1)^{x_j}, \chi(b_j) = (-1)^{y_j}, j = 1, 2.$ The kernel of f is precisely T^{\perp} , and we have just shown that $\hat{\gamma}_{\pm}$ factors through f. Note that $\beta(\cdot)$ is given by

$$\beta(t) = (-1)^{x_1y_1 + x_2y_2}$$
 where $t = a_1^{x_1} a_2^{x_2} b_1^{y_1} b_2^{y_2}$.

Also, $\chi_{b_j}(t) = x_j$ and $\chi_{a_j}(t) = y_j$, j = 1, 2. Now write $t_i = a_1^{x_1^{(i)}} a_2^{x_2^{(i)}} b_1^{y_1^{(i)}} b_2^{y_2^{(i)}}$, $i = 1, \dots, q$. Then Lemma 41 yields $\hat{\gamma}_{\pm}(\chi_{b_1},\chi_{b_2}) = (-1)^{\sum_{i=1}^q (x_1^{(i)}+1)(x_2^{(i)}+1)}, \quad \hat{\gamma}_{\pm}(\chi_{b_1},\chi_{a_2}) = (-1)^{\sum_{i=1}^q (x_1^{(i)}+1)(y_2^{(i)}+1)}, \\ \hat{\gamma}_{\pm}(\chi_{b_2},\chi_{a_1}) = (-1)^{\sum_{i=1}^q (x_2^{(i)}+1)(y_1^{(i)}+1)}, \quad \hat{\gamma}_{\pm}(\chi_{a_1},\chi_{a_2}) = (-1)^{\sum_{i=1}^q (y_1^{(i)}+1)(y_2^{(i)}+1)}.$

Let $\chi' = \chi_{b_1}$ and $\chi'' = \chi_{a_1}$. Since $\hat{\beta}(\chi', \chi'') = \beta(b_1, a_1) = -1$, Lemma 38 tells us that the group commutator of the corresponding invertible elements $s_{\chi'}$ and $s_{\chi''}$ of $\mathfrak{Cl}_{\bar{0}}(V)$ is a nonscalar central element of the spin group, so we may set

$$z := [s_{\chi'}, s_{\chi''}].$$

Then we have, by (26), that $\hat{\gamma}_{+}(\chi_{b_{1}},\chi_{a_{1}}) = [\pi_{+}(s_{\chi_{a}}),\pi_{+}(s_{\chi_{b}})] = 1$. Finally, using Lemma 41 again, we compute:

$$\hat{\gamma}_{+}(\chi_{b_{2}},\chi_{a_{2}}) = \hat{\gamma}_{+}(\chi_{b_{1}}\chi_{b_{2}},\chi_{a_{1}}\chi_{a_{2}})\hat{\gamma}_{+}^{-1}(\chi_{b_{1}},\chi_{a_{1}})\hat{\gamma}_{+}^{-1}(\chi_{b_{1}},\chi_{a_{2}})\hat{\gamma}_{+}^{-1}(\chi_{b_{2}},\chi_{a_{1}})$$
$$= (-1)^{\sum_{i=1}^{q} \left((x_{1}^{(i)} + x_{2}^{(i)} + 1)(y_{1}^{(i)} + y_{2}^{(i)} + 1) + (x_{1}^{(i)} + 1)(y_{2}^{(i)} + 1) + (x_{2}^{(i)} + 1)(y_{1}^{(i)} + 1) \right)}$$
$$= (-1)^{\sum_{i=1}^{q} (x_{1}^{(i)}y_{1}^{(i)} + x_{2}^{(i)}y_{2}^{(i)} + 1)} = (-1)^{q} \prod_{i=1}^{q} \beta(t_{i}) = (-1)^{q}.$$

To summarize our calculations, define a 4×4 matrix with entries in \mathbb{Z}_2 as follows:

$$M_{\Xi,g_0}^+ = \sum_{i=1}^q M^+(t_i).$$

where, for any $t = a_1^{x_1} a_2^{x_2} b_1^{y_1} b_2^{y_2}$ with $\beta(t) = 1$, the symmetric matrix $M^+(t)$ is

$$M^{+}(t) = \begin{bmatrix} 0 & (x_{1}+1)(x_{2}+1) & 0 & (x_{1}+1)(y_{2}+1) \\ 0 & (x_{2}+1)(y_{1}+1) & 1 \\ 0 & (y_{1}+1)(y_{2}+1) \\ \text{sym} & 0 \end{bmatrix}.$$

Then, for any $\chi_1, \chi_2 \in \widehat{G}, \, \widehat{\gamma}_+(\chi_1, \chi_2)$ is given by

(32)
$$\hat{\gamma}_{+}(\chi_{1},\chi_{2}) = (-1)^{t} (f(\chi_{1})) M^{+}_{\Xi,g_{0}} f(\chi_{2}).$$

Finally, $\hat{\gamma}_{-} = \hat{\beta}\hat{\gamma}_{+}$ by Proposition 39.

Remark 44. The supports of the graded division algebras representing $\hat{\gamma}_+$ and $\hat{\gamma}_$ are contained in T, so each of them is an elementary 2-group of rank 0, 2 or 4.

• Consider the case |T| > 16 (hence r is even).

Say, $|T| = 2^{2m}$ with m > 2, so $T = \langle a_1, \ldots, a_m \rangle \times \langle b_1, \ldots, b_m \rangle$, $\beta(a_j) = \beta(b_j) = 1$ for $j = 1, \ldots, m$. Using the same notation as in the previous case, we obtain from Lemma 41 that $T^{\perp} \subset \operatorname{rad} \hat{\gamma}_{\pm}$ and $\hat{\gamma}_{\pm}(\chi_u, \chi_v) = 1$ where u and v are any of the elements a_j , b_j except for the cases $u = a_j$, $v = b_j$ or $u = b_j$, $v = a_j$. Defining z appropriately, we may assume that $\hat{\gamma}_+(\chi_{a_1},\chi_{b_1}) = 1$. But then, using Lemma 41 again, we compute for j > 1:

 $\hat{\gamma}_{+}(\chi_{a_{j}},\chi_{b_{j}}) = \hat{\gamma}_{+}(\chi_{a_{1}}\chi_{a_{j}},\chi_{b_{1}}\chi_{b_{j}})\hat{\gamma}_{+}^{-1}(\chi_{a_{1}},\chi_{b_{1}})\hat{\gamma}_{+}^{-1}(\chi_{a_{1}},\chi_{b_{j}})\hat{\gamma}_{+}^{-1}(\chi_{a_{j}},\chi_{b_{1}}) = 1.$

Therefore, for all $\chi_1, \chi_2 \in \widehat{G}$, we obtain:

(33)
$$\hat{\gamma}_{+}(\chi_{1},\chi_{2}) = 1 \text{ and } \hat{\gamma}_{-}(\chi_{1},\chi_{2}) = \hat{\beta}(\chi_{1},\chi_{2}).$$

By examining the above cases, we see that if |T| > 1 then $\hat{\gamma}_+ \neq \hat{\beta}$. Recall also that $\hat{\gamma}_+ \neq \hat{\gamma}_-$ if and only if |T| > 1 (Corollary 40). In summary:

Theorem 45. Let $\mathcal{L} = \mathfrak{so}(V)$ be the simple Lie algebra of type D_r $(r \geq 3)$ over an algebraically closed field \mathbb{F} of characteristic 0. Suppose \mathcal{L} is given a grading Γ by an abelian group G, with parameters (T, β) , Ξ and g_0 , such that the image of \widehat{G} in Aut(\mathcal{L}) consists of inner automorphisms. If |T| > 1 then the Aut(\mathcal{L})-orbit (orbit under the stabilizer of the natural module if r = 4) of Γ consists of two Int(\mathcal{L})-orbits, which are distinguished by the Brauer invariants of the half-spin modules, $\hat{\gamma}_+$ and $\hat{\gamma}_-$, where $\hat{\gamma}_+$ is given by equations (30) through (33), while $\hat{\gamma}_- = \hat{\gamma}_+^{-1}$ for odd r and $\hat{\gamma}_- = \hat{\beta}\hat{\gamma}_+$ for even r, with $\hat{\beta} \colon \widehat{G} \times \widehat{G} \to \mathbb{F}^{\times}$ being the commutation factor associated to the parameters (T, β) . If |T| = 1 then $\hat{\gamma}_+ = \hat{\gamma}_-$ are given by equation (29), but we may have one or two Int(\mathcal{L})-orbits: the grading Γ on \mathcal{L} in this case is induced by a G-grading on the natural module V, and we get one Int(\mathcal{L})-orbit if and only if there exists an improper isometry $V \to V$ that is homogeneous of some degree with respect to the G-grading.

Theorem 46. Under the conditions of Theorem 45, consider a dominant integral weight $\lambda = \sum_{i=1}^{r} m_i \omega_i$. If |T| > 1, denote by Γ_+ a representative of the $\operatorname{Int}(\mathcal{L})$ -orbit for which $\operatorname{Br}(\omega_{r-1}) = \hat{\gamma}_-$, $\operatorname{Br}(\omega_r) = \hat{\gamma}_+$ and by Γ_- a representative of the orbit for which $\operatorname{Br}(\omega_{r-1}) = \hat{\gamma}_+$, $\operatorname{Br}(\omega_r) = \hat{\gamma}_-$. Then we have $H_{\lambda} = \{e\}$ and the following possibilities for $\operatorname{Br}(\lambda)$:

1) If $m_{r-1} \equiv m_r \pmod{2}$, then

$$\operatorname{Br}(\lambda) = \begin{cases} \hat{\beta} \sum_{i=1}^{r/2} m_{2i-1} & \text{if } r \text{ is even} \\ \hat{\beta} \sum_{i=1}^{(r-1)/2} m_{2i-1} - (m_{r-1} - m_r)/2 & \text{if } r \text{ is odd.} \end{cases}$$

2) If $m_{r-1} \not\equiv m_r \pmod{2}$, then

$$Br(\lambda) = \begin{cases} \hat{\beta} \sum_{i=1}^{r/2} m_{2i-1} \hat{\gamma}_{\pm} & \text{if } r \text{ is even} \\ \hat{\gamma}_{\pm}^{2 \sum_{i=1}^{(r-1)/2} m_{2i-1} - m_{r-1} + m_r} & \text{if } r \text{ is odd,} \end{cases}$$

where we take $\hat{\gamma}_+$ for Γ_+ and $\hat{\gamma}_-$ for Γ_- .

Proof. If |T| > 1, assume we are dealing with Γ_+ , the case of Γ_- being completely analogous. Since H_{ω_i} is trivial for all *i*, we can apply Proposition 10:

$$\operatorname{Br}(\lambda) = \prod_{i=1}^{r} \operatorname{Br}(\omega_i)^{m_i} = \hat{\beta}^{\sum_{i=1}^{r-2} i m_i} \hat{\gamma}_{-}^{m_{r-1}} \hat{\gamma}_{+}^{m_r}$$

The result follows from the fact $\hat{\beta}^2 = 1$ and the relations of Proposition 39: $\hat{\gamma}_- = \hat{\beta}\hat{\gamma}_+, \hat{\gamma}_+^2 = 1$ if r is even and $\hat{\gamma}_- = \hat{\gamma}_+^{-1}, \hat{\gamma}_+^2 = \hat{\beta}$ if r is odd.

Corollary 47. The simple \mathcal{L} -module V_{λ} admits a *G*-grading making it a graded \mathcal{L} -module if and only if one of the following conditions is satisfied:

- 1) $T = \{e\}$ and one of (i) $m_{r-1} \equiv m_r \pmod{2}$ or (ii) $\hat{\gamma}_+ = 1$;
- 2) T ≠ {e}, m_{r-1} ≡ m_r (mod 2) and one of (i) r is even and ∑^{r/2}_{i=1} m_{2i-1} is even or (ii) r is odd and ∑^{(r-1)/2}_{i=1} m_{2i-1} (m_{r-1} m_r)/2 is even;
 3) T ≠ {e}, m_{r-1} ≢ m_r (mod 2), r is even, γ̂₊ = 1 and one of (i) we are
- T ≠ {e}, m_{r-1} ≠ m_r (mod 2), r is even, γ̂₊ = 1 and one of (i) we are dealing with Γ⁺ and Σ^{r/2}_{i=1} m_{2i-1} is even or (ii) we are dealing with Γ⁻ and Σ^{r/2}_{i=1} m_{2i-1} is odd.

7.4. Outer gradings on $\mathfrak{so}_{2r}(\mathbb{F})$. Now suppose that the given grading on $\mathcal{L} = \mathfrak{so}_{2r}(\mathbb{F})$ is outer, i.e., the image of \widehat{G} in $\operatorname{Aut}(\mathcal{L})$ is not contained in $\operatorname{Int}(\mathcal{L})$. We remind the reader that, in the case r = 4, we assume that we have a matrix grading, i.e., the action of \widehat{G} fixes the natural module $V = V_{\omega_1}$. (This is automatic for $r \neq 4$.) So, the elements $\chi \in \widehat{G}$ act on \mathcal{L} as conjugations by similitudes u_{χ} of V and there exists $\chi \in \widehat{G}$ such that u_{χ} is improper. Recall that this can happen only for |T| = 1 or 4 and, moreover, there is a distinguished element $h \in G$ of order 2, which is characterized by the property that the corresponding \overline{G} -grading is inner, where $\overline{G} = G/\langle h \rangle$ (see Lemma 33 and Definition 34). Also recall that $H_{\omega_i} = \{e\}$ for i < r - 1 and $H_{\omega_{r-1}} = H_{\omega_r} = \langle h \rangle$. Denote $K = \langle h \rangle^{\perp} \subset \widehat{G}$, so $K = K_{\omega_{r-1}} = K_{\omega_r}$.

Since $\{\omega_{r-1}, \omega_r\}$ is a \widehat{G} -orbit, we have $\operatorname{Br}(\omega_{r-1}) = \operatorname{Br}(\omega_r)$ in the \overline{G} -graded Brauer group. This implies that $\operatorname{Br}(\omega_1)$ is trivial in the \overline{G} -graded Brauer group, i.e., the \overline{G} -grading on \mathcal{L} is induced from a \overline{G} -grading on V. (This can also be seen from the fact that $h \in T$ if |T| = 4.)

• Consider the case |T| = 1.

Already the *G*-grading on \mathcal{L} is induced by a *G*-grading on *V*, so it is determined by $g_0 \in G$ and a multiset $\Xi = \{g_1, \ldots, g_{2r}\}$ satisfying (14). For the \overline{G} -grading, we just have to take $g'_0 = \overline{g}_0 \in \overline{G}$ and $\Xi' = \{\overline{g}_1, \ldots, \overline{g}_{2r}\}$.

• Consider the case |T| = 4.

Here $T = \{e, a, b, c\}, \ \beta(a) = \beta(b) = 1$ (i.e., X_a and X_b are symmetric). The G-grading on \mathcal{L} is determined by $g_0 \in G$ and a multiset $\Xi = \{g_1, \ldots, g_r\}$ satisfying (22) with k = r. The operators $u_{\psi}, \psi \in K$, commute with each other, so they define a common eigenspace decomposition of V. Explicitly, writing $V = W \otimes_{\mathcal{D}} N$ and $u_{\psi} = \operatorname{diag}(\psi(g_1), \ldots, \psi(g_r)) \otimes X_t, t \in \{e, h\}$, we see that $v'_{2i-1} := v_i \otimes e_1$ and $v'_{2i} := v_i \otimes e_2, i = 1, \ldots, r$, are common eigenvectors, where $e_1, e_2 \in N$ are eigenvectors of X_h . Define a \overline{G} -grading on V by setting deg $v'_{2i-1} = \overline{g}_i$ and deg $v'_{2i} = \overline{g}_i \overline{h}'$, where h' is an element of T distinct from e and h. (There are two such elements but they yield the same element of \overline{G} .) One checks that the bilinear form $(\cdot, \cdot)_{\Phi}$ on V, with matrix (23), is homogeneous with respect to this \overline{G} -grading, of degree $g'_0 = \overline{g}_0$ if $h \in \{a, b\}$ and $g'_0 = \overline{g}_0 \overline{h}'$ if h = c. Hence the \overline{G} -grading on V induces a \overline{G} -grading on \mathcal{L} . Since diag $(\psi(g_1), \ldots, \psi(g_r)) \otimes \begin{pmatrix} 1 & 0 \\ 0 & \psi(h') \end{pmatrix}$ is a scalar multiple of u_{ψ} , for any $\psi \in K$, we see that this \overline{G} -grading on \mathcal{L} coincides with the one induced from the original G-grading by the quotient map $G \to \overline{G}$. The new multiset

$$\Xi' = \{\overline{g}_1, \overline{g}_1 h', \dots, \overline{g}_r, \overline{g}_r h'\},\$$

after reordering, satisfies (14) for g_0' indicated above, possibly with a different q.

In either case, we define $f_{\Xi',g'_0} \colon K \to \mathbb{Z}_2^{q'}/\langle \mathbf{1} \rangle$ similarly to equation (28) and set

(34)
$$\hat{\gamma}_0(\psi_1,\psi_2) = (-1)^{f_{\Xi',g_0'}(\psi_1) \bullet f_{\Xi',g_0'}(\psi_2)}$$

for all $\psi_1, \psi_2 \in K$, similarly to equation (29).

Theorem 48. Let \mathcal{L} be the simple Lie algebra of type D_r $(r \geq 3)$ over an algebraically closed field \mathbb{F} of characteristic 0. Suppose \mathcal{L} is graded by an abelian group G such that the image of \widehat{G} in $\operatorname{Aut}(\mathcal{L})$ contains outer automorphisms. If r = 4, assume further that this image is contained in the stabilizer of the natural module. Let $K = \langle h \rangle^{\perp}$, where $h \in G$ is the distinguished element. Then, for a dominant integral weight $\lambda = \sum_{i=1}^{r} m_i \omega_i$, we have the following possibilities:

- 1a) If $m_{r-1} \neq m_r$ but $m_{r-1} \equiv m_r \pmod{2}$, then $H_{\lambda} = \langle h \rangle$, $K_{\lambda} = K$, and $Br(\lambda) = 1$.
- 1b) If $m_{r-1} \neq m_r \pmod{2}$, then $H_{\lambda} = \langle h \rangle$, $K_{\lambda} = K$, and $\operatorname{Br}(\lambda) = \hat{\gamma}_0$, where $\hat{\gamma}_0 \colon K \times K \to \mathbb{F}^{\times}$ is given by equation (34).
- 2) If $m_{r-1} = m_r$, then $H_{\lambda} = \{e\}$ and $\operatorname{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{\lfloor r/2 \rfloor} m_{2i-1}}$, with $\hat{\beta} \colon \widehat{G} \times \widehat{G} \to \mathbb{F}^{\times}$ being the commutation factor associated to the parameters (T, β) of the grading on \mathcal{L} .

Proof. 1) Since we are dealing with an inner grading by $\overline{G} = G/\langle h \rangle$, we apply Theorem 46 and take into account that, with respect to our \overline{G} -grading, $\hat{\beta} = 1$ and hence $\hat{\gamma}_+ = \hat{\gamma}_-$ are given by equation (29), of which equation (34) is a restatement with the data pertaining to the \overline{G} -grading.

2) Here $\lambda = \sum_{i=1}^{r-2} m_i \omega_i + m_{r-1} (\omega_{r-1} + \omega_r)$. Since H_{ω_i} for i < r-1 and $H_{\omega_{r-1} + \omega_r}$ are trivial, we can apply Proposition 10:

$$\operatorname{Br}(\lambda) = \left(\prod_{i=1}^{r-2} \operatorname{Br}(\omega_i)^{m_i}\right) \operatorname{Br}(\omega_{r-1} + \omega_r)^{m_{r-1}} = \hat{\beta}^{\sum_{i=1}^{\lfloor (r-1)/2 \rfloor} m_{2i-1}} \operatorname{Br}(\omega_{r-1} + \omega_r)^{m_{r-1}}$$

Thus, it suffices to show that

$$Br(\omega_{r-1} + \omega_r) = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ \hat{\beta} & \text{if } r \text{ is even.} \end{cases}$$

In the \overline{G} -graded Brauer group, we can write $\operatorname{Br}(\omega_{r-1} + \omega_r) = \operatorname{Br}(\omega_{r-1})\operatorname{Br}(\omega_r)$ by Proposition 10, so $\operatorname{Br}(\omega_{r-1} + \omega_r) = 1$ by Proposition 39. (Recall that the restriction of $\hat{\beta}$ to K is trivial.) If r is odd, then we actually have $\operatorname{Br}(\omega_{r-1} + \omega_r) = 1$ in the G-graded Brauer group, thanks to Proposition 12, since $V_{\omega_{r-1}}$ and V_{ω_r} are dual to each other. So assume that r is even.

Denote $\hat{\gamma} = \operatorname{Br}(\omega_{r-1} + \omega_r)$. The fact that $\operatorname{Br}(\omega_{r-1} + \omega_r)$ is trivial in the \overline{G} -graded Brauer group means that $\hat{\gamma}(\psi_1, \psi_2) = 1$ for all $\psi_1, \psi_2 \in K$. Fix $\chi \in \widehat{G} \setminus K$ (i.e., $\chi(h) = -1$). It will be sufficient to show that $\hat{\gamma}(\chi, \psi) = \hat{\beta}(\chi, \psi)$ for all $\psi \in K$. We will again use the models S^{\pm} for the modules $V_{\omega_{r-1}}$ and V_{ω_r} . By our choice of χ , we have $S^{\pm} \cong (S^{\mp})^{\chi}$, and we are going to apply Proposition 11.

Let $S = S^+ \oplus S^-$. Then $\mathfrak{Cl}(V)$ can be identified with $\operatorname{End}(S)$ as a \mathbb{Z}_2 -graded algebra. Since the similitude u_{χ} is improper, the corresponding element s_{χ} is odd, i.e., it swaps S^+ and S^- . Recall that, by definition of the *G*-grading on $\mathfrak{Cl}_{\bar{0}}(V)$, we have $\chi * x = s_{\chi} x s_{\chi}^{-1}$ for all $x \in \mathfrak{Cl}_{\bar{0}}(V)$, where * denotes the \widehat{G} -action associated to the *G*-grading. Since the imbedding $\pi: \mathcal{L} \to \mathfrak{Cl}_{\bar{0}}(V)$ respects the *G*-grading,

36

 $s_{\chi}: S \to S^{\chi}$ is an isomorphism of \mathcal{L} -modules, so we may use $u' = s_{\chi}|_{S^+}$ and $u'' = s_{\chi}|_{S^-}$ in Proposition 11.

According to that proposition, $\hat{\gamma}$ is the commutation factor associated to the G-graded algebra $\mathcal{R} = \operatorname{End}(S^+ \otimes S^-) = \operatorname{End}(S^+) \otimes \operatorname{End}(S^-)$, where the G-grading is obtained by refining the \overline{G} -grading using the conjugation by the operator u = $(u'' \otimes u') \circ \tau$ on $S^+ \otimes S^-$, with $\tau \colon S^+ \otimes S^- \to S^- \otimes S^+$ being the flip. It is convenient to consider \mathcal{R} as imbedded into $\operatorname{End}(S \otimes S)$ by virtue of the decomposition $S \otimes S = (S^+ \otimes S^+) \oplus (S^+ \otimes S^-) \oplus (S^- \otimes S^+) \oplus (S^- \otimes S^-)$. Recall that the \overline{G} -grading on $\operatorname{End}(S^{\pm})$ is associated with the K-action $\psi * x = s_{\psi} x s_{\psi}^{-1}$ for $x \in \operatorname{End}(S^{\pm})$. It follows that the \overline{G} -grading on \mathcal{R} is associated with the K-action $\psi * x = \tilde{u}_{\psi} x \tilde{u}_{\psi}^{-1}$ where \tilde{u}_{ψ} is the restriction of $s_{\psi} \otimes s_{\psi}$ to $S^+ \otimes S^-$. By the definition of the refinement in Proposition 11, the G-grading on \mathcal{R} is associated to the extension of this K-action to a \widehat{G} -action obtained by setting $\chi * x = uxu^{-1}$. By definition of commutation factor, $\hat{\gamma}(\chi, \psi)$ is the group commutator of u and \tilde{u}_{ψ} . But u is the restriction of $(s_{\chi} \otimes s_{\chi}) \circ \tau$ to $S^+ \otimes S^-$, so it will be sufficient to compute the group commutator of $s_{\chi} \otimes s_{\chi}$ and $s_{\psi} \otimes s_{\psi}$ (since the flip clearly commutes with these operators). But by Lemma 38, we know that $[s_{\chi}, s_{\psi}] \in \{\pm 1\}$ if $\hat{\beta}(\chi, \psi) = 1$ and $[s_{\chi}, s_{\psi}] \in \{\pm z\}$ if $\hat{\beta}(\chi,\psi) = -1$. It follows that $[s_{\chi} \otimes s_{\chi}, s_{\psi} \otimes s_{\psi}]$ is 1 in the first case and $z \otimes z$ in the second case. But z acts as 1 on S^+ and -1 on S^- , so the restriction of $z \otimes z$ to $S^+ \otimes S^-$ is -1. Therefore, $\hat{\gamma}(\chi, \psi) = \hat{\beta}(\chi, \psi)$, as desired.

Corollary 49. The simple \mathcal{L} -module V_{λ} admits a *G*-grading making it a graded \mathcal{L} -module if and only if 1) $m_{r-1} = m_r$ and 2) $T = \{e\}$ or $\sum_{i=1}^{\lfloor r/2 \rfloor} m_{2i-1}$ is even.

Acknowledgments

The second author would like to thank the Instituto Universitario de Matemáticas y Aplicaciones and Departamento de Matemáticas of the University of Zaragoza for support and hospitality during his visit in January–April 2013.

References

- Bahturin, Y. and Kochetov, M. Classification of group gradings on simple Lie algebras of types A, B, C and D. J. Algebra 324 (2010), no. 11, 2971–2989.
- [2] Billig, Y. and Lau, M. Thin coverings of modules. J. Algebra **316** (2007), no. 1, 147–173.
- Bourbaki, N. Lie groups and Lie algebras. Chapters 7–9. Translated from the 1975 and 1982 French originals by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005.
- [4] Elduque, A. Fine gradings on simple classical Lie algebras. J. Algebra 324 (2010), no. 12, 3532–3571.
- [5] Elduque, A. and Kochetov, M. Gradings on simple Lie algebras. Mathematical Surveys and Monographs 189, American Mathematical Society, Providence, RI, 2013.
- [6] Knus, M.-A.; Merkurjev, A.; Rost, M.; Tignol, J.-P. The book of involutions. (With a preface in French by J. Tits.) American Mathematical Society Colloquium Publications 44, American Mathematical Society, Providence, RI, 1998.
- [7] Long, F. W. A generalization of the Brauer group of graded algebras. Proc. London Math. Soc. (3) 29 (1974), 237–256.
- [8] Năstăsescu, C. and van Oystaeyen, F. Graded ring theory. North-Holland Mathematical Library 28, North-Holland Publishing Co., Amsterdam, 1982.
- [9] Ram, A. and Ramagge, J. Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory. A tribute to C. S. Seshadri (Chennai, 2002), 428–466, Trends Math., Birkhäuser, Basel, 2003.

[10] Zolotykh, A. A. Classification of projective representations of finite abelian groups, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (2002), no. 3, 3–10, 70.

DEPARTAMENTO DE MATEMÁTICAS E INSTITUTO UNIVERSITARIO DE MATEMÁTICAS Y APLICA-CIONES, UNIVERSIDAD DE ZARAGOZA, 50009 ZARAGOZA, SPAIN *E-mail address:* elduque@unizar.es

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, A1C5S7, Canada

 $E\text{-}mail\ address: \texttt{mikhail@mun.ca}$

38