GRADINGS ON CLASSICAL CENTRAL SIMPLE REAL LIE ALGEBRAS

YURI BAHTURIN*, MIKHAIL KOCHETOV**, AND ADRIÁN RODRIGO-ESCUDERO†

Abstract. For any abelian group G, we classify up to isomorphism all G-gradings on the classical central simple Lie algebras, except those of type D_4, over the field of real numbers (or any real closed field).

1. Introduction

Let $\mathcal{R}$ be an algebra (not necessarily associative) over a field and let G be a group. A *G-grading* on $\mathcal{R}$ is a vector space decomposition $\Gamma: \mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ such that $\mathcal{R}_g\mathcal{R}_h \subseteq \mathcal{R}_{gh}$ for all $g, h \in G$. The nonzero elements $x \in \mathcal{R}_g$ are said to be *homogeneous of degree g*, which can be written as $\deg x = g$, and the *support* of the grading Γ (or of the graded algebra $\mathcal{R}$) is the set $\{g \in G \mid \mathcal{R}_g \neq 0\}$. The algebra $\mathcal{R}$ may have some additional structure, for example, an involution φ, in which case Γ is required to respect this structure: $\varphi(\mathcal{R}_g) = \mathcal{R}_g$ for all $g \in G$.

Group gradings have been extensively studied for many types of algebras — associative, Lie, Jordan, composition, etc. (see e.g. the recent monograph [11] and the references therein). In the case of gradings on simple Lie algebras, the support generates an abelian subgroup of G, so it is no loss of generality to assume G abelian. We will do so for all gradings considered in this paper.

The classification of fine gradings (up to equivalence) on all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0 has recently been completed by the efforts of many authors: see [11, Chapters 3–6], [19] and [10]. The classification of all G-gradings (up to isomorphism) is also known for these algebras, except for types E_6, E_7 and E_8, over an algebraically closed field of characteristic different from 2: see [11, Chapters 3–6] and [12].

On the other hand, group gradings on algebras over the field of real numbers have not yet been sufficiently studied. Fine gradings on real forms of the classical simple

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complex Lie algebras except D_4 were described in [14], but the equivalence problem remains open. Fine gradings on real forms of the simple complex Lie algebras of types G_2 and F_4 were classified up to equivalence in [7]; some partial results were obtained for type E_6 in [8].

Here we will classify up to isomorphism all G-gradings on real forms of the classical simple complex Lie algebras except D_4. We do not use the topology of $\mathbb{R}$, so in all of our results $\mathbb{R}$ can be replaced by an arbitrary real closed field. We will follow the approach that has already been used over algebraically closed fields [2, 11]: embed our Lie algebra $\mathcal{L}$ into an associative algebra $\mathcal{R}$ with involution φ and transfer the classification problem for gradings from $\mathcal{L}$ to $(\mathcal{R}, \varphi)$ using automorphism group schemes. Specifically:

- $\mathcal{R} = M_n(\mathbb{R})$, with φ orthogonal (that is, given by a symmetric bilinear form) and n odd, yield all real forms of series B by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi) := \{x \in \mathcal{R} \mid \varphi(x) = -x\}$;
- $\mathcal{R} = M_n(\mathbb{R})$, with φ symplectic (that is, given by a skew-symmetric bilinear form over $\mathbb{R}$, so n must be even), and $\mathcal{R} = M_n(\mathbb{H})$, with φ symplectic (that is, given by a hermitian form over $\mathbb{H}$), yield all real forms of series C by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)$;
- $\mathcal{R} = M_n(\mathbb{R})$, with φ orthogonal (that is, given by a symmetric bilinear form over $\mathbb{R}$) and n even, and $\mathcal{R} = M_n(\mathbb{H})$, with φ orthogonal (that is, given by a skew-hermitian form over $\mathbb{H}$), yield all real forms of series D by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)$;
- $\mathcal{R} = M_n(\mathbb{C})$, $M_n(\mathbb{R}) \times M_n(\mathbb{R})$ and $M_n(\mathbb{H}) \times M_n(\mathbb{H})$, with φ of the second kind (that is, nontrivial on the center of $\mathcal{R}$), yield all real forms of series A by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)'$, where prime denotes the derived Lie algebra.

With the exception of the last two, the above associative algebras $\mathcal{R}$ are simple, so for any G-grading on $\mathcal{R}$ we have $\mathcal{R} \cong \mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as a graded algebra, where $\mathcal{D}$ is a G-graded associative algebra which is a *graded division algebra* (that is, all nonzero homogeneous elements are invertible), and $\mathcal{V}$ is a graded right $\mathcal{D}$-module which is finite-dimensional over $\mathcal{D}$ (that is, has a finite basis consisting of homogeneous elements) — see details in Section 2. Note that the graded algebra $\mathcal{D}$ is determined up to isomorphism and the graded module $\mathcal{V}$ up to isomorphism and shift of grading. By a *shift of grading* we mean replacing $\mathcal{V}$ by $\mathcal{V}^{[g]}$ for some $g \in G$, where $\mathcal{V}^{[g]} = \mathcal{V}$ as a module, but with the elements of $\mathcal{V}_h$ now having degree gh, for all $h \in G$. Over an algebraically closed field $\mathbb{F}$, the identity component $\mathcal{D}_e$ (e being the identity element of G) must be $\mathbb{F}$, so the involution on $\mathcal{R}$ can be studied, as was done in [4, 2], using the tensor product decomposition $\mathcal{R} \cong \mathrm{End}_{\mathbb{F}}(V) \otimes_{\mathbb{F}} \mathcal{D}$ where V is the $\mathbb{F}$-span of a homogeneous $\mathcal{D}$-basis of $\mathcal{V}$. Here we will follow a graded version of the classical approach, as was done in [9, 11], and write the involution on $\mathcal{R}$ in terms of an involution on $\mathcal{D}$ and a sesquilinear form on $\mathcal{V}$.

As to the algebras $\mathcal{R} = M_n(\mathbb{R}) \times M_n(\mathbb{R})$ and $\mathcal{R} = M_n(\mathbb{H}) \times M_n(\mathbb{H})$, we have two types of G-gradings: if the center $Z(\mathcal{R})$ is trivially graded then we will say that the grading is of *Type I*, and otherwise of *Type II*. In the case of a Type II grading, $\mathcal{R}$ is *graded-simple* (that is, simple as a graded algebra: it has no nonzero proper graded ideals) and can apply the same approach as above. In the case of a Type I grading, we will use alternative models: $\mathcal{R} = M_n(\mathbb{R})$ and $\mathcal{R} = M_n(\mathbb{H})$, where $\mathcal{L} = \mathcal{R}'$ (traceless matrices). In these models, there is no involution on $\mathcal{R}$, so they are treated separately. Note that we also have Type I and Type II gradings for $M_n(\mathbb{C})$, but we do not have to use a different approach for Type I gradings.

Finite-dimensional graded division algebras over $\mathbb{R}$ that are simple as ungraded algebras and have an abelian grading group have recently been classified in [18], both up to isomorphism and up to equivalence, and independently in [5], up to equivalence (but note that one of the equivalence classes was overlooked). A classification up to equivalence has been obtained in [6] without assuming simplicity.

For our purposes, we will also need some information about (degree-preserving) involutions on a graded division algebra $\mathcal{D}$ that is isomorphic to $M_\ell(\mathbb{R})$, $M_\ell(\mathbb{C})$, $M_\ell(\mathbb{H})$, $M_\ell(\mathbb{R})\times M_\ell(\mathbb{R})$ or $M_\ell(\mathbb{H})\times M_\ell(\mathbb{H})$ as an ungraded algebra, where ℓ is a divisor of n. A classification of such graded division algebras with involution is given in [3].

The paper is structured as follows. In Section 2, we establish some properties of (associative) graded division algebras with involution over $\mathbb{R}$, which may be of independent interest. We do not restrict ourselves to the finite-dimensional case: we only assume that $\mathcal{D}_e$ is finite-dimensional. About the involution, we assume that the only symmetric elements in $Z(\mathcal{D})_e$ are the scalars (in other words, $\mathcal{D}$ is central as a graded algebra with involution). The main result (Theorem 18) is a structure theorem for central graded-simple associative algebras with involution $(\mathcal{R},\varphi)$ over $\mathbb{R}$, under certain finiteness assumptions. The structure of $(\mathcal{R},\varphi)$ is described by a graded division algebra with involution $(\mathcal{D},\varphi_0)$, an element $g_0\in G$, a sign $\delta\in\{\pm1\}$, and two functions, $\kappa: G/T\to\mathbb{Z}_{\geq 0}$ and $\sigma: G/T\to\mathbb{Z}$, satisfying certain conditions (see Definitions 16 and 17), where T is the support of $\mathcal{D}$ (a subgroup of G).

In Section 3, we solve the isomorphism problem for graded algebras with involution determined by these data (Theorem 24), under a certain technical restriction in the case $\mathbb{C}\cong\mathcal{D}_e\subseteq Z(\mathcal{D})$. In particular, if $\mathcal{D}$ is isomorphic to $M_\ell(\mathbb{R})$, $M_\ell(\mathbb{C})$, $M_\ell(\mathbb{H})$, $M_\ell(\mathbb{R})\times M_\ell(\mathbb{R})$ or $M_\ell(\mathbb{H})\times M_\ell(\mathbb{H})$ as an ungraded algebra and φ_0 is of the second kind in the cases with $Z(\mathcal{D})\neq\mathbb{R}$, then the classification boils down to orbits for an action of G on triples (g_0,κ,σ) (see Corollaries 26, 27 and 28). This action is associated with the shift of grading mentioned above.

In Section 4, we transfer the results to classical Lie algebras: Theorems 34 and 35 for Type I (that is, inner) gradings on real forms of series A, Theorems 36 and 37 for Type II (that is, outer) gradings on real forms of series A, and Theorems 38, 40 and 42 for series B, C and D, respectively. In particular, we determine which of the real forms arises from given data describing $(\mathcal{R},\varphi)$.

Finally, in the Appendix, we show, under a certain condition, how a G-grading of Type II on one of the special linear Lie algebras $\mathrm{Skew}(\mathcal{R},\varphi)'$, where $\mathcal{R}=M_n(\Delta)\times M_n(\Delta)$ and $\Delta\in\{\mathbb{R},\mathbb{H}\}$, can be realized in terms of the model $\mathfrak{sl}_n(\Delta)$. The condition is satisfied, for example, if G is an elementary 2-group. We also consider Type II gradings on simple Lie algebras of series A over an algebraically closed field $\mathbb{F}$, $\operatorname{char}\mathbb{F}\neq 2$, which were treated in [2, 11] in terms of the model $\mathfrak{sl}_n(\mathbb{F})$ (modulo the 1-dimensional center if $\operatorname{char}\mathbb{F}$ divides n). We show how the model $\mathrm{Skew}(\mathcal{R},\varphi)'$ (modulo the center), where $\mathcal{R}=M_n(\mathbb{F})\times M_n(\mathbb{F})$, gives a different — and simpler — parametrization of Type II gradings.

For any abelian group G, we will use the notation $G_{[2]}:=\{g\in G\mid g^2=e\}$ and $G^{[2]}:=\{g^2\mid g\in G\}$.

2. Involutions on graded-simple real associative algebras

2.1. **Graded-simple real associative algebras.** Let $\mathcal{R}$ be a G-graded real associative algebra that is graded-simple and satisfies the descending chain condition on graded left ideals. By [11, Theorem 2.6] (compare with [17, Theorem 2.10.10]), $\mathcal{R}$ is isomorphic to $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$, where $\mathcal{D}$ is a G-graded real associative algebra which is a graded division algebra, and $\mathcal{V}$ is a graded right $\mathcal{D}$-module which is finite-dimensional over $\mathcal{D}$. For an invertible $d \in \mathcal{D}$, we will denote by $\mathrm{Int}(d)$ the corresponding inner automorphism of $\mathcal{D}$: $\mathrm{Int}(d)(x) := dxd^{-1}$.

Let $T \subseteq G$ be the support of $\mathcal{D}$. We fix a transversal for the subgroup T in G, so we have a section $\xi : G/T \to G$ (which is not necessarily a group homomorphism) of the quotient map $\pi : G \to G/T$. We can write $\mathcal{V} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_s$ where $\mathcal{V}_i$ are the isotypic components of the graded right $\mathcal{D}$-module $\mathcal{V}$. Each of these components is determined by its support, which is a coset $x_i \in G/T$, and its $\mathcal{D}$-dimension k_i. Thus, $\mathcal{V}$ is determined by the multiset $\kappa : G/T \to \mathbb{Z}_{\geq 0}$ whose underlying set is $\{x_1, \ldots, x_s\}$ and the multiplicity of x_i is $\kappa(x_i) = k_i$. Let $g_i := \xi(x_i)$. Then $\mathcal{V}_{g_i}$ is a (right) vector space over the division algebra $\mathcal{D}_e$ and $\mathcal{V}_i = \mathcal{V}_{g_i} \otimes_{\mathcal{D}_e} \mathcal{D}$; hence $\mathcal{V} = V \otimes_{\mathcal{D}_e} \mathcal{D}$, with $V := \mathcal{V}_{g_1} \oplus \ldots \oplus \mathcal{V}_{g_s}$. Picking a basis $\mathcal{B}_i$ in each $\mathcal{V}_{g_i}$, we may identify $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$ with the algebra of matrices $M_k(\mathcal{D})$, where $k = |\kappa| := k_1 + \cdots + k_s$. The matrices are partitioned into s^2 blocks according to the partition of the basis $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_s$, and the G-grading is given by

$$\deg(E \otimes d) = g_i t g_j^{-1} \tag{1}$$

for any matrix unit E in the (i, j)-th block and any $d \in \mathcal{D}_t$. Here we are using the Kronecker product to identify $M_k(\mathcal{D})$ with $M_k(\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{D}$.

Therefore, the data that determines the graded algebra $\mathcal{R}$ is $(\mathcal{D}, \kappa)$. Conversely, any pair $(\mathcal{D}, \kappa)$, where $\mathcal{D}$ is a graded-division real associative algebra, and $\kappa : G/T \to \mathbb{Z}_{\geq 0}$ has finite support, determines, by means of Equation (1), a graded-simple real associative algebra $M_k(\mathcal{D})$ that satisfies the descending chain condition on graded left ideals (see [11, Section 2.1]).

We will assume throughout that the identity component $\mathcal{R}_e$ is finite-dimensional. From the above matrix description it is clear that $\mathcal{R}_e$ is a finite-dimensional vector space over the division algebra $\mathcal{D}_e$ (as $\mathcal{R}_e$ consists of the diagonal blocks with entries in $\mathcal{D}_e$). Thus, our assumption is tantamount to $\mathcal{D}_e$ having finite dimension, which makes it isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

2.2. **Properties of the graded-division algebra.** We will now record for future reference some elementary properties of $\mathcal{D}$. First of all, for each $t \in T$, pick $0 \neq X_t \in \mathcal{D}_t$. Then $\mathcal{D}_t = \mathcal{D}_e X_t = X_t \mathcal{D}_e$. If $\mathcal{D}_e \cong \mathbb{H}$, we can take $X_t \in C_{\mathcal{D}}(\mathcal{D}_e)$. Indeed, $\mathrm{Int}(X_t)$ restricts to an automorphism of $\mathcal{D}_e$, which is inner since $\mathcal{D}_e$ is central simple. Replacing X_t by $X_t q$ for a suitable $q \in \mathcal{D}_e$, we get the result. It also follows from the Double Centralizer Theorem (see for instance [15, Theorem 4.7]), which says that we have an isomorphism of algebras

$$\mathcal{D} \cong \mathcal{D}_e \otimes_{\mathbb{R}} C_{\mathcal{D}}(\mathcal{D}_e)$$

given by multiplication in $\mathcal{D}$. We will always assume that X_t are chosen in $C_{\mathcal{D}}(\mathcal{D}_e)$ in the case $\mathcal{D}_e \cong \mathbb{H}$. Of course, this is automatic in the case $\mathcal{D}_e = \mathbb{R}$. In these cases, define $\beta : T \times T \to Z(\mathcal{D}_e)^\times = \mathbb{R}^\times$ by

$$X_s X_t = \beta(s, t) X_t X_s. \tag{2}$$

Clearly, this is an alternating bicharacter that does not depend on the scaling of X_t and is therefore an invariant of the graded algebra $\mathcal{D}$.

As to the remaining case $\mathcal{D}_e \cong \mathbb{C}$, let

$$K := \{t \in T \mid \mathrm{Int}(X_t)|_{\mathcal{D}_e} = \mathrm{id}_{\mathcal{D}_e}\}.$$

This is a subgroup in T of index 1 or 2, and it does not depend on the choice of X_t. Equation (2) gives a well defined alternating bicharacter $\beta : K \times K \to \mathcal{D}_e^\times \cong \mathbb{C}^\times$. It is convenient to set $K := T$ in the cases $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$, so in all cases we have $\beta : K \times K \to Z(\mathcal{D}_e)^\times$.

Lemma 1 ([9, Lemma 3.3])**.** *Let α be an inner automorphism of $\mathcal{D}$ that preserves degree. Then, there exists a nonzero homogeneous $d \in \mathcal{D}$ such that $\alpha = \mathrm{Int}(d)$.*

Proof. By definition, there exists $d' \in \mathcal{D}^\times$ such that $\alpha = \mathrm{Int}(d')$, that is, $\alpha(x)d' = d'x$ for all $x \in \mathcal{D}$. Write $d' = d_1 + \cdots + d_n$ where the d_i are nonzero homogeneous elements of pairwise distinct degrees h_i. As α preserves dregree, if $x \in \mathcal{D}$ is homogeneous of degree h, so is $\alpha(x)$. Since G is abelian, if we consider the terms of degree hh_1 in $\alpha(x)d' = d'x$, we get $\alpha(x)d_1 = d_1x$. But d_1 is invertible because it is homogeneous, so $\alpha(x) = d_1xd_1^{-1}$ for all $x \in \mathcal{D}_h$ and $h \in G$. □

Lemma 2. *The group of automorphisms of the graded algebra $\mathcal{D}$ that act trivially on $\mathcal{D}_e$ is isomorphic to $Z^1(T, Z(\mathcal{D}_e)^\times)$, where $t \in T$ acts on $Z(\mathcal{D}_e)^\times$ on the right by $z^t = X_t^{-1}zX_t$ for all $z \in Z(\mathcal{D}_e)^\times$.*

Proof. Let ψ_0 be such an automorphism. Then $\psi_0(X_t) = X_t\nu(t)$, with $\nu(t) \in \mathcal{D}_e^\times$, and the condition $\psi_0|_{\mathcal{D}_e} = \mathrm{id}_{\mathcal{D}_e}$ implies $\mathrm{Int}(X_t)|_{\mathcal{D}_e} = \mathrm{Int}(\psi_0(X_t))|_{\mathcal{D}_e}$, hence $\nu(t) \in C_{\mathcal{D}}(\mathcal{D}_e)$. Therefore, we have $\nu : T \to Z(\mathcal{D}_e)^\times$. Applying ψ_0 to both sides of the equation $X_sX_t = X_{st}\lambda$ ($\lambda \in \mathcal{D}_e^\times$), we get $X_s\nu(s)X_t\nu(t) = X_{st}\nu(st)\lambda$, so

$$\nu(s)^t\nu(t) = \nu(st)$$

where the action is trivial for $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$, but may be nontrivial for $\mathcal{D}_e \cong \mathbb{C}$ (recall K). In other words, ν is a (right) 1-cocycle. The converse is also clear. Finally, if 1-cocycles ν and ν' correspond to ψ_0 and ψ_0', respectively, then the product $\nu\nu'$ corresponds to the composition $\psi_0\psi_0'$. □

If $\mathcal{D}_e = \mathbb{R}$, $\mathcal{D}_e \cong \mathbb{H}$ or $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$ (that is, $K = T$), then the action is trivial, so $Z^1(T, Z(\mathcal{D}_e)^\times) = \mathrm{Hom}(T, Z(\mathcal{D}_e)^\times)$.

Lemma 3. *If $\mathbb{C} \cong \mathcal{D}_e \not\subseteq Z(\mathcal{D})$, then there is a split short exact sequence:*

$$1 \longrightarrow U \longrightarrow Z^1(T, \mathcal{D}_e^\times) \xrightarrow{\mathrm{res}} \mathrm{Hom}^+(K, \mathbb{R}^\times) \longrightarrow 1$$

where $\mathrm{Hom}^+(K, \mathbb{R}^\times) := \{\chi \in \mathrm{Hom}(K, \mathbb{R}^\times) \mid \chi(t^2) > 0, \forall t \in T\}$*, U is the unit circle in $\mathbb{C}$, and* res *is the restriction map.*

Proof. Let $\nu \in Z^1(T, \mathcal{D}_e^\times)$ and pick $t \in T \setminus K$. For all $s \in K$, we have $\overline{\nu(s)}\nu(t) = \nu(st) = \nu(ts) = \nu(t)\nu(s)$, so $\nu(s) \in \mathbb{R}^\times$. For any $t \in T$, we have $\nu(t^2) = \nu(t)^t\nu(t) = |\nu(t)|^2 > 0$, since $\nu(t)^t = \nu(t) \in \mathbb{R}$ if $t \in K$ and $\nu(t)^t = \overline{\nu(t)}$ if $t \in T \setminus K$. Thus, the above restriction map is well defined.

Now take ν in the kernel of the restriction map, that is, such that $\nu(s) = 1$ for all $s \in K$. If $t \in T \setminus K$, then $\nu(ts) = \nu(t)^s\nu(s) = \nu(t)$, so ν is constant on $T \setminus K$. Moreover, if $t \in T \setminus K$ then $t^2 \in K$, so $1 = \nu(t^2) = |\nu(t)|^2$. Conversely, for any $z \in U$, the map ν, defined by $\nu(t) = 1$ if $t \in K$ and $\nu(t) = z$ if $t \in T \setminus K$, is a 1-cocycle. Therefore, the kernel is isomorphic to U.

Finally, to construct a section of the restriction map, we fix $t \in T \setminus K$. Then, for any $\chi \in \mathrm{Hom}^+(K, \mathbb{R}^\times)$ define $\nu_\chi(s) = \chi(s)$ and $\nu_\chi(ts) = \sqrt{\chi(t^2)}\chi(s)$ for all $s \in K$. It is easy to check that ν_χ is a 1-cocycle, and the mapping $\chi \mapsto \nu_\chi$ is a homomorphism of groups and a section of the restriction map. This proves that res is surjective and the above sequence splits (noncanonically). □

Lemma 4. *The center $Z(\mathcal{D})$ is a graded subalgebra with support*

$$\mathrm{rad}\beta := \{s \in K \mid \beta(s,t) = 1,\ \forall t \in K\}.$$

Except possibly in the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$, β takes values in $\mathbb{R}^\times$.

Proof. As G is abelian, the center $Z(\mathcal{D})$ is a graded subalgebra, because $x = \sum_{u\in G} x_u \in Z(\mathcal{D})$ if and only if $xy = yx$ for all $y \in \mathcal{D}_v$ and $v \in G$, if and only if $x_u y = y x_u$ for all $y \in \mathcal{D}_v$ and $u, v \in G$, if and only if $x_u \in Z(\mathcal{D})$ for all $u \in G$.

If $K = T$, the assertion about the support is clear from Equation (2); also $Z(\mathcal{D}_e) = \mathbb{R}$ unless $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$. Now assume that $K \neq T$ and consider $\mathrm{Int}(X_s)$ for some $s \in K$. This is an automorphism that acts trivially on $\mathcal{D}_e$, so by Lemma 2, $\mathrm{Int}(X_s)(X_t) = X_t \nu(t)$ for some 1-cocycle ν. In particular, $\beta(s,t) = \nu(t)$, $t \in K$, takes values in $\mathbb{R}^\times$ by Lemma 3. By definition of K and β, we have $\mathrm{supp}\, Z(\mathcal{D}) \subseteq \mathrm{rad}\beta$. To prove the opposite inclusion, assume that $s \in \mathrm{rad}\beta$. Then the corresponding ν is in the kernel of the restriction map, so $\nu(t) = z \in U$ for all $t \in T \setminus K$. Pick $w \in \mathcal{D}_e \cong \mathbb{C}$ such that $w\overline{w}^{-1} = z$, then $X_s w$ commutes with the homogeneous elements of degree t, so it lies in the center of $\mathcal{D}$. □

2.3. **Involutions.** Now suppose φ is an involution on the graded algebra $\mathcal{R}$. By [11, Theorem 2.57], φ corresponds to a pair (φ_0, B), where φ_0 is an antiautomorphism of the graded algebra $\mathcal{D}$, and $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$ is a nondegenerate homogeneous φ_0-sesquilinear form (that is, linear over $\mathcal{D}$ with respect to the second variable and φ_0-semilinear over $\mathcal{D}$ with respect to the first variable). The correspondence is given by

$$B(rv, w) = B(v, \varphi(r)w) \tag{3}$$

for all $v, w \in \mathcal{V}$ and $r \in \mathcal{R}$. Note that (φ_0, B) is unique up to the following operation: B can be replaced by dB, where d is a nonzero homogeneous element of $\mathcal{D}$, and φ_0 is simultaneously replaced by $\mathrm{Int}(d)\varphi_0$. The pair $(\varphi_0^{-1}, \overline{B})$, where $\overline{B}(v,w) := \varphi_0^{-1}(B(w,v))$ for all $v, w \in \mathcal{V}$, determines $\varphi^{-1} = \varphi$. Hence, there exists a nonzero homogeneous element $\delta \in \mathcal{D}$ such that $\overline{B} = \delta B$ and $\varphi_0^{-1} = \mathrm{Int}(\delta)\varphi_0$. Clearly $\delta \in \mathcal{D}_e$.

We will assume throughout that $\mathcal{R}$ is central as a graded algebra with involution, that is, $\mathrm{Sym}(Z(R)_e, \varphi) := \{r \in Z(R)_e \mid \varphi(r) = r\} = \mathbb{R}$.

Lemma 5. *B can be chosen so that $\delta \in \{\pm 1\}$ (hence $\varphi_0^2 = \mathrm{id}_{\mathcal{D}}$) and φ_0 restricts to the conjugation on $\mathcal{D}_e$.*

Proof. If we replace B by dB and φ_0 by $\mathrm{Int}(d)\varphi_0$, then δ is replaced by:

$$\delta' = \delta\varphi_0(d)d^{-1} \tag{4}$$

Indeed, we have $\overline{dB}(v,w) = \varphi_0^{-1}(\mathrm{Int}(d^{-1})(dB(w,v))) = \varphi_0^{-1}(d)\varphi_0^{-1}(B(w,v)) = \mathrm{Int}(\delta)(\varphi_0(d))(\delta B(v,w)) = (\delta\varphi_0(d)d^{-1})dB(v,w)$. Applying Equation (4) to $\delta B = \overline{B}$, we get: $B = \overline{\overline{B}} = \delta\varphi_0(\delta)\delta^{-1}\overline{B} = \delta\varphi_0(\delta)B$, so $\delta\varphi_0(\delta) = 1$. Consider the possibilities for $\mathcal{D}_e$:

- $\mathcal{D}_e = \mathbb{R}$. We know that φ_0 is the identity on $\mathcal{D}_e$, so $\delta^2 = 1$ and $\delta \in \{\pm 1\}$.

- $\mathcal{D}_e \cong \mathbb{H}$. Replacing B by a suitable qB with $q \in \mathcal{D}_e$, we may assume that φ_0 restricts to the standard involution (conjugation) on $\mathcal{D}_e$. Since $\delta \in \mathcal{D}_e$, φ_0 is involutive on $C_{\mathcal{D}}(\mathcal{D}_e)$, and hence $\varphi_0^2 = \mathrm{id}_{\mathcal{D}}$. It follows that $\delta \in Z(\mathcal{D}) \cap \mathcal{D}_e = \mathbb{R}$, and it has to be $+1$ or -1.
- $\mathcal{D}_e \cong \mathbb{C}$. If $\mathcal{D}_e$ is central in $\mathcal{D}$, then φ_0 must be nontrivial on $\mathcal{D}_e$ by assumption, because the restrictions of φ and φ_0 to $Z(\mathcal{R}) = Z(\mathcal{D})$ coincide by Equation (3). If $\mathcal{D}_e$ is not central and $\varphi_0|_{\mathcal{D}_e}$ is trivial, pick a nonzero homogeneous $d \in \mathcal{D}$ such that $\mathrm{Int}(d)|_{\mathcal{D}_e}$ is nontrivial and replace B with dB. Thus, we may assume that φ_0 is the complex conjugation on $\mathcal{D}_e$. So $1 = \delta\varphi_0(\delta) = |\delta|^2$; pick $z \in \mathcal{D}_e$ such that $z\overline{z}^{-1} = \delta$, then replacing B with zB as in Equation (4) yields $\delta' = 1$.

□

Remark 6. If $\mathcal{D}_e \cong \mathbb{C}$ then we can make δ equal to $+1$ or -1 as we wish, since multiplying B by an imaginary unit $\mathbf{i} \in \mathcal{D}_e$ changes δ to $-\delta$. Of course, we must simultaneously adjust φ_0 unless $\mathcal{D}_e \subseteq Z(\mathcal{D})$. In this latter case, we will always make the choice $\delta = 1$.

Equation (4) suggests the following equivalence relation:

Definition 7. Let φ_0 and φ_0' be degree-preserving involutions on a graded division algebra $\mathcal{D}$. We will write $\varphi_0 \sim \varphi_0'$ if $\varphi_0' = \mathrm{Int}(d)\varphi_0$ for some nonzero homogeneous $d \in \mathcal{D}$ such that $\varphi_0(d) \in \{\pm d\}$.

If B is φ_0-sesquilinear and satisfies $\overline{B} \in \{\pm B\}$ then, for any $\varphi_0' \sim \varphi_0$, the form B can be replaced by B' that is φ_0'-sesquilinear and also satisfies $\overline{B'} \in \{\pm B'\}$.

Notation 8. Fix B as in Lemma 5. Recall that in the case $\mathcal{D}_e \cong \mathbb{H}$, we take $X_t \in C_{\mathcal{D}}(\mathcal{D}_e)$. Then, as well as in the case $\mathcal{D}_e = \mathbb{R}$, we have $\varphi_0(X_t) = \eta(t)X_t$, where $\eta(t) \in \{\pm 1\}$ does not depend on the choice of X_t. In the case $\mathcal{D}_e \cong \mathbb{C}$, recall $K := \{t \in T \mid \mathrm{Int}(X_t)|_{\mathcal{D}_e} = \mathrm{id}_{\mathcal{D}_e}\}$. We have $\varphi_0(X_t) = X_t\eta(t)$, $\eta(t) \in \mathcal{D}_e$, and there are two possibilities:

- If $t \in T \setminus K$, then for all $z \in \mathcal{D}_e$, $\varphi_0(X_t z) = \overline{z}X_t\eta(t) = (X_t z)\eta(t)$. Hence $X_t = \varphi_0^2(X_t) = \varphi_0(X_t\eta(t)) = X_t\eta(t)^2$. So $\eta(t) \in \{\pm 1\}$, and $\eta(t)$ does not depend on the choice of X_t.
- If $t \in K$, then for all $z \in \mathcal{D}_e$, we have $\varphi_0(X_t z) = \overline{z}X_t\eta(t) = X_t\overline{z}\eta(t) = (X_t z)\eta(t)z^{-1}\overline{z}$. Then we can replace X_t by $X_t z$ so that $\eta(t) \in \mathbb{R}$ and, moreover, we can control the sign of $\eta(t)$. In fact $\eta(t) \in \{\pm 1\}$, because $X_t = \varphi_0^2(X_t) = X_t\eta(t)^2$. *We choose X_t so that $\eta(t) = \delta$ when $t \in K$.*

Thus, in all cases, we have

$$\varphi_0(X_t) = \eta(t)X_t \quad \text{and} \quad \eta(t) \in \{\pm 1\} \quad \text{for all} \quad t \in T.$$

Lemma 9. *For all $s, t \in T$, we have $\eta(st^2) = \eta(s)$.*

Proof. Consider the case $\mathcal{D}_e \cong \mathbb{C}$. First, note that $s \in K$ if and only if $st^2 \in K$. By our convention, if $s \in K$ then $\eta(st^2) = \delta = \eta(s)$. Suppose $s \notin K$. Then $\eta(st^2)$ does not depend on the choice of a nonzero element in the component $\mathcal{D}_{st^2}$, so we may use the element $X_tX_sX_t$, which gives $\varphi_0(X_tX_sX_t) = \varphi_0(X_t)\varphi_0(X_s)\varphi_0(X_t) = \eta(s)\eta(t)^2X_tX_sX_t = \eta(s)X_tX_sX_t$.

The same argument works in the cases $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$. □

Lemma 10. *$\varphi_0 \sim \varphi_0'$ if and only if $\varphi_0'\varphi_0^{-1}$ is an inner automorphism.*

Proof. The "only if" part is clear. For the "if" part, use Lemma 1 to find a nonzero homogeneous $d \in \mathcal{D}$ such that $\varphi_0' = \mathrm{Int}(d)\varphi_0$. Since $\varphi_0 \mathrm{Int}(d) = \mathrm{Int}(\varphi_0(d)^{-1})\varphi_0$ and both φ_0 and φ_0' are involutions, we obtain $\mathrm{Int}(d)\mathrm{Int}(\varphi_0(d)^{-1}) = \mathrm{id}_{\mathcal{D}}$, that is, $\varphi_0(d) = \lambda d$ for some $\lambda \in Z(\mathcal{D})$. Since we also have $\lambda \in \mathcal{D}_e$, we conclude that $\lambda \in \mathbb{R}$ except possibly in the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$. In this latter case, we can replace d by dz as in Notation 8, for a suitable $z \in \mathcal{D}_e$, so that $\lambda \in \mathbb{R}$, and this change does not affect $\mathrm{Int}(d)$. Since φ_0 is an involution, it follows that $\lambda \in \{\pm 1\}$. □

2.4. **More properties of the graded-division algebra.** The existence of the involution φ_0 as in Lemma 5 imposes restrictions on the graded division algebra $\mathcal{D}$.

Lemma 11. *The alternating bicharacter β takes values in the unit circle. Thus, in view of Lemma 4, β takes values in $\{\pm 1\}$ except possibly in the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$.*

Proof. Applying φ_0 to $X_s X_t = \beta(s,t) X_t X_s$, we get $X_t X_s = \overline{\beta(s,t)} X_s X_t$, so $|\beta(s,t)|^2 = \beta(s,t)\overline{\beta(s,t)} = 1$. □

Lemma 12. *If $\mathbb{C} \cong \mathcal{D}_e \nsubseteq Z(\mathcal{D})$, then X_t^2 is central for all $t \in T \setminus K$.*

Proof. Let $s \in K$, then $\beta(s,t^2)X_{t^2} = \mathrm{Int}(X_s)(X_{t^2}) = X_{t^2}\nu(t^2)$ as in Lemma 2. Because of Lemma 11, $\beta(s,t^2) \in \{\pm 1\}$. But $\nu(t^2) = |\nu(t)|^2 > 0$, therefore $t^2 \in \mathrm{rad}(\beta)$. So X_t^2 commutes with all X_s, $s \in K$. Of course, X_t^2 commutes with X_t, and with $\mathcal{D}_e$, so it is in the center. □

Lemma 13. *Every automorphism of the graded algebra $\mathcal{D}$ commutes with φ_0 except possibly in the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$.*

Proof. Recall that $\varphi_0(X_t) = \eta(t)X_t$, where η takes values in $\{\pm 1\}$. Let ψ_0 be an automorphism of the graded algebra $\mathcal{D}$. First, if $\psi_0|_{\mathcal{D}_e}$ is nontrivial, which can happen only if $\mathcal{D}_e \cong \mathbb{C}$ or $\mathbb{H}$, we may compose ψ_0 with $\mathrm{Int}(d)$ for a suitable nonzero homogeneous d so that $\mathrm{Int}(d)\psi_0|_{\mathcal{D}_e} = \mathrm{id}_{\mathcal{D}_e}$, namely, $d \in \mathcal{D}_e$ in the case $\mathcal{D}_e \cong \mathbb{H}$, and $d = X_t$, $t \in T \setminus K$, in the case $\mathcal{D}_e \cong \mathbb{C}$. Now, $\varphi_0(d)d$ is central: $\varphi_0(d)d = |d|^2$ in the case $\mathcal{D}_e \cong \mathbb{H}$, and $\varphi_0(d)d = \pm X_t^2$ in the case $\mathcal{D}_e \cong \mathbb{C}$, so we can use Lemma 12. Hence φ_0 commutes with $\mathrm{Int}(d)$.

Therefore, we may assume that $\psi_0|_{\mathcal{D}_e} = \mathrm{id}_{\mathcal{D}_e}$, and write ψ_0 as in Lemma 2, $\psi_0(X_t) = X_t\nu(t)$. As η takes values in $\{\pm 1\}$, and also $\nu(t) \in \mathbb{R}^\times$ for $t \in K$ (see Lemma 3), it is clear that $\varphi_0\psi_0 = \psi_0\varphi_0$ in the cases $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$ (where $K = T$). In the case $\mathcal{D}_e \cong \mathbb{C}$, for the same reasons, the restrictions of φ_0 and ψ_0 to $\mathcal{D}_K := \bigoplus_{s \in K} \mathcal{D}_s$ commute with each other. Finally, consider $t \in T \setminus K$ in the case $\mathcal{D}_e \cong \mathbb{C}$, then $\psi_0\varphi_0(X_t) = X_t\nu(t)\eta(t)$, and $\varphi_0\psi_0(X_t) = \varphi_0(X_t\nu(t)) = \overline{\nu(t)}X_t\eta(t) = X_t\nu(t)\eta(t)$. □

Lemma 14. *Suppose that $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$, that is, $\mathcal{D}$ can be given a complex structure (in two ways) that makes it a graded $\mathbb{C}$-algebra. Then the automorphisms of the graded algebra $\mathcal{D}$ are as follows:*

- *The $\mathbb{C}$-linear automorphisms ψ_0 (that is, those whose restriction to $\mathcal{D}_e$ is the identity) are given by $\psi_0(X_t) = X_t\nu(t)$, where $\nu \in \mathrm{Hom}(T, \mathbb{C}^\times)$.*
- *The $\mathbb{C}$-antilinear automorphisms ψ_0 (that is, those whose restriction to $\mathcal{D}_e$ is the conjugation) exist only if β takes values in $\{\pm 1\}$. They are given by $\psi_0(X_t) = X_t\nu(t)$, where the map $\nu : T \to \mathbb{C}^\times$ satisfies $\nu(st) = \nu(s)\nu(t)\beta(s,t)$ for all $s,t \in T$.*

In both cases, ψ_0 commutes with φ_0 if and only if ν takes values in $\mathbb{R}^\times$.

Proof. If ψ_0 is $\mathbb{C}$-linear then we already know from Lemma 2 that it has the form $\psi_0(X_t) = X_t\nu(t)$, where $\nu \in Z^1(T, \mathcal{D}_e^\times) = \mathrm{Hom}(T, \mathbb{C}^\times)$.

Assume that ψ_0 is $\mathbb{C}$-antilinear. Then $\psi_0 = \psi_0'\varphi_0$, where ψ_0' is a $\mathbb{C}$-linear anti-automorphism of the graded algebra $\mathcal{D}$. Clearly, ψ_0' is given by $\psi_0'(X_t) = X_t\nu(t)$, where $\nu(t) \in \mathcal{D}_e^\times = \mathbb{C}^\times$. Applying ψ_0' to Equation (2), we get $\psi_0'(X_t)\psi_0'(X_s) = \beta(s,t)\psi_0'(X_s)\psi_0'(X_t) = \beta(s,t)^2\psi_0'(X_t)\psi_0'(X_s)$, so β takes values in $\{\pm 1\}$. Recall that φ_0 restricts to the conjugation on $\mathcal{D}_e$ and $\varphi_0(X_t) = X_t$ for all $t \in T$ (since $T = K$ and we may assume $\delta = 1$ by Remark 6). Hence, we have $\psi_0(X_t) = X_t\nu(t)$.

Now let ψ_0' be a $\mathbb{C}$-linear map given by $\psi_0'(X_t) = X_t\nu(t)$, where $\nu(t) \in \mathbb{C}^\times$. Then, on the one hand, $\psi_0'(X_sX_t) = X_sX_t\nu(st)$ and, on the other hand, $\psi_0'(X_t)\psi_0'(X_s) = \beta(s,t)X_sX_t\nu(s)\nu(t)$ by Equation (2). Therefore, ψ_0' is an antiautomorphism of $\mathcal{D}$ if and only if $\nu(st) = \nu(s)\nu(t)\beta(s,t)$ for all $s,t \in T$.

Finally, whether ψ_0 is $\mathbb{C}$-linear or $\mathbb{C}$-antilinear, we have $\psi_0\varphi_0(X_t) = X_t\nu(t)$ and $\varphi_0\psi_0(X_t) = \overline{\nu(t)}X_t$. Therefore, $\varphi_0\psi_0 = \psi_0\varphi_0$ if and only if $\nu(t) = \overline{\nu(t)}$ for all $t \in T$. ☐

2.5. **The structure theorem.** Let $g_0 \in G$ be the degree of B. Define $(G/T)_{g_0} := \{gT \mid g_0g^2 \in T\}$ and $\tau : (G/T)_{g_0} \to T$ by $\tau(gT) = g_0g^2$ where g is in the fixed transversal for T in G. In other words, $\tau(x) = g_0\xi(x)^2$ whenever $g_0\xi(x)^2 \in T$.

Remark 15. If T is an elementary 2-group, then the definition of τ does not depend on the transversal.

Recall that the isomorphism class of $\mathcal{V}$ is determined by a function $\kappa : G/T \to \mathbb{Z}_{\geq 0}$ (a multiset) with a finite support $\{x_1, \ldots, x_s\}$. We will now see that the existence of B imposes restrictions on κ, and also determines another function, $\sigma : G/T \to \mathbb{Z}$.

Definition 16. We will say that a map $\kappa : G/T \to \mathbb{Z}_{\geq 0}$ is an *admissible multiplicity function* if it satisfies the following conditions:

(a) the support of κ is finite;
(b) $\kappa(g_0^{-1}x^{-1}) = \kappa(x)$ for all $x \in G/T$;
(c) for $\mathcal{D}_e = \mathbb{R}$:
$\kappa(x) \equiv 0 \pmod 2$ if $x \in (G/T)_{g_0}$ and $\eta(\tau(x)) = -\delta$;
(c′) for $\mathcal{D}_e \cong \mathbb{C}$:
$\kappa(x) \equiv 0 \pmod 2$ if $x \in (G/T)_{g_0}$, $\tau(x) \in T \setminus K$ and $\eta(\tau(x)) = -\delta$.

The set of all admissible multiplicity functions will be denoted by $\mathsf{K}(G, \mathcal{D}, \varphi_0, g_0, \delta)$. (Thanks to Lemma 9, it does not depend on our choice of transversal.) As before, we will write $k_i = \kappa(x_i)$ and $|\kappa| := \sum_{x \in G/T} \kappa(x) = k_1 + \cdots + k_s$.

Definition 17. For a given admissible multiplicity function κ, we will say that a map $\sigma : G/T \to \mathbb{Z}$ is a *signature function* if it satisfies the following conditions:

(i) $|\sigma(x)| \leq \kappa(x)$ for all $x \in G/T$;
(ii) for $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$:
$\sigma(x) = 0$ unless $x \in (G/T)_{g_0}$ and $\eta(\tau(x)) = \delta$;
$\sigma(x) \equiv \kappa(x) \pmod 2$ if $x \in (G/T)_{g_0}$ and $\eta(\tau(x)) = \delta$;
(ii′) for $\mathcal{D}_e \cong \mathbb{C}$:
$\sigma(x) = 0$ unless $x \in (G/T)_{g_0}$ and $\tau(x) \in K$;
$\sigma(x) \equiv \kappa(x) \pmod 2$ if $x \in (G/T)_{g_0}$ and $\tau(x) \in K$.

The set of all signature functions for a given κ will be denoted by $\Sigma(G, \mathcal{D}, \varphi_0, g_0, \kappa, \delta)$. (It does not depend on our choice of transversal.)

Recall the isotypic components $\mathcal{V}_i = \mathcal{V}_{g_i} \otimes_{\mathcal{D}_e} \mathcal{D}$, where $g_i = \xi(x_i)$, and observe that $B(\mathcal{V}_i, \mathcal{V}_j) = 0$ unless $g_0 g_i g_j \in T$. Hence B pairs $\mathcal{V}_i$ with itself if $x_i = g_i T \in (G/T)_{g_0}$ and with $\mathcal{V}_j$, $j \neq i$, otherwise. This shows that κ satisfies Property (b) of Definition 16. For i such that $x_i \in (G/T)_{g_0}$, we get a nondegenerate homogeneous φ_0-sesquilinear form $B|_{\mathcal{V}_i \times \mathcal{V}_i}$ whose values on $\mathcal{V}_{g_i} \times \mathcal{V}_{g_i}$ lie on $\mathcal{D}_{t_i}$, where $t_i := \tau(x_i) = g_0 g_i^2$. Define $B_i : \mathcal{V}_{g_i} \times \mathcal{V}_{g_i} \to \mathcal{D}_e$ by

$$B(v, w) = X_{t_i} B_i(v, w) \text{ for all } v, w \in \mathcal{V}_{g_i}. \tag{5}$$

Then B_i is a nondegenerate form on the $\mathcal{D}_e$-vector space $\mathcal{V}_{g_i}$ and it is $\mathrm{Int}(X_{t_i}^{-1})\varphi_0|_{\mathcal{D}_e}$-sesquilinear. Accordingly, we consider $\overline{B_i}(v, w) = \varphi_0(X_{t_i} B_i(w, v) X_{t_i}^{-1})$ for all $v, w \in \mathcal{V}_{g_i}$. Using Equation (4), we get $\overline{B_i} = \delta\varphi_0(X_{t_i}^{-1}) X_{t_i} B_i = \delta\eta(t_i) B_i$. Depending on the type of $\mathcal{D}_e$ we get the following:

- If $\mathcal{D}_e = \mathbb{R}$, then $B_i(w, v) = \delta\eta(t_i) B_i(v, w)$, so B_i is either symmetric or skew-symmetric.
- If $\mathcal{D}_e \cong \mathbb{H}$, then $\varphi_0(B_i(w, v)) = \delta\eta(t_i) B_i(v, w)$, so B_i is either hermitian or skew-hermitian.
- If $\mathcal{D}_e \cong \mathbb{C}$ and $t_i \in T \setminus K$, then $B_i(w, v) = \delta\eta(t_i) B_i(v, w)$, so B_i is either symmetric or skew-symmetric.
- If $\mathcal{D}_e \cong \mathbb{C}$ and $t_i \in K$, then $\varphi_0(B_i(w, v)) = \delta\eta(t_i) B_i(v, w) = B_i(v, w)$, so B_i is hermitian.

Since nondegenerate skew-symmetric forms (over $\mathbb{R}$ or $\mathbb{C}$) exist only in even dimension, κ satisfies conditions (c)-(c$'$) of Definition 16. Also, the form B_i is hermitian (over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) and therefore has inertia (p_i, q_i) in the following cases:

- for $\mathcal{D}_e = \mathbb{R}$ or $\mathcal{D}_e \cong \mathbb{H}$, when $\eta(t_i) = \delta$;
- for $\mathcal{D}_e \cong \mathbb{C}$, when $t_i \in K$.

So we can define a map $\sigma : G/T \to \mathbb{Z}$ as $\sigma(x_i) = p_i - q_i$ if B_i has inertia, and $\sigma(x) = 0$ for all other $x \in G/T$. Since $p_i + q_i = k_i = \kappa(x_i)$, this map σ satisfies the conditions of Definition 17, that is, it is a signature function.

Recall that, if we choose a homogeneous $\mathcal{D}$-basis of $\mathcal{V}$, then the elements of $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(\mathcal{V})$ can be identified with matrices in $M_k(\mathcal{D})$, where $k = |\kappa|$. Also, Equation (3) becomes

$$\varphi(X) = \Phi^{-1} \varphi_0(X^T) \Phi \tag{6}$$

for all $X \in M_k(\mathcal{D})$, where Φ is the matrix in $M_k(\mathcal{D})$ representing B with respect to the chosen basis, and φ_0 acts entrywise.

We can relabel $(g_1, \dots, g_s)$ so that the first m entries satisfy $g_0 g_i^2 \in T$ and

$$g_{m+1} g_{m+2} \equiv \dots \equiv g_{m+2r-1} g_{m+2r} \equiv g_0^{-1} \pmod{T}$$

where $m + 2r = s$. Write, as in Equation (5), $B(v, w) = X_{t_{m+j}} B_{m+j}(v, w)$, for all $v \in \mathcal{V}_{g_{m+2j-1}}$ and $w \in \mathcal{V}_{g_{m+2j}}$, where $t_{m+j} := g_0 g_{m+2j-1} g_{m+2j}$. We can repeat the same argument to get $\overline{B_{m+j}} = \delta\eta(t_{m+j}) B_{m+j}$. Choosing $\mathcal{D}_e$-bases in $\mathcal{V}_{g_1}, \dots, \mathcal{V}_{g_m}$ (which are then $\mathcal{D}$-bases in $\mathcal{V}_1, \dots, \mathcal{V}_m$) to bring B_i to canonical form, and choosing $\mathcal{D}_e$-bases in $\mathcal{V}_{g_{m+2j-1}}$ and $\mathcal{V}_{g_{m+2j}}$ that are dual with respect to B_{m+j}, we obtain:

$$\Phi = X_{t_1} S_1 \oplus \dots \oplus X_{t_m} S_m \oplus X_{t_{m+1}} S_{m+1} \oplus \dots \oplus X_{t_{m+r}} S_{m+r} \tag{7}$$

where the matrices S_i are as follows. For $i = m+j$, $j = 1, \dots, r$, we have $k_{m+2j-1} = k_{m+2j}$ and

$$S_i = \begin{pmatrix} 0 & I_{k_{m+2j}} \\ \delta\eta(t_{m+j}) I_{k_{m+2j}} & 0 \end{pmatrix},$$

whereas for $i = 1, \ldots, m$ we have the following cases:

- For $\mathcal{D}_e = \mathbb{R}$ or $\mathcal{D}_e \cong \mathbb{H}$ with $\eta(t_i) = \delta$, and for $\mathcal{D}_e \cong \mathbb{C}$ with $t_i \in K$: $S_i = I_{p_i,q_i}$ (the diagonal matrix with ± 1 on the main diagonal: the first p_i entries are 1 and the remaining q_i entries are -1).
- For $\mathcal{D}_e = \mathbb{R}$ with $\eta(t_i) = -\delta$, and for $\mathcal{D}_e \cong \mathbb{C}$ with $t_i \in T \backslash K$ and $\eta(t_i) = -\delta$:
$$S_i = \begin{pmatrix} 0 & I_{k_{i/2}} \\ -I_{k_{i/2}} & 0 \end{pmatrix}.$$
- For $\mathcal{D}_e \cong \mathbb{C}$ with $t_i \in T \setminus K$ and $\eta(t_i) = \delta$: $S_i = I_{k_i}$.
- For $\mathcal{D}_e \cong \mathbb{H}$ with $\eta(t_i) = -\delta$: $S_i = \mathbf{i} I_{k_i}$ ($\mathbf{i}$ is a fixed imaginary unit in $\mathcal{D}_e$).

To summarize:

Theorem 18. *Let $\mathcal{R}$ be a G-graded real associative algebra that is graded-simple and satisfies the descending chain condition on graded left ideals, and such that the identity component $\mathcal{R}_e$ has finite dimension. Let φ be an involution on the graded algebra $\mathcal{R}$ such that $\mathcal{R}$ is central as a graded algebra with involution. Then there exists a real graded division algebra $\mathcal{D}$ with $\mathcal{D}_e$ isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and a degree-preserving involution φ_0 on $\mathcal{D}$ restricting to the conjugation on $\mathcal{D}_e$ such that, as a graded algebra with involution, $\mathcal{R}$ is isomorphic to $M_k(\mathcal{D})$, where the grading is given by Equation* (1), *and the involution is given by Equations* (6) *and* (7). $\square$

Remark 19. It may be convenient to choose the transversal for T in G so that, for all $x' \neq x''$ in G/T satisfying $g_0 x' x'' = T$, we have $g_0 \xi(x') \xi(x'') = e$. Then, in Equation (7), each of the $t_{m+1}, \ldots, t_{m+r}$ is just e. However, this choice of transversal depends on g_0.

Suppose we are given $(\mathcal{D}, \varphi_0)$ as in Theorem 18 and $\delta \in \{\pm 1\}$. Fix a transversal for T in G and, for each $t \in T$, an element $X_t \in \mathcal{D}_t$ as described in Notation 8. Then the data that determine $(\mathcal{R}, \varphi)$ are (g_0, κ, σ). Conversely, let $g_0 \in G$, $\kappa \in \mathsf{K}(G, \mathcal{D}, \varphi_0, g_0, \delta)$ and $\sigma \in \Sigma(G, \mathcal{D}, \varphi_0, g_0, \kappa, \delta)$. Then we know that, by means of Equation (1), κ makes $\mathcal{R} := M_k(\mathcal{D})$, with $k = |\kappa|$, a graded-simple algebra that satisfies the descending chain condition on graded left ideals and such that $\mathcal{R}_e$ has finite dimension. We can construct $\Phi \in M_k(\mathcal{D})$ as in Equation (7), with $p_i = \frac{1}{2}(\kappa(x_i) + \sigma(x_i))$ and $q_i = \frac{1}{2}(\kappa(x_i) - \sigma(x_i))$ where appropriate, and define $\varphi : M_k(\mathcal{D}) \to M_k(\mathcal{D})$ by Equation (6), which is equivalent to Equation (3) for the φ_0-sesquilinear form B represented by Φ. Since $\varphi_0(\Phi^T) = \delta\Phi$ (equivalently, $\overline{B} = \delta B$), φ is an involution. Since B is homogeneous, $\varphi(\mathcal{R}_g) = \mathcal{R}_g$. Finally, $M_k(\mathcal{D})$ is central as a graded algebra with involution because $Z(\mathcal{R})_e = Z(\mathcal{D})_e$ and φ_0 restricts to the conjugation on $\mathcal{D}_e$.

Definition 20. We will denote the graded algebra with involution $(M_k(\mathcal{D}), \varphi)$ constructed above by $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, \delta)$.

3. Classification of associative algebras with involution up to isomorphism

Let $(\mathcal{R}, \varphi)$ and $(\mathcal{R}', \varphi')$ be graded algebras with involution as in Theorem 18, which are isomorphic to $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, \delta)$ and $M(\mathcal{D}', \varphi'_0, g'_0, \kappa', \sigma', \delta')$, respectively, as in Definition 20. The purpose of this section is to determine conditions under which these two objects are isomorphic to each other.

3.1. **Preliminary remarks.** Recall that, by [11, Theorem 2.10], an isomorphism $\psi : \mathcal{R} \to \mathcal{R}'$ between the G-graded algebras $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(\mathcal{V})$ and $\mathcal{R}' = \mathrm{End}_{\mathcal{D}'}(\mathcal{V}')$ (for now, without regard to involutions) corresponds to a pair (ψ_0, ψ_1) where $\psi_0 : \mathcal{D} \to \mathcal{D}'$ is an isomorphism of G-graded algebras and $\psi_1 : \mathcal{V}^{[g]} \to \mathcal{V}'$, for some $g \in G$, is an isomorphism of G-graded spaces (in other words, an invertible linear map $\mathcal{V} \to \mathcal{V}'$ of degree g) such that $\psi_1(vd) = \psi_1(v)\psi_0(d)$ for all $v \in \mathcal{V}$ and $d \in \mathcal{D}$. The correspondence is given by $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in \mathcal{R}$ and $v \in \mathcal{V}$. Moreover, (ψ_0, ψ_1) is unique up to replacing ψ_0 by $\psi_0' = \mathrm{Int}(d^{-1})\psi_0$, where d is a nonzero homogeneous element in $\mathcal{D}'$, and simultaneously replacing ψ_1 by ψ_1', where $\psi_1'(v) = \psi_1(v)d$ for all $v \in \mathcal{V}$.

Since it is necessary that $\mathcal{D} \cong \mathcal{D}'$, we will suppose that $\mathcal{D} = \mathcal{D}'$ and thus ψ_0 is an automorphism of the graded algebra $\mathcal{D}$, while ψ_1 is ψ_0-semilinear as a map between $\mathcal{D}$-modules. We will denote the group of automorphisms of the graded algebra $\mathcal{D}$ by $\mathrm{Aut}^G(\mathcal{D})$. Also, we will write $\mathcal{D}_{\mathrm{gr}}^{\times}$ for the group of nonzero homogeneous elements of $\mathcal{D}$. Note that if ψ_0 is inner then, in view of Lemma 1, we have $\psi_0 = \mathrm{Int}(d)$ for some $d \in \mathcal{D}_{\mathrm{gr}}^{\times}$, so we can adjust the pair (ψ_0, ψ_1) so that $\psi_0 = \mathrm{id}_{\mathcal{D}}$ and still obtain the same isomorphism ψ.

In the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$ (in other words, $\mathcal{D}_e \cong \mathbb{C}$ and $K = T$), we will restrict ourselves to the following situation: we will assume that $\mathcal{D}_e = Z(\mathcal{D})$ and T is finite. This simplifies the arguments and will be sufficient for our applications to classical Lie algebras (but see Subsection 3.4 for the general situation). Since here $\mathcal{D}$ is a twisted group algebra of T, we see by a generalization of Maschke's Theorem (e.g. [16, Corollary 10.2.5]) that our assumption is tantamount to $\mathcal{D}$ (equivalently, $\mathcal{R}$) being a finite-dimensional central simple algebra over $\mathbb{C}$ (which is graded as a $\mathbb{C}$-algebra). We will refer to this as the $\mathbb{C}$-*central case*. Note that in this case the alternating bicharacter β must be nondegenerate (see Lemma 4).

The next step is to take into account the involutions. By [11, Lemma 3.32], if we want ψ to be an isomorphism of graded algebras with involution, that is, $\varphi' = \psi\varphi\psi^{-1}$, it is necessary and sufficient that there exist $d_0 \in \mathcal{D}_{\mathrm{gr}}^{\times}$ such that

$$B'(\psi_1(v), \psi_1(w)) = \psi_0(d_0 B(v, w)) \tag{8}$$

for all $v, w \in \mathcal{V}$. Automatically, $\mathrm{Int}(d_0)\varphi_0 = \psi_0^{-1}\varphi_0'\psi_0$. Note that, since both φ_0 and φ_0' restrict to the conjugation on $\mathcal{D}_e$, we have $d_0 \in C_{\mathcal{D}}(\mathcal{D}_e)$ and hence $\deg d_0 \in K$. Also, it follows from Equations (4) and (8) that $\delta' = \delta\varphi_0(d_0)d_0^{-1}$ and, in particular, $\varphi_0(d_0) \in \{\pm d_0\}$. Indeed, consider the form $B'' : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$ given by $B''(v, w) = \psi_0^{-1}(B'(\psi_1(v), \psi_1(w)))$. One checks that B'' is $(\psi_0^{-1}\varphi_0'\psi_0)$-sesquilinear and satisfies $\overline{B''} = \delta' B''$. Now Equation (8) reads $B'' = d_0 B$, so we may apply Equation (4) (with d_0 in place of d).

If we are not in the $\mathbb{C}$-central case, then ψ_0 commutes with φ_0 and φ_0' by Lemma 13, so the relation between φ_0 and φ_0' simplifies: $\mathrm{Int}(d_0)\varphi_0 = \varphi_0'$. Moreover, adjusting the pair (ψ_0, ψ_1), we may assume without loss of generality that $\psi_0|_{\mathcal{D}_e} = \mathrm{id}_{\mathcal{D}_e}$ (see the proof of Lemma 13). Then, by Lemma 2, we have $\psi_0(X_t) = X_t\nu(t)$, where $\nu \in Z^1(T, Z(\mathcal{D}_e)^{\times})$. We define for future reference:

$$A := \{\psi_0 \in \mathrm{Aut}^G(\mathcal{D}) \mid \psi_0|_{\mathcal{D}_e} = \mathrm{id}_{\mathcal{D}_e}\} \cong Z^1(T, Z(\mathcal{D}_e)^{\times}). \tag{9}$$

If we are in the $\mathbb{C}$-central case, then ψ_0 does not always commute with the involutions — see Lemma 14. However, if ψ_0 is $\mathbb{C}$-linear then, by Skolem–Noether Theorem, it is inner and hence can be eliminated. The existence of $\mathbb{C}$-antilinear ψ_0 forces β to take values in $\{\pm 1\}$. Since β is nondegenerate, this implies that

T is an elementary 2-group (compare with [11, Lemma 2.50]). Conversely, if T is an elementary 2-group then, by [11, Proposition 2.51], there exists $\nu : T \to \{\pm 1\}$ satisfying $\nu(st) = \nu(s)\nu(t)\beta(s,t)$, so we obtain a $\mathbb{C}$-antilinear automorphism ψ_0 sending $X_t \mapsto X_t\nu(t)$, which generates the group $\mathrm{Aut}^G(\mathcal{D})$ modulo the inner automorphisms. Fix such ν and the corresponding ψ_0. (Note that, if we regard T as a vector space over the field of two elements, then ν is a quadratic form on T with polar form β.) Since $\nu(t) \in \mathbb{R}^\times$ for all $t \in T$, ψ_0 commutes with φ_0. It also commutes with $\mathrm{Int}(d)$ for all $d \in \mathcal{D}^\times_{\mathrm{gr}}$, so in this case, too, the relation between φ_0 and φ_0' simplifies: $\mathrm{Int}(d_0)\varphi_0 = \varphi_0'$. We define for future reference:

$$A := \begin{cases} \langle \psi_0 \rangle \cong \mathbb{Z}_2 & \text{if } T \text{ is an elementary 2-group;} \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

We have seen that in all cases $\mathrm{Int}(d_0)\varphi_0 = \varphi_0'$. Thus, for $(\mathcal{R}, \varphi)$ and $(\mathcal{R}', \varphi')$ to be isomorphic, it is necessary that $\varphi_0 \sim \varphi_0'$ in the sense of Definition 7. Fixing a representative for each equivalence class, we may suppose that $\varphi_0 = \varphi_0'$, so both B and B' are φ_0-sesquilinear forms, and they are related by Equation (8) where $d_0 \in Z(\mathcal{D})$.

There are two possibilities:

- If $\varphi_0|_{Z(\mathcal{D})} = \mathrm{id}_{Z(\mathcal{D})}$, then necessarily $\varphi_0(d_0) = d_0$ and hence $\delta' = \delta$ whenever (φ_0, B) and (φ_0, B') yield isomorphic $(\mathcal{R}, \varphi)$ and $(\mathcal{R}', \varphi')$. Thus, $\delta \in \{\pm 1\}$ is an invariant in this case, and it suffices to consider φ_0-sesquilinear forms with a fixed δ.
- If $\varphi_0|_{Z(\mathcal{D})} \neq \mathrm{id}_{Z(\mathcal{D})}$, then there exists a nonzero homogeneous $d_0 \in Z(\mathcal{D})$ such that $\varphi_0(d_0) = -d_0$, and hence we may adjust (φ_0, B) to make $\delta = 1$ or $\delta = -1$ as we wish. *We choose $\delta = 1$ and thus consider only hermitian φ_0-sesquilinear forms over $\mathcal{D}$ in this case.*

In either case, we will have $\varphi_0(d_0) = d_0$. We define for future reference:

$$C := \{c \in \mathcal{D}^\times_{\mathrm{gr}} \mid c \in Z(\mathcal{D}) \text{ and } \varphi_0(c) = c\}. \tag{11}$$

To summarize: we may suppose without loss of generality that $\mathcal{D} = \mathcal{D}'$, $\varphi_0 = \varphi_0'$ and $\delta = \delta'$. Under this assumption, we will obtain conditions on (g_0, κ, σ) and (g_0', κ', σ') that are necessary and sufficient for $(\mathcal{R}, \varphi)$ and $(\mathcal{R}', \varphi')$ to be isomorphic.

3.2. **Extended signature functions.** Recall that the signature function $\sigma \in \Sigma(G, \mathcal{D}, \varphi_0, g_0, \kappa, \delta)$ corresponding to a nondegenerate φ_0-sesquilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$, with $\deg B = g_0$ and $\overline{B} = \delta B$, assigns to each $x \in G/T$ the signature of the form $X^{-1}_{\tau(x)} B : \mathcal{V}_{\xi(x)} \times \mathcal{V}_{\xi(x)} \to \mathcal{D}_e$ if this signature is defined (that is, if $X^{-1}_{\tau(x)} B|_{\mathcal{V}_{\xi(x)} \times \mathcal{V}_{\xi(x)}}$ is a nondegenerate hermitian form over $\mathcal{D}_e$) and zero otherwise. Here $\xi : G/T \to G$ is the section given by our fixed transversal and $\tau(x) = g_0 \xi(x)^2$. It is convenient to define $\tilde{\sigma} : G \to \mathbb{Z}$ by

$$\tilde{\sigma}(h) := \begin{cases} \mathrm{signature}(X^{-1}_{g_0h^2} B|_{\mathcal{V}_h \times \mathcal{V}_h}) & \text{if } g_0h^2 \in K \text{ and } \eta(g_0h^2) = \delta; \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

(Recall that if $\mathcal{D}_e = \mathbb{R}$ or $\mathcal{D}_e \cong \mathbb{H}$ then $K = T$, and if $\mathcal{D}_e \cong \mathbb{C}$ then $\eta(t) = \delta$ for all $t \in K$.) The relation between $\sigma : G/T \to \mathbb{Z}$ and $\tilde{\sigma} : G \to \mathbb{Z}$ is simply $\sigma = \tilde{\sigma}\xi$.

Proposition 21. *For all $h \in G$ such that $t := g_0h^2 \in K$ and for all $u \in T$,*

$$\tilde{\sigma}(hu) = \eta(u)\mathrm{sign}\big(X^{-1}_{tu^2} X_u X_t X_u\big)\tilde{\sigma}(h) \tag{13}$$

Proof. First, observe that $X_{tu^2}^{-1} X_u X_t X_u \in \mathbb{R}^\times$, so the sign above makes sense. Indeed, this element belongs to $\mathcal{D}_e \cap C_{\mathcal{D}}(\mathcal{D}_e) = Z(\mathcal{D}_e)$, so the result is clear in the cases $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$. In the case $\mathcal{D}_e \cong \mathbb{C}$, it suffices to show that $\varphi_0(X_{tu^2}^{-1} X_u X_t X_u) = X_{tu^2}^{-1} X_u X_t X_u$. We compute:

$$\begin{aligned}\varphi_0(X_{tu^2}^{-1} X_u X_t X_u) &= \varphi_0(X_u)\varphi_0(X_t)\varphi_0(X_u)\varphi_0(X_{tu^2})^{-1}\\ &= \eta(u)\eta(t)\eta(u)\eta(tu^2)^{-1}(X_u X_t X_u) X_{tu^2}^{-1}\\ &= X_{tu^2}^{-1}(X_u X_t X_u),\end{aligned}$$

where we have used that η takes values ± 1 and $\eta(t) = \eta(tu^2) = \delta$ because t and tu^2 are in K; we have also used that $X_u X_t X_u$ and X_{tu^2} commute because they are in the same component $\mathcal{D}_{tu^2}$.

Now, we have to compare $\mathcal{D}_e$-valued forms $X_t^{-1} B|_{\mathcal{V}_h \times \mathcal{V}_h}$ and $X_{tu^2}^{-1} B|_{\mathcal{V}_{hu} \times \mathcal{V}_{hu}}$. First of all, they either both have signature or both do not. In the case $\mathcal{D}_e \cong \mathbb{C}$, t and tu^2 are in K. In the cases $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$, we have to check that $\eta(t) = \eta(tu^2)$. Indeed, $X_t X_u^2 = \lambda X_{tu^2}$ for some $\lambda \in \mathbb{R}$, hence $\varphi_0(X_u)^2 \varphi_0(X_t) = \lambda \varphi_0(X_{tu^2})$, so $\eta(t) X_u^2 X_t = \lambda \eta(tu^2) X_{tu^2} = \eta(tu^2) X_t X_u^2 = \eta(tu^2) X_u^2 X_t$, where we have used the fact that β takes values ± 1 (Lemma 11).

Finally, suppose that the signatures are defined. We compute, for all $v, w \in \mathcal{V}_h$:

$$\begin{aligned}X_{tu^2}^{-1} B(vX_u, wX_u) &= X_{tu^2}^{-1} \varphi_0(X_u) B(v,w) X_u\\ &= \eta(u) X_{tu^2}^{-1} X_u X_t (X_t^{-1} B(v,w)) X_u;\end{aligned}$$

the last expression equals $\eta(u) X_{tu^2}^{-1} X_u X_t X_u \overline{(X_t^{-1} B(v,w))}$ if $\mathcal{D}_e \cong \mathbb{C}$ and $u \in T \setminus K$ and $\eta(u) X_{tu^2}^{-1} X_u X_t X_u (X_t^{-1} B(v,w))$ otherwise. The result follows because the map $v \mapsto vX_u$ is a $\mathcal{D}_e$-linear isomorphism from $\mathcal{V}_h$ to $\mathcal{V}_{hu}$, respectively from $\overline{\mathcal{V}_h}$ to $\mathcal{V}_{hu}$ if $\mathcal{D}_e \cong \mathbb{C}$ and $u \in T \setminus K$. (Here we use the bar to denote the conjugate complex vector space: it has the same addition of vectors, but the scalar multiplication is twisted by complex conjugation.) □

Definition 22. For a given admissible multiplicity function κ, we will say that a map $\tilde{\sigma} : G \to \mathbb{Z}$ is an *extended signature function* if it satisfies Equation (13) and the following conditions:

(i) $|\tilde{\sigma}(h)| \leq \kappa(hT)$ for all $h \in G$;
(ii) for $\mathcal{D}_e = \mathbb{R}$ and $\mathcal{D}_e \cong \mathbb{H}$:
$\tilde{\sigma}(h) = 0$ unless $g_0 h^2 \in T$ and $\eta(g_0 h^2) = \delta$;
$\tilde{\sigma}(h) \equiv \kappa(hT) \pmod 2$ if $g_0 h^2 \in T$ and $\eta(g_0 h^2) = \delta$;
(ii′) for $\mathcal{D}_e \cong \mathbb{C}$:
$\tilde{\sigma}(h) = 0$ unless $g_0 h^2 \in K$;
$\tilde{\sigma}(h) \equiv \kappa(hT) \pmod 2$ if $g_0 h^2 \in K$.

The set of all extended signature functions will be denoted by $\tilde{\Sigma}(G, \mathcal{D}, \varphi_0, g_0, \kappa, \delta)$.

From Definition 17 and Proposition 21, it is clear that $\tilde{\sigma}$ determined by B via Equation (12) is an extended signature function. Conversely, any $\tilde{\sigma} \in \tilde{\Sigma}(G, \mathcal{D}, \varphi_0, g_0, \kappa, \delta)$ is uniquely determined by the signature function $\sigma := \tilde{\sigma}\xi$, so $\tilde{\sigma}$ comes from some form B.

3.3. **The isomorphism theorem.** Let ψ_0 be an automorphism of $(\mathcal{D}, \varphi_0)$ as a graded algebra with involution and let $\psi_1 : \mathcal{V}^{[g]} \to \mathcal{V}'$ be a ψ_0-semilinear isomorphism of G-graded vector spaces. The isomorphism ψ_1 becomes linear over $\mathcal{D}$ if we regard it as a map from $(\mathcal{V}^{[g]})^{\psi_0^{-1}}$ to $\mathcal{V}'$, where the superscript ψ_0^{-1} refers to the twisted $\mathcal{D}$-module structure: for all $v \in \mathcal{V}$ and $d \in \mathcal{D}$, $v \cdot d := v\psi_0^{-1}(d)$. Consequently, the map $(v, w) \mapsto B'(\psi_1(v), \psi_1(w))$, which appears in the left-hand side of Equation (8), is a φ_0-sesquilinear form on $(\mathcal{V}^{[g]})^{\psi_0^{-1}}$ that has the same parameters $(g_0', \kappa', \tilde{\sigma}')$ as the form B' on $\mathcal{V}'$. Let us compute the parameters $(\underline{g_0}, \underline{\kappa}, \underline{\tilde{\sigma}})$ of the right-hand side,

$$\underline{B}(v, w) := \psi_0(d_0 B(v, w)) = \psi_0(d_0)\psi_0(B(v, w)),$$

regarded as a φ_0-sesquilinear form on $(\mathcal{V}^{[g]})^{\psi_0^{-1}}$, in terms of the parameters $(g_0, \kappa, \tilde{\sigma})$ of B on $\mathcal{V}$.

Since $\mathcal{V}_{hg}^{[g]} = \mathcal{V}_h$ for all $h \in G$, we have:

$$\underline{g_0} = \deg \underline{B} = g^{-2} g_0 t_0 \tag{14}$$

where $t_0 := \deg d_0 \in K$. Next, for all $h \in G$, we have $\kappa(hT) = \dim_{\mathcal{D}_e} \mathcal{V}_h$, so

$$\underline{\kappa}(hT) = \dim_{\mathcal{D}_e} \mathcal{V}_h^{[g]} = \dim_{\mathcal{D}_e} \mathcal{V}_{g^{-1}h} = \kappa(g^{-1}hT). \tag{15}$$

Finally, we have to compute the signature of the $\mathcal{D}_e$-valued form

$$X_{\underline{g_0}h^2}^{-1} \underline{B}|_{\mathcal{V}_h^{[g]} \times \mathcal{V}_h^{[g]}} = X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(X_t)\psi_0\big(X_t^{-1} B|_{\mathcal{V}_{g^{-1}h} \times \mathcal{V}_{g^{-1}h}}\big)$$

where $t := g^{-2} g_0 h^2 \in K$ (which is equivalent to $t_0 t = \underline{g_0} h^2$ being in K). Using the fact that the inertia of a $\mathcal{D}_e$-valued form is not affected by an automorphism of $\mathcal{D}_e$ (as an $\mathbb{R}$-algebra), we get:

$$\underline{\tilde{\sigma}}(h) = \operatorname{sign}\big(X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(X_t)\big)\tilde{\sigma}(g^{-1}h). \tag{16}$$

Note that $X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(X_t) \in \mathbb{R}^\times$, so the sign above makes sense. Indeed, as $d_0 \in Z(\mathcal{D})$, we have $X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(X_t) \in \mathcal{D}_e \cap C_{\mathcal{D}}(\mathcal{D}_e) = Z(\mathcal{D}_e)$, so the only case that needs attention is $\mathcal{D}_e \cong \mathbb{C}$. Then, as the elements $X_{t_0 t}$, $\psi_0(d_0)$ and $\psi_0(X_t)$ commute with each other, and $\varphi_0\psi_0 = \psi_0\varphi_0$, we have:

$$\begin{aligned}\varphi_0(X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(X_t)) &= \varphi_0(X_{t_0 t}^{-1})\varphi_0(\psi_0(d_0))\varphi_0(\psi_0(X_t)) = \delta X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(\delta X_t) \\ &= X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(X_t),\end{aligned}$$

proving that $X_{t_0 t}^{-1} \psi_0(d_0)\psi_0(X_t) \in \mathbb{R}$.

We can express Equations (14), (15) and (16) in terms of group action. Denote by $\operatorname{Aut}^G(\mathcal{D}, \varphi_0)$ the group of automorphisms of $(\mathcal{D}, \varphi_0)$ as a graded algebra with involution, and recall the subgroup $C \subseteq Z(\mathcal{D})^\times$ defined by Equation (11). Then $\operatorname{Aut}^G(\mathcal{D}, \varphi_0)$ acts naturally on C, so we can form their semidirect product $C \rtimes \operatorname{Aut}^G(\mathcal{D}, \varphi_0)$. Consider the collection of pairs $(\mathcal{V}, B)$ where $\mathcal{V}$ is a graded right $\mathcal{D}$-module of finite dimension over $\mathcal{D}$ and B is a nondegenerate homogeneous φ_0-sesquilinear form on $\mathcal{V}$ satisfying $\overline{B} = \delta B$. The groups $\operatorname{Aut}^G(\mathcal{D}, \varphi_0)$ and C act on this collection as follows: for $\psi_0 \in \operatorname{Aut}^G(\mathcal{D}, \varphi_0)$, define $\psi_0 \cdot (\mathcal{V}, B) := (\mathcal{V}^{\psi_0^{-1}}, \psi_0 B)$ and, for $c \in C$, define $c \cdot (\mathcal{V}, B) := (\mathcal{V}, cB)$. Since $\psi_0(cB) = \psi_0(c)\psi_0 B$, this gives rise to an action of $C \rtimes \operatorname{Aut}^G(\mathcal{D}, \varphi_0)$. Finally, G acts by shift of grading: $g \cdot (\mathcal{V}, B) := (\mathcal{V}^{[g]}, B)$, and this action commutes with that of $C \rtimes \operatorname{Aut}^G(\mathcal{D}, \varphi_0)$. Then $((\mathcal{V}^{[g]})^{\psi_0^{-1}}, \underline{B})$ is obtained from $(\mathcal{V}, B)$ by the action of the element $(g, \psi_0(d_0), \psi_0) \in G \times (C \rtimes \operatorname{Aut}^G(\mathcal{D}, \varphi_0))$.

Recall that if d is a nonzero homogeneous element of $\mathcal{D}$, $\psi_0' = \operatorname{Int}(d^{-1})\psi_0$ and $\psi_1'(v) = \psi_1(v)d$ for all $v \in \mathcal{V}$, then (ψ_0', ψ_1') and (ψ_0, ψ_1) yield the same isomorphism $\psi : \mathcal{R} \to \mathcal{R}'$. In particular, it is sufficient to consider the subgroup $A \subseteq \operatorname{Aut}^G(\mathcal{D}, \varphi_0)$

defined by Equation (9) or (10), depending on whether or not we are in the $\mathbb{C}$-central case.

Remark 23. We can say more: let $\mathrm{Int}^G(\mathcal{D},\varphi_0)$ be the group of inner automorphisms of $(\mathcal{D},\varphi_0)$ as a graded algebra with involution, so, by Lemma 1, $\mathrm{Int}^G(\mathcal{D},\varphi_0) = \{\mathrm{Int}(d) \mid d \in \mathcal{D}_{\mathrm{gr}}^\times \text{ such that } \varphi_0(d)d \in Z(\mathcal{D})\}$. Then, $\mathrm{Int}^G(\mathcal{D},\varphi_0)$ acts trivially on C, and the $G \times (C \times \mathrm{Int}^G(\mathcal{D},\varphi_0))$-orbits in the set of isomorphism classes of pairs $(\mathcal{V},B)$ are the same as the $G \times C$-orbits.

Equations (14), (15) and (16) show the effect of the above actions on the parameters $(g_0,\kappa,\tilde\sigma) \in \tilde{\mathsf{X}}(G,\mathcal{D},\varphi_0,\delta)$ of $(\mathcal{V},B)$, where

$$\tilde{\mathsf{X}}(G,\mathcal{D},\varphi_0,\delta) := \coprod_{g_0\in G}\coprod_{\kappa\in\mathsf{K}(g_0)} \tilde\Sigma(g_0,\kappa) \subseteq G \times \mathbb{Z}_{\geq 0}^{G/T} \times \mathbb{Z}^G,$$

$\mathsf{K}(g_0) := \mathsf{K}(G,\mathcal{D},\varphi_0,g_0,\delta)$ and $\tilde\Sigma(g_0,\kappa) := \tilde\Sigma(G,\mathcal{D},\varphi_0,g_0,\kappa,\delta)$. Recall that if we are not in the $\mathbb{C}$-central case then $\psi_0 \in A$ is given by $\psi_0(X_t d) = X_t\nu(t)d$ for all $t \in T$ and $d \in \mathcal{D}_e$, whereas if we are in the $\mathbb{C}$-central case then either $\psi_0 = \mathrm{id}_{\mathcal{D}}$ or $\psi_0(X_t d) = X_t\nu(t)\bar d$ for all $t \in T$ and $d \in \mathcal{D}_e = \mathbb{C}$, and also $C = \mathbb{R}^\times$. Hence, A acts on the group C by $\psi_0(c) = c\nu(\deg c)$ and on the set $\tilde{\mathsf{X}}(G,\mathcal{D},\varphi_0,\delta)$ through its action on each "fiber" $\tilde\Sigma(g_0,\kappa)$:

$$\psi_0 \cdot (g_0,\kappa,\tilde\sigma) = (g_0,\kappa,\underline{\tilde\sigma}) \text{ where } \underline{\tilde\sigma}(h) = \mathrm{sign}\big(\nu(g_0h^2)\big)\tilde\sigma(h).$$

(The factor $\mathrm{sign}(\nu(g_0h^2))$ is defined for $g_0h^2 \in K$, but $\tilde\sigma(h) = 0$ for $g_0h^2 \notin K$, so the above formula makes sense.) Also, C acts on the set $\tilde{\mathsf{X}}(G,\mathcal{D},\varphi_0,\delta)$ as follows:

$$c \cdot (g_0,\kappa,\tilde\sigma) = (g_0(\deg c),\kappa,\underline{\tilde\sigma}) \text{ where } \underline{\tilde\sigma}(h) = \mathrm{sign}\big(X_{g_0h^2(\deg c)}^{-1}X_{g_0h^2}c\big)\tilde\sigma(h).$$

(Again, the formula makes sense because $\tilde\sigma(h) = 0$ for $g_0h^2 \notin K$.) Finally, G acts on the set $\tilde{\mathsf{X}}(G,\mathcal{D},\varphi_0,\delta)$ through its componentwise action on $G \times \mathbb{Z}_{\geq 0}^{G/T} \times \mathbb{Z}^G$:

$$g \cdot g_0 = g^{-2}g_0, \qquad (g\cdot\kappa)(hT) = \kappa(g^{-1}hT), \qquad (g\cdot\tilde\sigma)(h) = \tilde\sigma(g^{-1}h).$$

The action of $G \times (C \rtimes A)$ can be reformulated in terms of the parameters $(g_0,\kappa,\sigma) \in \mathsf{X}(G,\mathcal{D},\varphi_0,\delta)$, where we use $\Sigma(g_0,\kappa) := \Sigma(G,\mathcal{D},\varphi_0,g_0,\kappa,\delta)$ instead of $\tilde\Sigma(g_0,\kappa)$ to define the set $\mathsf{X}(G,\mathcal{D},\varphi_0,\delta) \subseteq G \times \mathbb{Z}_{\geq 0}^{G/T} \times \mathbb{Z}^{G/T}$. Then, for all $\psi_0 \in A$, we have

$$\psi_0 \cdot (g_0,\kappa,\sigma) = (g_0,\kappa,\underline{\sigma}) \text{ where } \underline{\sigma}(x) = \mathrm{sign}\big(\nu(g_0\xi(x)^2)\big)\sigma(x),$$

and, similarly, for all $c \in C$, we have

$$c \cdot (g_0,\kappa,\sigma) = (g_0(\deg c),\kappa,\underline{\sigma}) \text{ where } \underline{\sigma}(x) = \mathrm{sign}\big(X_{g_0\xi(x)^2(\deg c)}^{-1}X_{g_0\xi(x)^2}c\big)\sigma(x).$$

Finally, Equation (13) gives us a formula for the action of $g \in G$, which sends $\sigma \in \Sigma(g_0,\kappa)$ to $\underline{\sigma} \in \Sigma(g\cdot g_0, g\cdot\kappa)$:

$$\begin{aligned}\underline{\sigma}(x) &= (g\cdot\tilde\sigma)(\xi(x)) = \tilde\sigma(g^{-1}\xi(x)) = \tilde\sigma(\xi(g^{-1}x)u)\\ &= \eta(u)\mathrm{sign}\big(X_{tu^2}^{-1}X_uX_tX_u\big)\sigma(g^{-1}x)\end{aligned} \tag{17}$$

where $x \in G/T$, $t := g_0\xi(g^{-1}x)^2$ (which is in K or else $\sigma(g^{-1}x) = 0$) and $u := \xi(g^{-1}x)^{-1}g^{-1}\xi(x)$ (which is in T).

Theorem 24. *Let $M(\mathcal{D},\varphi_0,g_0,\kappa,\sigma,\delta)$ and $M(\mathcal{D},\varphi_0,g_0',\kappa',\sigma',\delta)$ be graded algebras with involution as in Definition 20. If $\mathbb{C}\cong\mathcal{D}_e\subseteq Z(\mathcal{D})$, assume that T is finite and $\mathcal{D}_e=Z(\mathcal{D})$ (equivalently, β is nondegenerate). Then the graded algebras with involution are isomorphic if and only if (g_0,κ,σ) and (g_0',κ',σ') lie in the same $G\times(C\rtimes A)$-orbit.*

Proof. We have already proved that if $M(\mathcal{D},\varphi_0,g_0,\kappa,\sigma,\delta)$ and $M(\mathcal{D},\varphi_0,g_0',\kappa',\sigma',\delta)$ are isomorphic as graded algebras with involution then (g_0,κ,σ) and (g_0',κ',σ') lie in the same orbit. Conversely, suppose that $(g_0',\kappa',\sigma')=(g,c,\psi_0)\cdot(g_0,\kappa,\sigma)$ and let $d_0=\psi_0^{-1}(c)$. Then the φ_0-sesquilinear forms $\underline{B}=\psi_0(d_0B)$ on $(\mathcal{V}^{[g]})^{\psi_0^{-1}}$ and B' on $\mathcal{V}'$ are represented by the same matrix Φ, as in Equation (7), with respect to appropriate homogeneous $\mathcal{D}$-bases, whose respective elements have the same degrees. Therefore, there exists an isomorphism of graded $\mathcal{D}$-modules $\psi_1:(\mathcal{V}^{[g]})^{\psi_0^{-1}}\to\mathcal{V}'$ such that $\underline{B}(v,w)=B'(\psi_1(v),\psi_1(w))$ for all $v,w\in\mathcal{V}$. The result follows. $\square$

Remark 25. If $K\neq T$ (that is, $\mathbb{C}\cong\mathcal{D}_e\nsubseteq Z(\mathcal{D})$), then only the values $\nu(t)$ for $t\in K$ are relevant for the actions of A on C and on $\mathsf{X}(G,\mathcal{D},\varphi_0,\delta)$, so we can replace $A\cong Z^1(T,\mathcal{D}_e^\times)$ by its quotient $\mathrm{Hom}^+(K,\mathbb{R}^\times)$ (see Lemma 3). Further, only the sign of $\nu(t)$ matters, so we can replace $\mathrm{Hom}^+(K,\mathbb{R}^\times)$ by its image under the homomorphism $\mathrm{Hom}(K,\mathbb{R}^\times)\to\mathrm{Hom}(K,\{\pm1\})$ induced by $\mathrm{sign}:\mathbb{R}^\times\to\{\pm1\}$, which is naturally isomorphic to $\mathrm{Hom}(K/T^{[2]},\{\pm1\})$ where $T^{[2]}:=\{t^2\mid t\in T\}$. This latter reduction is also valid in the cases $\mathcal{D}_e=\mathbb{R}$ and $\mathcal{D}_e\cong\mathbb{H}$. Therefore, except in the $\mathbb{C}$-central case, the group A can be replaced by $\bar{A}:=\mathrm{Hom}(K/T^{[2]},\{\pm1\})$. Similarly, the group C can always be replaced by $\bar{C}:=C/\mathbb{R}_{>0}$.

3.4. **Interpretation in terms of groupoids.** The classification we have just obtained can be expressed in the language of groupoids (that is, categories whose morphisms are invertible). This interpretation will not be used in the remainder of the paper, but may elucidate what we have done so far.

For a given abelian group G, let $\mathfrak{R}$ be the groupoid whose objects are the pairs $(\mathcal{R},\varphi)$ as in Theorem 18 and whose morphisms are the isomorphisms of G-graded algebras with involution. Our classification problem is to parametrize the connected components of $\mathfrak{R}$. We begin by translating the problem to another groupoid, $\mathfrak{M}$, defined as follows.

Let $\mathfrak{D}$ be the groupoid whose objects are the real graded division algebras with support in G and the identity component isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and whose morphisms are the isomorphisms of G-graded algebras. For a fixed object $\mathcal{D}$ in $\mathfrak{D}$ and a degree-preserving involution φ_0 of $\mathcal{D}$ that restricts to the conjugation on $\mathcal{D}_e$, let $\mathfrak{Q}(\mathcal{D},\varphi_0)$ be the groupoid whose objects are the pairs $(\mathcal{V},B)$, where $\mathcal{V}$ is a graded right $\mathcal{D}$-module of finite dimension over $\mathcal{D}$ and $B:\mathcal{V}\times\mathcal{V}\to\mathcal{D}$ is a nondegenerate homogeneous φ_0-sesquilinear form satisfying $\overline{B}\in\{\pm B\}$, and whose morphisms $(\mathcal{V},B)\to(\mathcal{V}',B')$ are the isomorphisms $\psi_1:\mathcal{V}\to\mathcal{V}'$ of graded $\mathcal{D}$-modules such that $B=B'(\psi_1\times\psi_1)$. Finally, let $\mathfrak{M}$ be the groupoid whose objects are the quadruples $(\mathcal{D},\varphi_0,\mathcal{V},B)$, where $\mathcal{D}$ is an object in $\mathfrak{D}$, φ_0 is an involution of $\mathcal{D}$ as above and $(\mathcal{V},B)$ is an object in $\mathfrak{Q}(\mathcal{D},\varphi_0)$, and whose morphisms $(\mathcal{D},\varphi_0,\mathcal{V},B)\to(\mathcal{D}',\varphi_0',\mathcal{V}',B')$ are the pairs (ψ_0,ψ_1), where $\psi_0:\mathcal{D}\to\mathcal{D}'$ is a morphism in $\mathfrak{D}$ and there exist $g\in G$ and $d\in\mathcal{D}_{\mathrm{gr}}^\times$ such that ψ_1 is a morphism $\big((\mathcal{V}^{[g]})^{\psi_0^{-1}},\psi_0(dB)\big)\to(\mathcal{V}',B')$ in $\mathfrak{Q}(\mathcal{D}',\varphi_0')$, that is, $\psi_1:(\mathcal{V}^{[g]})^{\psi_0^{-1}}\to\mathcal{V}'$ is an isomorphism of graded $\mathcal{D}'$-modules and $\psi_0(dB) = B'(\psi_1 \times \psi_1)$. Note that these conditions determine the elements g and d uniquely and imply that $\psi_0^{-1}\varphi_0'\psi_0 = \mathrm{Int}(d)\varphi_0$.

There is a functor $E : \mathfrak{M} \to \mathfrak{R}$ that maps $(\mathcal{D}, \varphi_0, \mathcal{V}, B) \mapsto (\mathrm{End}_{\mathcal{D}}(\mathcal{V}), \varphi)$ and $(\psi_0, \psi_1) \mapsto \psi$ where φ is given by Equation (3) and $\psi(r) = \psi_1 r \psi_1^{-1}$ for all $r \in \mathrm{End}_{\mathcal{D}}(\mathcal{V})$. The functor E is full and essentially surjective (although not faithful, hence not an equivalence), so it gives a bijection between the connected components of $\mathfrak{M}$ and those of $\mathfrak{R}$.

Next, we partition the groupoid $\mathfrak{M}$. Let $F_1 : \mathfrak{M} \to \mathfrak{D}$ be the projection $(\mathcal{D}, \varphi_0, \mathcal{V}, B) \mapsto \mathcal{D}$ and $(\psi_0, \psi_1) \mapsto \psi_0$. Then the object class of $\mathfrak{M}$ is partitioned into the inverse images of the connected components of $\mathfrak{D}$ under F_1. Moreover, if $\mathcal{D}$ and $\mathcal{D}'$ are in the same connected component of $\mathfrak{D}$, that is, there exists a morphism $\psi_0 : \mathcal{D} \to \mathcal{D}'$, and if $(\mathcal{D}, \varphi_0, \mathcal{V}, B)$ is an object in $\mathfrak{M}$, then $(\psi_0, \iota_{\mathcal{V}}^{\psi_0^{-1}})$ is a morphism $(\mathcal{D}, \varphi_0, \mathcal{V}, B) \to (\mathcal{D}', \psi_0\varphi_0\psi_0^{-1}, \mathcal{V}^{\psi_0^{-1}}, \psi_0 B)$ in $\mathfrak{M}$, where $\iota_{\mathcal{V}}^{\psi_0^{-1}}$ is the identity map from $\mathcal{V}$ to $\mathcal{V}^{\psi_0^{-1}}$ (which have the same underlying set). It follows that every connected component of $\mathfrak{M}$ mapped by F_1 to the connected component $[\mathcal{D}]$ in $\mathfrak{D}$ has a representative mapped to $\mathcal{D}$. Therefore, we may fix $\mathcal{D}$ and work in the "fiber" over $\mathcal{D}$, that is, the full subgroupoid $\mathfrak{M}(\mathcal{D}) \subseteq \mathfrak{M}$ defined by the object class $F_1^{-1}(\mathcal{D})$. Also, the group of automorphisms of $\mathcal{D}$ acts on $F_1^{-1}(\mathcal{D})$, assuming the latter is nonempty: for $\psi_0 \in \mathrm{Aut}^G(\mathcal{D})$, we let $\psi_0 \cdot (\mathcal{D}, \varphi_0, \mathcal{V}, B) := (\mathcal{D}, \psi_0\varphi_0\psi_0^{-1}, \mathcal{V}^{\psi_0^{-1}}, \psi_0 B)$.

Further, we partition $\mathfrak{M}(\mathcal{D})$ as follows. The group $\mathcal{D}_{\mathrm{gr}}^{\times} \rtimes \mathrm{Aut}^G(\mathcal{D})$ acts on the set of degree-preserving antiautomorphisms of $\mathcal{D}$ by the formula $(c, \psi_0) \cdot \varphi_0 := \mathrm{Int}(c)\psi_0\varphi_0\psi_0^{-1}$, so we get the corresponding action groupoid $\mathfrak{A}(\mathcal{D})$. Let $\mathfrak{I}(\mathcal{D}) \subseteq \mathfrak{A}(\mathcal{D})$ be the full subgroupoid whose objects are the degree-preserving involutions of $\mathcal{D}$ that restrict to the conjugation on $\mathcal{D}_e$. Then we can define another projection, $F_2 : \mathfrak{M}(\mathcal{D}) \to \mathfrak{I}(\mathcal{D})$, sending an object $(\mathcal{D}, \varphi_0, \mathcal{V}, B)$ to φ_0 and a morphism $(\psi_0, \psi_1) : (\mathcal{D}, \varphi_0, \mathcal{V}, B) \to (\mathcal{D}, \varphi_0', \mathcal{V}', B')$ to the arrow $\varphi_0 \to \varphi_0'$ labeled by the element $(\psi_0(d), \psi_0) \in \mathcal{D}_{\mathrm{gr}}^{\times} \rtimes \mathrm{Aut}^G(\mathcal{D})$ where d is determined by (ψ_0, ψ_1) as above. Moreover, if φ_0 and φ_0' are in the same connected component of $\mathfrak{I}(\mathcal{D})$, that is, there exist $c \in \mathcal{D}_{\mathrm{gr}}^{\times}$ and $\psi_0 \in \mathrm{Aut}^G(\mathcal{D})$ such that $(c, \psi_0) \cdot \varphi_0 = \varphi_0'$ then we have $\psi_0^{-1}\varphi_0'\psi_0 = \mathrm{Int}(d)\varphi_0$, where $d = \psi_0^{-1}(c)$, and the proof of Lemma 10 (with $\psi_0^{-1}\varphi_0'\psi_0$ playing the role of φ_0') shows that c can be chosen to satisfy $\varphi_0(d) \in \{\pm d\}$. Then $(\mathcal{V}, dB)$ is an object of $\mathfrak{Q}(\mathcal{D}, \mathrm{Int}(d)\varphi_0)$ and hence $(\psi_0, \iota_{\mathcal{V}}^{\psi_0^{-1}})$ is a morphism $(\mathcal{D}, \varphi_0, \mathcal{V}, B) \to (\mathcal{D}, \varphi_0', \mathcal{V}^{\psi_0^{-1}}, \psi_0(dB))$ in $\mathfrak{M}(\mathcal{D})$. Therefore, we may fix φ_0 and work in the "fiber" over φ_0, that is, the full subgroupoid $\mathfrak{M}(\mathcal{D}, \varphi_0) \subseteq \mathfrak{M}(\mathcal{D})$ defined by the object class $F_2^{-1}(\varphi_0)$. Also, let H be the subgroup of the stabilizer of φ_0 in $\mathcal{D}_{\mathrm{gr}}^{\times} \rtimes \mathrm{Aut}^G(\mathcal{D})$ consisting of all $h = (\psi_0(d), \psi_0)$ such that $h \cdot \varphi_0 = \varphi_0$ and $\varphi_0(d) \in \{\pm d\}$. Then, any morphism (ψ_0, ψ_1) in $\mathfrak{M}(\mathcal{D}, \varphi_0)$ is mapped by F_2 to an arrow $\varphi_0 \to \varphi_0$ whose label is in H. Moreover, H acts on $F_2^{-1}(\varphi_0)$, which is always nonempty, as follows: $h \cdot (\mathcal{D}, \varphi_0, \mathcal{V}, B) := (\mathcal{D}, \varphi_0, \mathcal{V}^{\psi_0^{-1}}, \psi_0(dB))$, and the corresponding action groupoid can be regarded as a subgroupoid of $\mathfrak{M}(\mathcal{D}, \varphi_0)$: it has the same objects, and the arrow $(\mathcal{D}, \varphi_0, \mathcal{V}, B) \to (\mathcal{D}, \varphi_0, \mathcal{V}^{\psi_0^{-1}}, \psi_0(dB))$ labeled by h can be identified with $(\psi_0, \iota_{\mathcal{V}}^{\psi_0^{-1}})$.

Except in the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$, all elements of $\mathrm{Aut}^G(\mathcal{D})$ commute with our involutions, so the situation simplifies: the connected components of $\mathfrak{I}(\mathcal{D})$ are given by the equivalence relation in Definition 7 and the elements of H have

$d \in Z(\mathcal{D})$. If $\varphi_0|_{Z(\mathcal{D})} = \mathrm{id}_{Z(\mathcal{D})}$ then there are no morphisms in $\mathfrak{M}(\mathcal{D}, \varphi_0)$ between objects with $\overline{B} = +B$ and those with $\overline{B} = -B$, hence we partition $\mathfrak{M}(\mathcal{D}, \varphi_0)$ into two full subgroupoids: $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$, $\delta \in \{\pm 1\}$, by the condition $\overline{B} = \delta B$. If $\varphi_0|_{Z(\mathcal{D})} \neq \mathrm{id}_{Z(\mathcal{D})}$ then each connected component of $\mathfrak{M}(\mathcal{D}, \varphi_0)$ has a representative in $\mathfrak{M}(\mathcal{D}, \varphi_0, 1)$, so it suffices to study this latter. In either case, the morphisms in $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$ are mapped to the subgroup $H^+ \subseteq H$ defined by the condition $\varphi_0(d) = d$, that is,

$$H^+ := \{(\psi_0(d), \psi_0) \in \mathcal{D}_{\mathrm{gr}}^\times \rtimes \mathrm{Aut}^G(\mathcal{D}) \mid \varphi_0(d) = d \text{ and } \psi_0 \mathrm{Int}(d) \varphi_0 \psi_0^{-1} = \varphi_0\}.$$

This subgroup acts on the objects of $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$, and the corresponding action groupoid can be regarded as a subgroupoid of $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$. In the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$, each connected component of $\mathfrak{M}(\mathcal{D}, \varphi_0)$ has a representative in $\mathfrak{M}(\mathcal{D}, \varphi_0, 1)$, so the same remarks about H^+ apply. Note that, whenever all elements of $\mathrm{Aut}^G(\mathcal{D})$ commute with φ_0, we have $H^+ = C \rtimes \mathrm{Aut}^G(\mathcal{D})$.

The lack of faithfulness of the functor E can be exploited to replace the groupoid $\mathfrak{M}(\mathcal{D})$ by a subgroupoid with the same objects, but whose morphisms have some restrictions on ψ_0. This allowed us to replace the group H^+ by its subgroup $C \rtimes A$, but we had to make a simplifying assumption in the case $\mathbb{C} \cong \mathcal{D}_e \subseteq Z(\mathcal{D})$. (Also, under the said assumption, the connected components of $\mathfrak{I}(\mathcal{D})$ are given by the equivalence relation in Definition 7.) Now we proceed in full generality.

In addition to H^+, the group G also acts on the objects of $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$: for all $g \in G$, let $g \cdot (\mathcal{D}, \varphi_0, \mathcal{V}, B) := (\mathcal{D}, \varphi_0, \mathcal{V}^{[g]}, B)$, and the corresponding action groupoid can be regarded as a subgroupoid of $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$: it has the same objects, and the arrow $(\mathcal{D}, \varphi_0, \mathcal{V}, B) \to (\mathcal{D}, \varphi_0, \mathcal{V}^{[g]}, B)$ labeled by g can be identified with $(\mathrm{id}_{\mathcal{D}}, \iota_{\mathcal{V}}^{[g]})$, where $\iota_{\mathcal{V}}^{[g]}$ is the identity map from $\mathcal{V}$ to $\mathcal{V}^{[g]}$ (which have the same underlying set). Since the actions of G and H^+ commute, we get an action of $G \times H^+$, and the corresponding action groupoid embeds in $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$.

Recall the groupoid $\mathfrak{Q}(\mathcal{D}, \varphi_0)$. Its full subgroupoid $\mathfrak{Q}(\mathcal{D}, \varphi_0, \delta)$, determined by the condition $\overline{B} = \delta B$, also embeds in $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$, by sending $(\mathcal{V}, B) \mapsto (\mathcal{D}, \varphi_0, \mathcal{V}, B)$ and $\psi_1 \mapsto (\mathrm{id}_{\mathcal{D}}, \psi_1)$. For any morphism θ in $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$, we have (unique) factorizations: $\theta = \theta' \theta'' = \tilde{\theta}'' \tilde{\theta}'$ where θ' and $\tilde{\theta}'$ are morphisms in the action groupoid of $G \times H^+$, and θ'' and $\tilde{\theta}''$ are morphisms in $\mathfrak{Q}(\mathcal{D}, \varphi_0, \delta)$. It follows that $G \times H^+$ acts on the connected components of $\mathfrak{Q}(\mathcal{D}, \varphi_0, \delta)$, and each connected component of $\mathfrak{M}(\mathcal{D}, \varphi_0, \delta)$ is the union of an orbit of this action.

It remains to parametrize the connected components of $\mathfrak{Q} := \mathfrak{Q}(\mathcal{D}, \varphi_0, \delta)$, which we did by means of the set $\mathsf{X} := \mathsf{X}(G, \mathcal{D}, \varphi_0, \delta)$ or, alternatively, $\tilde{\mathsf{X}} := \tilde{\mathsf{X}}(G, \mathcal{D}, \varphi_0, \delta)$, and calculate the action of $G \times H^+$ (or its subgroup that has the same orbits) on X (or $\tilde{\mathsf{X}}$). The parametrization, that is, a mapping P from X (or $\tilde{\mathsf{X}}$) to the object class of $\mathfrak{Q}$ that selects a unique representative in each connected component of $\mathfrak{Q}$, depends on the choice of a transversal for T in G and on the choice of the elements $X_t \in \mathcal{D}_t$ for $t \in T$. The first is tantamount to selecting representatives for the isomorphism classes of graded right $\mathcal{D}$-modules of dimension 1 over $\mathcal{D}$, and the second affects the matrix Φ representing B in Theorem 18. Note that $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, \delta) \cong E(P(g_0, \kappa, \sigma))$.

To summarize: a parametrization of the isomorphism classes of G-graded algebras with involution as in Theorem 18 is given by $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, \delta)$, where (g_0, κ, σ) ranges over a set of representatives of the $G \times H^+$-orbits in $\mathsf{X}(G, \mathcal{D}, \varphi_0, \delta)$, $\mathcal{D}$ ranges over a set of representatives of the isomorphism classes of real graded division algebras with $\mathcal{D}_e$ isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, φ_0 ranges over a set of representatives of the equivalence classes of degree-preserving involutions on $\mathcal{D}$ that restrict to the conjugation on $\mathcal{D}_e$, and finally $\delta \in \{\pm 1\}$ if φ_0 restricts to the identity on $Z(\mathcal{D})$ and $\delta = 1$ otherwise. The equivalence relation for the involutions is given by $\varphi_0 \sim \varphi_0'$ if $\psi_0^{-1}\varphi_0'\psi_0 = \mathrm{Int}(d)\varphi_0$ for some $\psi_0 \in \mathrm{Aut}^G(\mathcal{D})$ and $d \in \mathcal{D}_{\mathrm{gr}}^\times$ satisfying $\varphi_0(d) \in \{\pm d\}$.

3.5. **Central simple algebras with involution.** We now specialize to the setting needed for applications to classical central simple Lie algebras: $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, \delta)$ is finite-dimensional and central simple as an ungraded algebra with involution. In other words, $\mathcal{D}$ is finite-dimensional (hence semisimple because of the generalization of Maschke's Theorem), its center $Z(\mathcal{D})$ is either $\mathbb{R}$, $\mathbb{C}$ or $\tilde{\mathbb{C}} := \mathbb{R} \times \mathbb{R}$, and in the latter two cases φ_0 is an involution of the second kind, that is, $\varphi_0|_{Z(\mathcal{D})} \neq \mathrm{id}_{Z(\mathcal{D})}$. Note that if $Z(\mathcal{D}) = \tilde{\mathbb{C}}$ then it must be nontrivially graded. Also, in all cases we have $C = \mathbb{R}^\times$ and hence, by Remark 25, we may replace C with $\{\pm 1\}$, where $\epsilon \in \{\pm 1\}$ acts on $\mathsf{X}(G, \mathcal{D}, \varphi_0, \delta)$ in the obvious way: $\epsilon \cdot (g_0, \kappa, \sigma) = (g_0, \kappa, \epsilon\sigma)$.

It is shown in [3] that T must be an elementary 2-group, except possibly in the case $\mathbb{C} = \mathcal{D}_e = Z(\mathcal{D})$. We give a proof here for completeness and to introduce some notation for future use. First, recall that $\mathrm{rad}\beta = \mathrm{supp}\, Z(\mathcal{D})$ by Lemma 4. Since β takes values in $\{\pm 1\}$ by Lemma 11, $t^2 \in \mathrm{rad}\beta$ for all $t \in K$. Moreover, if $K \neq T$ then, for any $t \in T \setminus K$, we also have $t^2 \in \mathrm{rad}\beta$ by Lemma 12, so we conclude that $t^2 \in \mathrm{rad}\beta$ for all $t \in T$. If $Z(\mathcal{D})$ is trivially graded, this immediately implies that T is an elementary 2-group. Also note that β is nondegenerate in this case. If $Z(\mathcal{D})$ is nontrivially graded then $Z(\mathcal{D})$ is $\mathbb{C}$ or $\tilde{\mathbb{C}}$ and $\mathrm{rad}\beta = \mathrm{supp}\, Z(\mathcal{D}) = \{e, f\}$ where $f \in K$ is an element of order 2, which will be referred to as the *distinguished element*. (Indeed, f is the degree of an imaginary unit $\mathbf{i}$, where $\mathbf{i}^2 = -1$ in the case of $\mathbb{C}$ and $\mathbf{i}^2 = 1$ in the case of $\tilde{\mathbb{C}}$.) We claim that f is not a square in T. Indeed, if $t^2 = f$ then X_t^2 generates $Z(\mathcal{D})$ as an $\mathbb{R}$-algebra and at the same time $\varphi_0(X_t^2) = \varphi_0(X_t)^2 = \eta(t)^2 X_t^2 = X_t^2$, which contradicts the fact $\varphi_0|_{Z(\mathcal{D})} \neq \mathrm{id}_{Z(\mathcal{D})}$. It follows that $t^2 = e$ for all $t \in T$, that is, T is an elementary 2-group.

In particular, Remark 25 implies that $A \cong Z^1(T, Z(\mathcal{D}_e)^\times)$ (see Equation (9)) acts on $\mathsf{X}(G, \mathcal{D}, \varphi_0, \delta)$ through its quotient $\mathrm{Hom}(K, \{\pm 1\})$. To be precise, for $\nu \in \mathrm{Hom}(K, \{\pm 1\})$, we have

$$\nu \cdot (g_0, \kappa, \sigma) = (g_0, \kappa, \nu\sigma) \text{ where } (\nu\sigma)(x) := \nu(g_0\xi(x)^2)\sigma(x).$$

The same formula applies to the "quadratic form" $\nu : T \to \{\pm 1\}$ in the case $\mathbb{C} = \mathcal{D}_e = Z(\mathcal{D})$, if T happens to be an elementary 2-group (see Equation (10)).

Corollary 26. *Let $\mathcal{D}$ be finite-dimensional with $Z(\mathcal{D}) = \mathbb{R}$. Then the graded algebras with involution $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, \delta)$ and $M(\mathcal{D}, \varphi_0, g_0', \kappa', \sigma', \delta)$ are isomorphic if and only if (g_0, κ, σ) and (g_0', κ', σ') are in the same $G \times \{\pm 1\}$-orbit.*

Proof. By Skolem–Noether Theorem, we have $A \subseteq \mathrm{Int}^G(\mathcal{D}, \varphi_0)$, so the result follows from Theorem 24 and Remark 23. □

In the next two corollaries, since $\varphi_0|_{Z(\mathcal{D})} \neq \mathrm{id}_{Z(\mathcal{D})}$, we necessarily have $\delta = 1$.

Corollary 27. *Suppose that $\mathcal{D}$ is finite-dimensional and $Z(\mathcal{D})$ is $\mathbb{C}$ or $\tilde{\mathbb{C}}$, nontrivially graded. Pick $\nu \in \mathrm{Hom}(K, \{\pm 1\})$ such that $\nu(f) = -1$. Then the graded algebras with involution $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, 1)$ and $M(\mathcal{D}, \varphi_0, g_0', \kappa', \sigma', 1)$ are isomorphic if and only if (g_0', κ', σ') lies in the same $G \times \{\pm 1\}$-orbit as (g_0, κ, σ) or $(g_0, \kappa, \nu\sigma)$.*

Proof. Let $A_{\mathrm{in}} = A \cap \mathrm{Int}^G(\mathcal{D}, \varphi_0)$. By Skolem–Noether Theorem, ψ_0 is inner if and only if $\psi_0|_{Z(\mathcal{D})} = \mathrm{id}_{Z(\mathcal{D})}$, which happens if and only if $\nu(f) = 1$. Therefore, the group A is generated by A_{in} and a single ψ_0 with $\nu(f) = -1$. The result follows from Theorem 24 and Remark 23. $\square$

Finally, let us analyse the remaining case $\mathbb{C} = \mathcal{D}_e = Z(\mathcal{D})$.

Corollary 28. *Suppose that $\mathcal{D}$ is finite-dimensional and $Z(\mathcal{D})$ is $\mathbb{C}$, trivially graded. If T is an elementary 2-group, pick $\nu : T \to \{\pm 1\}$ such that $\nu(st) = \nu(s)\nu(t)\beta(s,t)$ for all $s, t \in T$. Then the graded algebras with involution $M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, 1)$ and $M(\mathcal{D}, \varphi_0, g_0', \kappa', \sigma', 1)$ are isomorphic if and only if*

- *when T is not an elementary 2-group: (g_0', κ', σ') lies in the same $G \times \{\pm 1\}$-orbit as (g_0, κ, σ);*
- *when T is an elementary 2-group: (g_0', κ', σ') lies in the same $G \times \{\pm 1\}$-orbit as (g_0, κ, σ) or $(g_0, \kappa, \nu\sigma)$.*

$\square$

Thus, we see that, in the setting at hand, the isomorphism problem essentially boils down to the G-action on $\mathsf{X}(G, \mathcal{D}, \varphi_0, \delta)$. Recall that the action is the following: $g \cdot (g_0, \kappa, \sigma) = (g^{-2} g_0, g \cdot \kappa, g \cdot \sigma)$ where $(g \cdot \kappa)(x) = \kappa(g^{-1}x)$ for all $x \in G/T$ and $g \cdot \sigma$ is $\underline{\sigma}$ given by Equation (17). (There is abuse of notation in writing $g \cdot \sigma$ because, in general, $\underline{\sigma}$ depends not only on σ but also on g_0.) The sign factor in Equation (17) can be simplified in our setting, but we first need to recall some facts about $\mathcal{D}$ and φ_0.

When $Z(\mathcal{D})$ is $\mathbb{C}$ with trivial grading, $\mathcal{D}$ is a graded division algebra over $\mathbb{C}$, so it is determined up to isomorphism of graded algebras by the pair (T, β) where the alternating bicharacter $\beta : T \times T \to \mathbb{C}^\times$ is nondegenerate (see e.g. [11, Theorem 2.15]). In the remaining cases, we know that T must be an elementary 2-group and $\beta : K \times K \to \{\pm 1\}$.

If $\mathcal{D}_e = \mathbb{R}$ or $\mathcal{D}_e \cong \mathbb{H}$ then $K = T$ and $X_t^2 \in \mathbb{R}$ for all $t \in T$, hence each X_t can be normalized with a real scalar so that $X_t^2 \in \{\pm 1\}$. This gives us a function $\mu : T \to \{\pm 1\}$ defined by $\mu(t) = X_t^2$. Moreover, each normalized element X_t is unique up to sign, so the function μ is uniquely determined. It is shown in [18, Theorems 15, 16 and 19] (see [3, §10] for the semisimple case) that $\mu : T \to \{\pm 1\}$ is a quadratic form with polar form β and that the pair (T, μ) determines the graded algebra $\mathcal{D}$ up to isomorphism.

If $\mathcal{D}_e \cong \mathbb{C}$ then $K \neq T$ (since $Z(\mathcal{D})$ is nontrivially graded) and, for any $t \in T \setminus K$, we have $X_t^2 \in Z(\mathcal{D})_e = \mathbb{R}$ (using Lemma 12). Moreover, for all $z \in \mathcal{D}_e$, we have $(X_t z)^2 = X_t^2 |z|^2$, hence each X_t can be normalized so that $X_t^2 \in \{\pm 1\}$, and this gives a uniquely determined function $\mu : T \setminus K \to \{\pm 1\}$. As to the elements X_t with $t \in K$, we have $(X_t z)^2 = X_t^2 z^2$ for all $z \in \mathcal{D}_e$, hence they can also be normalized (with complex scalars) to satisfy $X_t^2 \in \{\pm 1\}$, but the values $+1$ or -1 can be chosen arbitrarily. It is shown in [18, Theorems 22 and 23] (see [3, §10] for the semisimple case) that $\mu : T \setminus K \to \{\pm 1\}$ is what is called there a *nice map*, that is, for some (and hence any) $u \in T \setminus K$, the function $\mu_u(t) := \mu(ut)\mu(u)$ is a quadratic form $K \to \{\pm 1\}$ with polar form β, and that (T, μ) determines the graded algebra $\mathcal{D}$ up to isomorphism.

Thanks to Skolem–Noether Theorem and Lemma 10, all degree-preserving involutions on $\mathcal{D}$ (of the second kind if $Z(\mathcal{D})$ is $\mathbb{C}$ or $\tilde{\mathbb{C}}$) are equivalent in the sense of Definition 7, hence φ_0 can be fixed arbitrarily. However, in all cases except

$Z(\mathcal{D}) = \tilde{\mathbb{C}}$, the classification in [3] suggests especially nice choices, which will be referred to as the *distinguished involutions* (see [3, §11]). If $Z(\mathcal{D}) = \mathbb{R}$ or $Z(\mathcal{D}) = \mathbb{C}$ with nontrivial grading, then there is a unique distinguished involution, whereas in the case $Z(\mathcal{D}) = \mathbb{C}$ with trivial grading, there is an isomorphism class of distinguished involutions. They are defined as follows.

When $Z(\mathcal{D})$ is $\mathbb{C}$ with trivial grading, we have $\mathcal{D}_e = Z(\mathcal{D})$ and $K = T$, and we agreed to choose X_t so that $\eta(t) = 1$ for all $t \in T$ (see Remark 6 and Notation 8) or, in other words, $\varphi_0(X_t) = X_t$ for all $t \in T$. Since $\varphi_0(X_t z) = X_t \bar{z}$ for all $z \in \mathcal{D}_e$, each element X_t is determined up to a real scalar. We can choose this scalar so that $|X_t^{o(t)}| = 1$, where $o(t)$ denotes the order of t, and this determines X_t up to sign. Since $\varphi_0(X_t^{o(t)}) = X_t^{o(t)}$, we have $X_t^{o(t)} \in \{\pm 1\}$. If $o(t)$ is odd then there is a unique choice of X_t such that $X_t^{o(t)} = 1$, whereas if $o(t)$ is even then $X_t^{o(t)}$ is the same for both choices of X_t. The distinguished involutions are characterized by the property that these normalized elements X_t satisfy $X_t^{o(t)} = 1$ for all $t \in T$.

In the remaining cases, the distinguished involution is obtained by setting $\eta = \mu$. This requires some comment. If $\mathcal{D}_e = \mathbb{R}$ or $\mathcal{D}_e \cong \mathbb{H}$ then $K = T$ and μ is a quadratic form with polar form β, hence the mapping $X_t \mapsto \mu(t) X_t$ uniquely extends to an involution φ_0 such that $\varphi_0|_{\mathcal{D}_e}$ is the conjugation. If $\mathcal{D}_e \cong \mathbb{C}$ (hence $K \neq T$), the nice map μ is defined only on $T \setminus K$. However, we agreed to choose the elements X_t for $t \in K$ so that $\varphi_0(X_t) = \delta X_t$ (Notation 8), which determines them up to a real scalar. Moreover, for these elements we have $\varphi_0(X_t^2) = X_t^2$, so $X_t^2 \in \mathbb{R}$ and hence X_t can be normalized with real scalars so that $X_t^2 \in \{\pm 1\}$. This determines a unique extension of μ to a function $T \to \{\pm 1\}$, which we still denote by μ, by setting $\mu(t) := X_t^2$ for all $t \in T$. The distinguished involution is characterized by the property that $\eta(t) = \mu(t)$ for all $t \in T$; in fact, it is sufficient to require $\eta(t) = \mu(t)$ for all $t \in T \setminus K$ (see Lemma 29 below), that is, $\varphi_0(X_t) = \mu(t) X_t$ for all $t \in T \setminus K$. It turns out that φ_0 thus defined is of the second kind if $Z(\mathcal{D}) = \mathbb{C}$, but of the first kind if $Z(\mathcal{D}) = \tilde{\mathbb{C}}$, so it is not suitable for our purposes in this latter case.

We will need one more ingredient in the case $K \neq T$, namely, the extension of $\beta : K \times K \to \{\pm 1\}$ to a function $T \times K \to \{\pm 1\}$, which we still denote by β. As we already observed, the elements X_t with $t \in K$ are determined by the condition $\varphi_0(X_t) = \delta X_t$ up to a real scalar, hence Equation (2) defines $\beta(s, t) \in \mathcal{D}_e$ uniquely as long as at least one of the elements s and t is in K.

Lemma 29. *If $\mathcal{D}_e \cong \mathbb{C}$ and $K \neq T$ then the function β defined on $T \times K$ by the equation $X_u X_t = \beta(u, t) X_t X_u$ takes values in $\{\pm 1\}$ and is multiplicative in the first variable. It also satisfies*

$$\beta(u, t) = \mu(ut)\mu(u)\mu(t) = \eta(ut)\eta(u)\delta$$

for all $u \in T \setminus K$ and $t \in K$. In particular, if η and μ coincide on $T \setminus K$ then they coincide on the whole T.

Proof. We already know that β takes values in $\{\pm 1\}$ on $K \times K$, so let $u \in T \setminus K$ and $t \in K$. Applying φ_0 to $X_u X_t = \beta(u, t) X_t X_u$, we get $X_t X_u = X_u X_t \overline{\beta(u, t)} = \beta(u, t) X_u X_t$, so $\beta(u, t)^2 = 1$. Hence, $\beta(u, t) = \beta(u, t)^{-1}$ and we get $\mathrm{Int}(X_t)(X_u) = \beta(u, t) X_u$, so Lemma 2 implies that $\beta(\cdot, t) \in Z^1(T, \{\pm 1\}) = \mathrm{Hom}(T, \{\pm 1\})$.

We have $X_u X_t = X_{ut} \lambda$ for some $\lambda \in \mathcal{D}_e^{\times}$. Squaring both sides and taking into account that $ut \in T \setminus K$, we get $X_u^2 X_t^2 \beta(u, t) = X_{ut}^2 |\lambda|^2$, hence $\mu(u)\mu(t)\beta(u, t) =$

$\mu(ut)$. Finally, applying φ_0 to $X_uX_t = X_{ut}\lambda$, we get $X_tX_u\eta(u)\eta(t) = \bar{\lambda}X_{ut}\eta(ut) = X_{ut}\lambda\eta(ut)$, hence $\eta(ut) = \beta(u,t)\eta(u)\eta(t) = \beta(u,t)\eta(u)\delta$. □

We will now obtain more explicit formulas for the action of G on the set of parameters $\mathsf{X}(G,\mathcal{D},\varphi_0,\delta)$. We emphasize that, in the next result, whenever $K \neq T$, μ and β stand for the extensions of, respectively, the nice map $\mu : T \setminus K \to \{\pm 1\}$ and the bicharacter $\beta : K \times K \to \{\pm 1\}$, as defined above.

Proposition 30. *Suppose that $\mathcal{D}$ is finite-dimensional and $Z(\mathcal{D})$ is $\mathbb{R}$ or either $\mathbb{C}$ or $\tilde{\mathbb{C}}$ with nontrivial grading. Then the action of G on $\mathsf{X}(G,\mathcal{D},\varphi_0,\delta)$ is the following: $g\cdot(g_0,\kappa,\sigma) = (g^{-2}g_0, g\cdot\kappa, g\cdot\sigma)$ where $(g\cdot\kappa)(x) = \kappa(g^{-1}x)$ and*

$$(g\cdot\sigma)(x) = \eta(u)\mu(u)\beta(u,t)\sigma(g^{-1}x),$$

with $u := \xi(g^{-1}x)^{-1}g^{-1}\xi(x) \in T$ and $t := g_0\xi(g^{-1}x)^2 = g^{-2}g_0\xi(x)^2$ (which is in K or else $\sigma(g^{-1}x) = 0$). Furthermore, if $Z(\mathcal{D})$ is not $\tilde{\mathbb{C}}$ and φ_0 is the distinguished involution then

$$(g\cdot\sigma)(x) = \beta(u,t)\sigma(g^{-1}x) = \beta(\xi(g^{-1}x)^{-1}g^{-1}\xi(x), g^{-2}g_0\xi(x)^2)\,\sigma(g^{-1}x).$$

Proof. Taking into account that T is an elementary 2-group and using Lemma 29, we can compute the sign factor in Equation (17) as follows:

$$\mathrm{sign}(X_{tu^2}^{-1}X_uX_tX_u) = \mathrm{sign}(X_t^{-1}X_uX_tX_u) = \beta(u,t)\mathrm{sign}(X_u^2) = \beta(u,t)\mu(u).$$

Note that $\xi(g^{-1}x)g\xi(x)^{-1}$ is always in T and hence $\xi(g^{-1}x)^2 = g^{-2}\xi(x)^2$. The result follows. □

If $Z(\mathcal{D}) = \mathbb{C} = \mathcal{D}_e$ (hence $K = T$), the sign factor in Equation (17) will, in general, depend on the choice of the elements X_t, because if $tu^2 \neq t$ then X_t and X_{tu^2} can be independently normalized with real scalars. (This problem does not arise if T happens to be an elementary 2-group.) It is shown in [3, §11] that, if φ_0 is a distinguished involution, then the elements X_t can be chosen in such a way that, in addition to the condition $X_t^{o(t)} = 1$, they also satisfy $X_uX_sX_u = X_{su^2}$ for all $s \in T^{[2]}$ and $u \in T$. Moreover, the elements X_s for $s \in T^{[2]}$ are uniquely determined, while the elements X_t for $t \notin T^{[2]}$ are determined up to sign. We will make the following choice. Fix a transversal for the subgroup $T^{[2]}$ in T containing e and let $\xi' : T/T^{[2]} \to T$ be the corresponding section of the quotient map $\pi' : T \to T/T^{[2]}$. Also pick a transversal for the subgroup $T_{[2]}$ in T containing e and let $\xi'' : T^{[2]} \to T$ be the corresponding section of the epimorphism $T \to T^{[2]}$ given by $t \mapsto t^2$ (in other words, $\xi''(s)^2 = s$ for all $s \in T^{[2]}$, and $\xi''(e) = e$). Make an arbitrary choice of X_t (out of two options) for each $t \neq e$ in the transversal of $T^{[2]}$, and then define

$$X_t := \beta(\xi''(s), \xi'\pi'(t))\,X_{\xi'\pi'(t)}X_s \text{ where } s := t(\xi'\pi'(t))^{-1}. \tag{18}$$

Note that $\xi'\pi'(t)$ belongs to the transversal of $T^{[2]}$ and s belongs to $T^{[2]}$, so the elements $X_{\xi'\pi'(t)}$ and X_s were defined earlier. The above formula is consistent with those definitions because, for t in the transversal of $T^{[2]}$, we have $\xi'\pi'(t) = t$ and hence $s = e$, while for $t \in T^{[2]}$, we have $\xi'\pi'(t) = e$ and hence $s = t$.

Lemma 31. *The elements defined by Equation (18) satisfy $\varphi_0(X_t) = X_t$ and $X_t^{o(t)} = 1$ for all $t \in T$. Moreover, $X_uX_sX_u = X_{su^2}$ for all $s \in T^{[2]}$ and $u \in T$.*

Proof. Using the fact $s = \xi''(s)^2$, we obtain:

$$\begin{aligned}\varphi_0(X_t) &= \overline{\beta(\xi''(s), \xi'\pi'(t)}\, X_s X_{\xi'\pi'(t)} \\ &= \beta(\xi''(s), \xi'\pi'(t)^{-1} \beta(s, \xi'\pi'(t))\, X_{\xi'\pi'(t)} X_s = X_t.\end{aligned}$$

It is clear that $|X_t^{o(t)}| = 1$, and the fact that $\varphi_0(X_t) = X_t$ implies $X_t^{o(t)} \in \{\pm 1\}$. If $o(t)$ is odd then $t \in T^{[2]}$ and hence $X_t^{o(t)} = 1$. If $o(t)$ is even then $X_t^{o(t)} = 1$ because φ_0 is a distinguished involution. This completes the proof of the first assertion. The second assertion follows by the choice of X_s for $s \in T^{[2]}$. □

Proposition 32. *Suppose that $\mathcal{D}$ is finite-dimensional and $Z(\mathcal{D})$ is $\mathbb{C}$ with trivial grading. Assume that φ_0 is a distinguished involution and the elements X_t are defined by Equation* (18). *Then the action of G on $\mathsf{X}(G, \mathcal{D}, \varphi_0, 1)$ is the following: $g \cdot (g_0, \kappa, \sigma) = (g^{-2} g_0, g \cdot \kappa, g \cdot \sigma)$ where $(g \cdot \kappa)(x) = \kappa(g^{-1}x)$ and*

$$(g \cdot \sigma)(x) = \beta(\xi''(su^2)^{-1} \xi''(s)u, \xi'\pi'(t))\, \sigma(g^{-1}x),$$

with $t := g_0 \xi(g^{-1}x)^2$ (which is in T or else $\sigma(g^{-1}x) = 0$), $u := \xi(g^{-1}x)^{-1} g^{-1} \xi(x) \in T$, and $s := t(\xi'\pi'(t))^{-1} \in T^{[2]}$.

Proof. Since $\pi'(tu^2) = \pi'(t)$, we have $tu^2(\xi'\pi'(tu^2))^{-1} = su^2$, and hence

$$\begin{aligned}&X_{tu^2}^{-1} X_u X_t X_u \\ &= (\beta(\xi''(su^2), \xi'\pi'(t))\, X_{\xi'\pi'(t)} X_{su^2})^{-1} X_u (\beta(\xi''(s), \xi'\pi'(t))\, X_{\xi'\pi'(t)} X_s) X_u \\ &= \beta(\xi''(su^2)^{-1} \xi''(s), \xi'\pi'(t))\, X_{su^2}^{-1} X_{\xi'\pi'(t)}^{-1} X_u X_{\xi'\pi'(t)} X_s X_u \\ &= \beta(\xi''(su^2)^{-1} \xi''(s)u, \xi'\pi'(t))\, X_{su^2}^{-1} X_{\xi'\pi'(t)}^{-1} X_{\xi'\pi'(t)} X_u X_s X_u \\ &= \beta(\xi''(su^2)^{-1} \xi''(s)u, \xi'\pi'(t)),\end{aligned}$$

in view of Lemma 31. Note that $\xi''(su^2)^{-1} \xi''(s)u \in T_{[2]}$ and hence

$$\beta(\xi''(su^2)^{-1} \xi''(s)u, \xi'\pi'(t)) \in \{\pm 1\}.$$

It remains to apply Equation (17) and the fact $\eta(u) = 1$. □

Corollary 33. *If T is an elementary 2-group then $(g \cdot \sigma)(x) = \beta(u, t)\, \sigma(g^{-1}x)$. If $|T|$ is odd then $(g \cdot \sigma)(x) = \sigma(g^{-1}x)$.*

Proof. If T is an elementary 2-group then $T^{[2]} = \{e\}$ and hence $\xi'\pi'(t) = t$ for all $t \in T$. If $|T|$ is odd then $T^{[2]} = T$ and hence $\xi'\pi'(t) = e$ for all $t \in T$. □

4. Classical central simple real Lie algebras

It is well known that a simple real Lie algebra is either a real form of a simple complex Lie algebra or a simple complex Lie algebra regarded as real. In the finite-dimensional case, the former algebras have centroid $\mathbb{R}$ (that is, they are central) and the latter have centroid $\mathbb{C}$. Here we are interested in the real forms of classical simple complex Lie algebras. As recalled in the introduction, every such Lie algebra $\mathcal{L}$ can be realized as $\mathrm{Skew}(\mathcal{R}, \varphi)'$ where $(\mathcal{R}, \varphi)$ is a semisimple finite-dimensional associative algebra with involution such that $\mathrm{Sym}(Z(\mathcal{R}), \varphi) = \mathbb{R}$, that is, $(\mathcal{R}, \varphi)$ is central simple as an algebra with involution.

Consider the affine group schemes $\mathbf{Aut}(\mathcal{L})$ and $\mathbf{Aut}(\mathcal{R}, \varphi)$. The restriction map gives a homomorphism $\theta : \mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$, which is actually an isomorphism except when $\mathcal{L}$ has type A_1 or D_4. Indeed, since we are in characteristic 0, it is sufficient to consider the homomorphism of the groups of points in the algebraic closure of the ground field, namely, $\theta_{\mathbb{C}} : \mathrm{Aut}_{\mathbb{C}}(\mathcal{R} \otimes_{\mathbb{R}} \mathbb{C}, \varphi \otimes \mathrm{id}) \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{L} \otimes_{\mathbb{R}} \mathbb{C})$, which is well known to be an isomorphism except for the said types. (It fails to be injective for type A_1 and surjective for type D_4.)

It follows (see e.g. Theorems 1.38 and 1.39 and also Remark 1.40 in [11]) that the restriction gives a bijection between the abelian group gradings on $(\mathcal{R}, \varphi)$ and on $\mathcal{L}$, which induces a bijection between the isomorphism classes of G-gradings on $(\mathcal{R}, \varphi)$ and on $\mathcal{L}$, as well as between the equivalence classes of fine gradings.

We will not consider type D_4 in this paper. As to type A_1, we will use the alternative models: $\mathcal{L} = \mathcal{R}'$ where $\mathcal{R}$ is $M_2(\mathbb{R})$ or $\mathbb{H}$, which give the two real forms of $\mathfrak{sl}_2(\mathbb{C})$, namely, $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{sl}_1(\mathbb{H}) \cong \mathfrak{su}_2$. There are analogous models for some real forms of type A_r with $r > 1$: $\mathcal{L} = \mathcal{R}'$ where $\mathcal{R}$ is $M_n(\mathbb{R})$ with $n = r+1$ or $M_n(\mathbb{H})$ with $2n = r + 1$. These models give the real forms $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{H})$, respectively, and, since the restriction map $\mathbf{Aut}(\mathcal{R}) \to \mathbf{Aut}(\mathcal{L})$ is a closed embedding with image $\mathbf{Int}(\mathcal{L})$, they will help us deal with the inner gradings on these real forms. A G-grading on $\mathcal{L}$ is said to be *inner* if the image of the corresponding homomorphism $G^D \to \mathbf{Aut}(\mathcal{L})$ (see e.g. [11, §1.4]) is contained in $\mathbf{Int}(\mathcal{L})$; otherwise the grading is called *outer*. In other words, the grading is inner if an only if the action of the character group $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^{\times})$ on the complexification $\mathcal{L}_{\mathbb{C}} := \mathcal{L} \otimes_{\mathbb{R}} \mathbb{C}$ is by inner automorphisms. (The action of the group $\widehat{G}$ on a G-graded complex algebra $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ is defined by $\chi \cdot a = \chi(g)a$ for all $\chi \in \widehat{G}$, $a \in \mathcal{A}_g$ and $g \in G$.) Therefore, if $\mathcal{L} = \mathcal{R}'$ as above, the inner G-gradings on $\mathcal{L}$ are precisely the restrictions of the G-gradings on the associative algebra $\mathcal{R}$. Of course, all gradings are inner if $\mathcal{L}$ is of type A_1.

In the models $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)'$ for type A_r with $r > 1$, the inner (respectively, outer) gradings on $\mathcal{L}$ are the restrictions of the Type I (respectively, Type II) gradings on $(\mathcal{R}, \varphi)$. Note that $\mathcal{R}$ is a graded-simple algebra in all cases except for the inner gradings on $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{H})$, which correspond to $(\mathcal{R}, \varphi)$ isomorphic $M_n(\mathbb{R}) \times M_n(\mathbb{R})$ and $M_n(\mathbb{H}) \times M_n(\mathbb{H})$, respectively, with $\varphi : (X, Y) \mapsto (Y^*, X^*)$, where $X^* := \overline{X}^T$. In these latter cases, we are going to use the alternative models.

4.1. **The global signature.** In Subsection 3.5, we classified up to isomorphism the finite-dimensional G-graded real associative algebras with involution $(\mathcal{R}, \varphi)$ that are graded-simple and central simple as algebras with involution (disregarding the grading). This gives a classification up to isomorphism of the G-graded classical central simple real Lie algebras, except those mentioned above, by restricting the gradings from $\mathcal{R}$ to $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)'$. However, we still have to identify the isomorphism class of $\mathcal{L}$ in terms of our classification parameters.

Recall the graded algebra with involution $(\mathcal{R}, \varphi) = M(\mathcal{D}, \varphi_0, g_0, \kappa, \sigma, \delta)$ from Definition 20 and assume that $(\mathcal{D}, \varphi_0)$ is central simple as an algebra with involution. If we disregard the grading, then $(\mathcal{R}, \varphi)$ is isomorphic to $M_k(\mathcal{D})$ with $\varphi(X) = \Phi^{-1}\varphi_0(X^T)\Phi$, where $k = |\kappa|$ and Φ is given by Equation (7). Hence, the type of $\mathcal{L}$ is easy to identify: it is determined by the isomorphism class of $\mathcal{D}$ as an ungraded algebra and, in the case $Z(\mathcal{R}) = \mathbb{R}$, also by δ and the type of φ_0 (orthogonal or symplectic). The isomorphism class of $\mathcal{D}$ also gives us information about which real form of a given type we obtain, but in some cases when $\mathcal{D}$ is simple, that is, $M_\ell(\Delta)$ with $\Delta \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, additional information is required to pin down the isomorphism class of $\mathcal{L}$, namely, the signature of φ as an involution on $\mathcal{R} \cong M_{k\ell}(\Delta)$. To be precise, if φ is given in matrix form by $Z \mapsto \Psi^{-1}Z^*\Psi$ where $\Psi^* = \Psi$ (that is, Ψ is a hermitian matrix, which is determined up to a real scalar) then the signature of φ is defined as the absolute value of the signature of Ψ. We will refer to this parameter as the *global signature*, and we will compute it now in terms of our signature function σ assuming that φ_0 is a distinguished involution (see Subsection 3.5).

We know from [3, §11] that the involution φ_0 on $\mathcal{D}$ is orthogonal if $\Delta = \mathbb{R}$ and symplectic if $\Delta = \mathbb{H}$. Hence, if we identify $\mathcal{D}$ with $M_\ell(\Delta)$, then $\varphi_0(Y) = \Lambda^{-1}Y^*\Lambda$ for all $Y \in M_\ell(\Delta)$ where $\Lambda \in M_\ell(\Delta)$ is hermitian. The same is true in the case $\Delta = \mathbb{C}$ because φ_0 is of the second kind. Using the Kronecker product to identify $M_k(\mathcal{D})$ with $M_k(\mathbb{R}) \otimes_\mathbb{R} M_\ell(\Delta)$, we obtain:

$$\varphi(X \otimes Y) = \Psi^{-1}(X^T \otimes Y^*)\Psi$$

for all $X \in M_k(\mathbb{R})$ and $Y \in M_\ell(\Delta)$, where $\Psi = \Psi_1 \oplus \cdots \oplus \Psi_{m+r}$ with the blocks given by $\Psi_i = S_i \otimes \Lambda X_{t_i}$ unless $\mathcal{D}_e \cong \mathbb{H}$ and $\eta(t_i) = -\delta$, and in this latter case $\Psi_i = I_{k_i} \otimes \Lambda \mathbf{i} X_{t_i}$ (recall that $\mathbf{i}$ and X_{t_i} commute).

Note that $S_i^T = \delta\eta(t_i)S_i$ and $(\Lambda X_{t_i})^* = X_{t_i}^*\Lambda = \eta(t_i)\Lambda X_{t_i}$, hence a block $S_i \otimes \Lambda X_{t_i} \in M_{k\ell}(\Delta)$ is hermitian if and only if $\delta = 1$. In the case $\mathcal{D}_e \cong \mathbb{H}$ and $\eta(t_i) = -\delta$, we have $(\Lambda\mathbf{i}X_{t_i})^* = \delta\Lambda\mathbf{i}X_{t_i}$, because $-\eta(t_i)\mathbf{i}X_{t_i} = \varphi_0(\mathbf{i}X_{t_i}) = \Lambda^{-1}(\mathbf{i}X_{t_i})^*\Lambda$. Hence, the global signature is defined if and only if $\delta = 1$. So let us assume that $\delta = 1$ and compute the signature of the matrix Ψ as the sum of the signatures of the blocks Ψ_i.

If S_i is skew-symmetric, then the bilinear form that it defines has a totally isotropic subspace of maximal dimension, so the same happens to the hermitian form defined by $\Psi_i = S_i \otimes \Lambda X_{t_i}$, which means that the signature is 0. If S_i is symmetric, or if $\mathcal{D}_e \cong \mathbb{H}$ and $\eta(t_i) = -1$, then the first factor of Ψ_i is real symmetric and the second is hermitian, hence the signature of Ψ_i is the product of their signatures. In this case, note that the signature of the involution on $M_\ell(\Delta)$ given by $Y \mapsto X_{t_i}^{-1}\Lambda^{-1}Y^*\Lambda X_{t_i}$ (respectively $Y \mapsto X_{t_i}^{-1}\mathbf{i}^{-1}\Lambda^{-1}Y^*\Lambda\mathbf{i}X_{t_i}$) is the absolute value of the signature of ΛX_{t_i} (respectively $\Lambda\mathbf{i}X_{t_i}$). The signs depend on the choice of the elements X_t.

Assume now that we are not in the case $\Delta = \mathbb{C} = \mathcal{D}_e$. If $\mathcal{D}_e \cong \mathbb{H}$ and $\eta(t_i) = -1$, we know from [3, §11] that the signature of the involution on $M_\ell(\Delta)$ given by $Y \mapsto X_{t_i}^{-1}\mathbf{i}^{-1}\varphi_0(Y)\mathbf{i}X_{t_i}$ is 0. Hence we only have to consider the blocks of the form $S_i \otimes \Lambda X_{t_i}$. We know, again from [3, §11], that the signature of the involution on $M_\ell(\Delta)$ given by $Y \mapsto X_{t_i}^{-1}\varphi_0(Y)X_{t_i}$ is ℓ if $t_i = e$, and 0 otherwise. In particular, the signs mentioned above are the same for all terms that give nonzero contribution to the sum. Therefore, the signature of Ψ equals $\pm\sum_{t_i=e}(p_i - q_i)\ell$ and hence the global signature is given by the following formula:

$$\text{signature}(\varphi) = \ell \,\Big| \sum_{\substack{x \in (G/T)_{g_0} \\ \tau(x)=e}} \sigma(x) \Big|. \tag{19}$$

In the case $\Delta = \mathbb{C} = \mathcal{D}_e$ (hence $K = T$ and $\eta = 1$), we know from [3, §11] that the distinguished involutions on $\mathcal{D}$ are those that have nonzero signature, namely, $\sqrt{|T_{[2]}|}$. Moreover, the signature of the (second kind) involution on $M_\ell(\mathbb{C})$ given by $Y \mapsto X_t^{-1}\varphi_0(Y)X_t$ is $\sqrt{|T_{[2]}|}$ if $t \in T^{[2]}$, and 0 otherwise. *We choose X_t for $t \in T^{[2]}$ in such a way that $X_uX_tX_u \in \mathbb{R}_{>0}X_{tu^2}$ for all $u \in T$.* It is shown in [3, §11] that the same elements X_t, $t \in T^{[2]}$, have the property that signature(ΛX_t) = signature(Λ).

It follows that the signature of Ψ equals $\pm\sum_{t_i\in T^{[2]}}(p_i-q_i)\sqrt{|T_{[2]}|}$ and hence the global signature is given by the following formula:

$$\operatorname{signature}(\varphi)=\sqrt{|T_{[2]}|}\,\Big|\sum_{\substack{x\in(G/T)_{g_0}\\ \tau(x)\in T^{[2]}}}\sigma(x)\Big|. \tag{20}$$

4.2. **Notation.** Whenever possible (that is, except in the case $Z(\mathcal{D})=\tilde{\mathbb{C}}$), we will use distinguished involutions φ_0. In our notation $M(\mathcal{D},\varphi_0,g_0,\kappa,\sigma,\delta)$, we will substitute for $\mathcal{D}$ the parameters that determine its isomorphism class (hence also the isomorphism class of $(\mathcal{D},\varphi_0)$ if φ_0 is distinguished) and omit the parameters that are clear from the context:

(1) If $Z(\mathcal{D})=\mathbb{R}$ then we will write $M(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ where $\mathcal{D}$ has the identity component $\Delta_0\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and is determined by (T,μ), and φ_0 is the distinguished involution.
(2) If $Z(\mathcal{D})=\mathbb{C}$, nontrivially graded, then we will write $M(\Delta_0,T,\mu,g_0,\kappa,\sigma)$ where $\mathcal{D}$ has the identity component $\Delta_0\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and is determined by (T,μ), φ_0 is the distinguished involution, and $\delta=1$.
(3) If $Z(\mathcal{D})=\mathbb{C}$, trivially graded, then we will write $M(T,\beta,g_0,\kappa,\sigma)$ where $\mathcal{D}$ has the identity component $\mathbb{C}$ and is determined by (T,β), φ_0 is a (fixed) distinguished involution, and $\delta=1$.
(4) If $Z(\mathcal{D})=\tilde{\mathbb{C}}$ then we will write $M(\Delta_0,T,\mu,\eta,g_0,\kappa,\sigma)$ where $\mathcal{D}$ has the identity component $\Delta_0\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and is determined by (T,μ), φ_0 is determined by η, and $\delta=1$.

Recall that, except in Case (3), T is an elementary 2-group and μ is either a quadratic form $T\to\{\pm1\}$ with polar form $\beta:T\times T\to\{\pm1\}$ (for $\Delta_0\in\{\mathbb{R},\mathbb{H}\}$) or a nice map $T\setminus K\to\{\pm1\}$, whose associated quadratic forms $\mu_u:K\to\{\pm1\}$, with $u\in T\setminus K$, have the same polar form $\beta:K\times K\to\{\pm1\}$ (for $\Delta_0=\mathbb{C}$). The radical of β is the support of the grading on $Z(\mathcal{D})$, which is the trivial subgroup $\{e\}$ in Cases (1) and (3) and the subgroup $\{e,f\}\subseteq K$ in Cases (2) and (4), where f is the distinguished element. Note that, if μ is a quadratic form (respectively, nice map) then $\mu(f)=-1$ (respectively, $\mu_u(f)=-1$ for any $u\in T\setminus K$) in Case (2) and $\mu(f)=+1$ (respectively, $\mu_u(f)=+1$ for any $u\in T\setminus K$) in Case (4).

The isomorphism class of $\mathcal{D}$ as an ungraded algebra is determined as follows, using the notation $\operatorname{Arf}(\mu)$ for the value $\epsilon\in\{\pm1\}$ that μ takes more often, that is, $\operatorname{Arf}(\mu)=\epsilon$ if and only if $|\mu^{-1}(\epsilon)|>|\mu^{-1}(-\epsilon)|$. If μ takes values -1 and $+1$ equally often then $\operatorname{Arf}(\mu)$ is undefined. It is defined in Cases (1) and (4) and boils down to the classical Arf invariant as follows. In Case (1), if μ is a quadratic form (respectively, nice map) then $\operatorname{Arf}(\mu)=(-1)^\alpha$ (respectively, $\operatorname{Arf}(\mu)=\mu(u)(-1)^\alpha$) where α is the Arf invariant in the field of order 2 of the quadratic form μ (respectively, μ_u). In Case (4), the only difference is that we have to consider the quadratic forms induced from μ (respectively, μ_u) modulo the radical $\{e,f\}$.

(1) If $Z(\mathcal{D})=\mathbb{R}$ then $\mathcal{D}\cong M_\ell(\Delta)$ where
   - for $\Delta_0\in\{\mathbb{R},\mathbb{C}\}$: if $\operatorname{Arf}(\mu)=+1$ then $\Delta=\mathbb{R}$ and $\ell^2=|T|\dim\Delta_0$, and if $\operatorname{Arf}(\mu)=-1$ then $\Delta=\mathbb{H}$ and $\ell^2=\frac14|T|\dim\Delta_0$;
   - for $\Delta_0=\mathbb{H}$: if $\operatorname{Arf}(\mu)=+1$ then $\Delta=\mathbb{H}$ and $\ell^2=\frac14|T|\dim\Delta_0=|T|$, and if $\operatorname{Arf}(\mu)=-1$ then $\Delta=\mathbb{R}$ and $\ell^2=|T|\dim\Delta_0=4|T|$.

(2) If $Z(\mathcal{D})=\mathbb{C}$, nontrivially graded, then $\mathcal{D}\cong M_\ell(\mathbb{C})$ where $\ell^2=\frac12|T|\dim\Delta_0$.
(3) If $Z(\mathcal{D})=\mathbb{C}$, trivially graded, then $\mathcal{D}\cong M_\ell(\mathbb{C})$ where $\ell^2=|T|$.

(4) If $Z(\mathcal{D}) = \tilde{\mathbb{C}}$ then $\mathcal{D} \cong M_\ell(\Delta) \times M_\ell(\Delta)$ where
- for $\Delta_0 \in \{\mathbb{R}, \mathbb{C}\}$: if $\operatorname{Arf}(\mu) = +1$ then $\Delta = \mathbb{R}$ and $\ell^2 = \frac{1}{2}|T| \dim \Delta_0$, and if $\operatorname{Arf}(\mu) = -1$ then $\Delta = \mathbb{H}$ and $\ell^2 = \frac{1}{8}|T| \dim \Delta_0$;
- for $\Delta_0 = \mathbb{H}$: if $\operatorname{Arf}(\mu) = +1$ then $\Delta = \mathbb{H}$ and $\ell^2 = \frac{1}{8}|T| \dim \Delta_0 = \frac{1}{2}|T|$, and if $\operatorname{Arf}(\mu) = -1$ then $\Delta = \mathbb{R}$ and $\ell^2 = \frac{1}{2}|T| \dim \Delta_0 = 2|T|$.

Recall that κ belongs to the set $\mathsf{K}(G, \mathcal{D}, \varphi_0, g_0, \delta)$ of admissible multiplicity functions $G/T \to \mathbb{Z}_{\geq 0}$ as in Definition 16, where we substitute $\mathcal{D}_e \cong \mathbb{C}$ in Case (3) and $\mathcal{D}_e \cong \Delta_0$ in all other cases, $\delta = 1$ except in Case (1), and also $\eta = 1$ in Case (3) and $\eta = \mu$ in Cases (1) and (2). Note that, in Case (3), we have $K = T$ and the definition boils down to the following: $|\kappa| < \infty$ and $\kappa(g_0^{-1}x^{-1}) = \kappa(x)$ for all $x \in G/T$. Also, σ belongs to the set $\Sigma(G, \mathcal{D}, \varphi_0, g_0, \kappa, \delta)$ of signature functions as in Definition 17, where we make the same substitutions.

We will also need the graded matrix algebra $M_k(\mathcal{D})$ without involution, where $\mathcal{D}$ is as in Case (1) and the grading is determined according to Equation (1) by an arbitrary multiplicity function $\kappa : G/T \to \mathbb{Z}_{\geq 0}$ satisfying $|\kappa| = k$. We will denote this graded algebra by $M(\Delta_0, T, \mu, \kappa)$.

The group G acts on multiplicity functions in the natural way: $(g \cdot \kappa)(x) := \kappa(g^{-1}x)$ for all $x \in G/T$. It also acts on the triples $(g_0, \kappa, \sigma) \in \mathsf{X}(G, \mathcal{D}, \varphi_0, \delta)$ as described by Proposition 30 in Cases (1), (2) and (4), and by Proposition 32 in Case (3). Since the triples belonging to the same G-orbit produce isomorphic graded algebras with involution and since the action of $g \in G$ replaces g_0 by $g^{-2}g_0$, we can fix a transversal Θ for the subgroup $G^{[2]}$ in G and insist that $g_0 \in \Theta$.

For a given g_0, its stabilizer $G_{[2]}$ acts on the set of pairs (κ, σ) where $\kappa \in \mathsf{K}(G, \mathcal{D}, \varphi_0, g_0, \delta)$ and $\sigma \in \Sigma(G, \mathcal{D}, \varphi_0, g_0, \kappa, \delta)$. Taking into account our substitutions, we will denote this set of pairs by $\mathsf{X}(G, \Delta_0, T, \mu, g_0, \delta)$ in Case (1), $\mathsf{X}(G, \Delta_0, T, \mu, g_0)$ in Case (2), $\mathsf{X}(G, T, g_0)$ in Case (3), and $\mathsf{X}(G, \Delta_0, T, \eta, g_0)$ in Case (4). The action of $G_{[2]}$ on this set is given by $g \cdot (\kappa, \sigma) = (g \cdot \kappa, g \cdot \sigma)$ where $g \cdot \kappa$ is as above, but $g \cdot \sigma$ is more complicated: the absolute value $|\sigma| : G/T \to \mathbb{Z}_{\geq 0}$ is transformed in the same way as κ, but there are sign changes given by Propositions 30 and 32. For example, in Cases (1) and (2), the first of these propositions gives

$$(g \cdot \sigma)(x) = \beta(\xi(g^{-1}x)^{-1}g^{-1}\xi(x), \tau(x))\, \sigma(g^{-1}x)$$

for all $g \in G_{[2]}$ and $x \in (G/T)_{g_0}$. (Note that the set $(G/T)_{g_0}$ and the function $\tau : (G/T)_{g_0} \to T$ are invariant under $G_{[2]}$ and, by definition, $\sigma(x) = 0$ unless $x \in (G/T)_{g_0}$.) Finally, the group $\{\pm 1\}$ acts on the pairs (κ, σ) by $\epsilon \cdot (\kappa, \sigma) := (\kappa, \epsilon\sigma)$ for $\epsilon \in \{\pm 1\}$.

4.3. **Series A: inner gradings on special linear Lie algebras.** Let $\mathcal{R} = M(\Delta_0, T, \mu, \kappa)$ as defined above. The G-grading of $\mathcal{R}$ restricts to an inner grading on its Lie subalgebra $\mathcal{R}'$, which will be denoted by $\Gamma^{(\mathrm{I})}_{\mathfrak{sl}}(\Delta_0, T, \mu, \kappa)$. Fixing an isomorphism $\mathcal{D} \cong M_\ell(\Delta)$, $\Delta \in \{\mathbb{R}, \mathbb{H}\}$, we may identify $\mathcal{R}$ with $M_n(\Delta)$, where $n = |\kappa|\ell$, and hence regard $\Gamma^{(\mathrm{I})}_{\mathfrak{sl}}(\Delta_0, T, \mu, \kappa)$ as a grading on $\mathfrak{sl}_n(\Delta)$. Note that this identification depends on the choice of the isomorphism $\mathcal{D} \cong M_\ell(\Delta)$, but the isomorphism class of the grading on $\mathfrak{sl}_n(\Delta)$ does not.

Theorem 34. *Let $\mathcal{L}$ be one of the real special linear Lie algebras of type A_r, namely, $\mathfrak{sl}_{r+1}(\mathbb{R})$ or $\mathfrak{sl}_{(r+1)/2}(\mathbb{H})$ (if r is odd). Then any inner G-grading on $\mathcal{L}$ is isomorphic to $\Gamma^{(\mathrm{I})}_{\mathfrak{sl}}(\Delta_0, T, \mu, \kappa)$ where $r = |\kappa|\sqrt{|T| \dim \Delta_0} - 1$ and, in the first case, $\operatorname{Arf}(\mu) = 1$ if $\Delta_0 \in \{\mathbb{R}, \mathbb{C}\}$ and $\operatorname{Arf}(\mu) = -1$ if $\Delta_0 = \mathbb{H}$, while in the second* *case,* $\mathrm{Arf}(\mu) = -1$ *if* $\Delta_0 \in \{\mathbb{R}, \mathbb{C}\}$ *and* $\mathrm{Arf}(\mu) = 1$ *if* $\Delta_0 = \mathbb{H}$. *Moreover, two such gradings,* $\Gamma^{(\mathrm{I})}_{\mathfrak{sl}}(\Delta_0, T, \mu, \kappa)$ *and* $\Gamma^{(\mathrm{I})}_{\mathfrak{sl}}(\Delta'_0, T', \mu', \kappa')$*, are isomorphic if and only if* $\Delta_0 = \Delta'_0$*,* $T = T'$*,* $\mu = \mu'$*, and* κ' *is in the union of the* G*-orbits of* κ *and* $\bar{\kappa}$*, where* $\bar{\kappa}(x) := \kappa(x^{-1})$ *for all* $x \in G/T$.

Proof. Any inner grading on $\mathcal{L}$ uniquely extends to $M_n(\Delta)$ where $n = r+1$ if $\Delta = \mathbb{R}$ and $2n = r + 1$ if $\Delta = \mathbb{H}$. The resulting graded algebra is isomorphic to some $\mathcal{R} = M(\Delta_0, T, \mu, \kappa)$ where the parameters must satisfy the indicated conditions. Finally, two gradings on $\mathcal{L}$ are isomorphic if and only if the corresponding graded algebras $\mathcal{R}$ and $\mathcal{R}'$ are either isomorphic or anti-isomorphic. $\square$

4.4. **Series A: inner gradings on special unitary Lie algebras.** Let $(\mathcal{R}, \varphi) = M(T, \beta, g_0, \kappa, \sigma)$ as defined in Subsection 4.2, so $\mathcal{R} \cong M_n(\mathbb{C})$ as an ungraded algebra, $n = |\kappa|\ell$, and φ is of the second kind. The restriction of the G-grading of $\mathcal{R}$ to its Lie subalgebra $\mathrm{Skew}(\mathcal{R}, \varphi)'$ is inner and will be denoted by $\Gamma^{(\mathrm{I})}_{\mathfrak{su}}(T, \beta, g_0, \kappa, \sigma)$.

Theorem 35. *Let* $\mathcal{L}$ *be one of the special unitary Lie algebras of type* A_r *for* $r \geq 2$*, namely,* $\mathfrak{su}(p, q)$ *where* $p + q = r + 1$*,* $p \geq q$*. Then any inner* G*-grading on* $\mathcal{L}$ *is isomorphic to* $\Gamma^{(\mathrm{I})}_{\mathfrak{su}}(T, \beta, g_0, \kappa, \sigma)$ *where* $g_0 \in \Theta$*,* $r = |\kappa|\sqrt{|T|} - 1$*, and* $p - q$ *equals the right-hand side of Equation* (20)*. Moreover, two such gradings,* $\Gamma^{(\mathrm{I})}_{\mathfrak{su}}(T, \beta, g_0, \kappa, \sigma)$ *and* $\Gamma^{(\mathrm{I})}_{\mathfrak{su}}(T', \beta', g'_0, \kappa', \sigma')$*, are isomorphic if and only if* $T = T'$*,* $\beta = \beta'$*,* $g_0 = g'_0$*, and*

- *when* T *is not an elementary* 2*-group:* (κ', σ') *is in the* $G_{[2]} \times \{\pm 1\}$*-orbit of* (κ, σ) *in the set* $\mathsf{X}(G, T, g_0)$*;*
- *when* T *is an elementary* 2*-group:* (κ', σ') *is in the union of the* $G_{[2]} \times \{\pm 1\}$*-orbits of* (κ, σ) *and* $(\kappa, \nu\sigma)$*, where* $\nu : T \to \{\pm 1\}$ *is a quadratic form with polar form* β *and* $(\nu\sigma)(x) := \nu(\tau(x))\sigma(x)$ *for all* $x \in G/T$.

Proof. Any inner grading on $\mathcal{L}$ uniquely extends to a grading on $M_n(\mathbb{C})$ compatible with the (second kind) involution defining $\mathcal{L}$ and giving the center $\mathbb{C}$ the trivial grading. By Theorem 18, the resulting graded algebra with involution must be isomorphic to some $M(T, \beta, g_0, \kappa, \sigma)$, since we are in Case (3) of Subsection 4.2. This proves the first assertion. The second assertion follows from Corollary 28. $\square$

4.5. **Series A: outer gradings on special linear Lie algebras.** Let $(\mathcal{R}, \varphi) = M(\Delta_0, T, \mu, \eta, g_0, \kappa, \sigma)$ as defined in Subsection 4.2, so $\mathcal{R} \cong M_n(\Delta) \times M_n(\Delta)$ as an ungraded algebra, $\Delta \in \{\mathbb{R}, \mathbb{H}\}$, $n = |\kappa|\ell$, and φ is of the second kind. Recall that we may use an arbitrary second kind involution φ_0 on $\mathcal{D}$. *For each* (Δ_0, T, μ)*, we fix a quadratic form (respectively, nice map)* η *if* $\Delta_0 \in \{\mathbb{R}, \mathbb{H}\}$ *(respectively, if* $\Delta_0 = \mathbb{C}$*) such that* $\eta(f) = -1$ *(respectively,* $\eta_u(f) = -1$*).* The restriction of the G-grading of $\mathcal{R}$ to its Lie subalgebra $\mathrm{Skew}(\mathcal{R}, \varphi)' \cong \mathfrak{sl}_n(\Delta)$ is outer and will be denoted by $\Gamma^{(\mathrm{II})}_{\mathfrak{sl}}(\Delta_0, T, \mu, \eta, g_0, \kappa, \sigma)$.

Theorem 36. *Let* $\mathcal{L}$ *be one of the real special linear Lie algebras of type* A_r *for* $r \geq 2$*, namely,* $\mathfrak{sl}_{r+1}(\mathbb{R})$ *or* $\mathfrak{sl}_{(r+1)/2}(\mathbb{H})$ *(if* r *is odd). Then any outer* G*-grading on* $\mathcal{L}$ *is isomorphic to* $\Gamma^{(\mathrm{II})}_{\mathfrak{sl}}(\Delta_0, T, \mu, \eta, g_0, \kappa, \sigma)$ *where* $g_0 \in \Theta$*,* $r = |\kappa|\sqrt{\frac{1}{2}|T| \dim \Delta_0} - 1$ *and, in the first case,* $\mathrm{Arf}(\mu) = 1$ *if* $\Delta_0 \in \{\mathbb{R}, \mathbb{C}\}$ *and* $\mathrm{Arf}(\mu) = -1$ *if* $\Delta_0 = \mathbb{H}$*, while in the second case,* $\mathrm{Arf}(\mu) = -1$ *if* $\Delta_0 \in \{\mathbb{R}, \mathbb{C}\}$ *and* $\mathrm{Arf}(\mu) = 1$ *if* $\Delta_0 = \mathbb{H}$*. Moreover, two such gradings,* $\Gamma^{(\mathrm{II})}_{\mathfrak{sl}}(\Delta_0, T, \mu, \eta, g_0, \kappa, \sigma)$ *and* $\Gamma^{(\mathrm{II})}_{\mathfrak{sl}}(\Delta'_0, T', \mu', \eta', g'_0, \kappa', \sigma')$*, are isomorphic if and only if* $\Delta_0 = \Delta'_0$*,* $T = T'$*,* $\mu = \mu'$ *(hence* $\eta = \eta'$*),* $g_0 = g'_0$*, and*

(κ',σ') is in the union of the $G_{[2]}\times\{\pm1\}$-orbits of (κ,σ) and $(\kappa,\nu\sigma)$ in the set $\mathsf{X}(G,\Delta_0,T,\eta,g_0)$, where $\nu: K\to\{\pm1\}$ is a homomorphism satisfying $\nu(f)=-1$ and $(\nu\sigma)(x):=\nu(\tau(x))\sigma(x)$ for all $x\in G/T$.

Proof. Any outer grading on $\mathcal{L}$ uniquely extends to a grading on $M_n(\Delta)\times M_n(\Delta)$ compatible with the (second kind) involution defining $\mathcal{L}$ and giving the center $\tilde{\mathbb{C}}$ a nontrivial grading. By Theorem 18, the resulting graded algebra with involution must be isomorphic to some $M(\Delta_0,T,\mu,\eta,g_0,\kappa,\sigma)$, since we are in Case (4) of Subsection 4.2. This proves the first assertion. The second assertion follows from Corollary 27. ☐

4.6. **Series A: outer gradings on special unitary Lie algebras.** Let $(\mathcal{R},\varphi)=M(\Delta_0,T,\mu,g_0,\kappa,\sigma)$ as defined in Subsection 4.2, so $\mathcal{R}\cong M_n(\mathbb{C})$ as an ungraded algebra, $n=|\kappa|\ell$, and φ is of the second kind. The restriction of the G-grading of $\mathcal{R}$ to its Lie subalgebra $\mathrm{Skew}(\mathcal{R},\varphi)'$ is outer and will be denoted by $\Gamma^{(\mathrm{II})}_{\mathfrak{su}}(\Delta_0,T,\mu,g_0,\kappa,\sigma)$.

The proof of the next result is analogous to Theorem 36, with Case (2) instead of Case (4).

Theorem 37. *Let $\mathcal{L}$ be one of the special unitary Lie algebras of type A_r for $r\geq 2$, namely, $\mathfrak{su}(p,q)$ where $p+q=r+1$, $p\geq q$. Then any outer G-grading on $\mathcal{L}$ is isomorphic to $\Gamma^{(\mathrm{II})}_{\mathfrak{su}}(\Delta_0,T,\mu,g_0,\kappa,\sigma)$ where $g_0\in\Theta$, $r=|\kappa|\sqrt{\frac{1}{2}|T|\dim\Delta_0}-1$, and $p-q$ equals the right-hand side of Equation (19). Moreover, two such gradings, $\Gamma^{(\mathrm{II})}_{\mathfrak{su}}(\Delta_0,T,\mu,g_0,\kappa,\sigma)$ and $\Gamma^{(\mathrm{II})}_{\mathfrak{sl}}(\Delta_0',T',\mu',g_0',\kappa',\sigma')$, are isomorphic if and only if $\Delta_0=\Delta_0'$, $T=T'$, $\mu=\mu'$, $g_0=g_0'$, and (κ',σ') is in the union of the $G_{[2]}\times\{\pm1\}$-orbits of (κ,σ) and $(\kappa,\nu\sigma)$ in the set $\mathsf{X}(G,\Delta_0,T,\mu,g_0)$, where $\nu: K\to\{\pm1\}$ is a homomorphism satisfying $\nu(f)=-1$ and $(\nu\sigma)(x):=\nu(\tau(x))\sigma(x)$ for all $x\in G/T$.* ☐

4.7. **Series B.** For series B, C and D, we deal with central simple associative algebras over $\mathbb{R}$, so we are in Case (1) of Subsection 4.2. The special feature of series B is that the degree of this algebra is odd, so it must be $M_n(\mathbb{R})$ with odd n. This simplifies the situation dramatically.

Let $(\mathcal{R},\varphi)=M(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ such that $\mathcal{R}\cong M_n(\mathbb{R})$ as an ungraded algebra and $n=|\kappa|\ell$ is odd. We must have $\Delta_0=\mathbb{R}$ and, since T is an elementary 2-group, we must also have $T=\{e\}$. The involution is orthogonal, so $\delta=1$. Moreover, since $|\kappa|$ is odd and $\kappa: G\to\mathbb{Z}_{\geq 0}$ is an admissible multiplicity function, part (b) of Definition 16 implies that $g_0g^2=e$ for some $g\in G$. Therefore, we can make $g_0=e$ using the action of G. Note that the definition of signature function $\sigma: G\to\mathbb{Z}$ becomes: $\sigma(x)=0$ for all $x\notin G_{[2]}$, while $|\sigma(x)|\leq\kappa(x)$ and $\sigma(x)\equiv\kappa(x)\pmod 2$ for all $x\in G_{[2]}$. The action of $G_{[2]}$ on signature functions is just $(g\cdot\sigma)(x)=\sigma(g^{-1}x)$ for all $x\in G$.

The restriction of the G-grading of $M(\mathbb{R},\{e\},1,e,\kappa,\sigma,1)$, with odd $|\kappa|$, to its Lie subalgebra $\mathrm{Skew}(\mathcal{R},\varphi)$ will be denoted by $\Gamma_B(\kappa,\sigma)$.

Theorem 38. *Let $\mathcal{L}$ be one of the real forms of type B_r for $r\geq 2$, namely, $\mathfrak{so}_{p,q}(\mathbb{R})$ where $p+q=2r+1$, $p\geq q$. Then any G-grading on $\mathcal{L}$ is isomorphic to $\Gamma_B(\kappa,\sigma)$ where $r=\frac{1}{2}(|\kappa|-1)$ and $p-q=|\sum_{g\in G_{[2]}}\sigma(g)|$. Moreover, two such gradings, $\Gamma_B(\kappa,\sigma)$ and $\Gamma_B(\kappa',\sigma')$, are isomorphic if and only if (κ,σ) and (κ',σ') are in the same $G_{[2]}\times\{\pm1\}$-orbit.*

Proof. Any grading on $\mathcal{L}$ uniquely extends to a grading on $M_{2r+1}(\mathbb{R})$ compatible with the involution defining $\mathcal{L}$. By Theorem 18 and the above discussion, the resulting graded algebra with involution must be isomorphic to $M(\mathbb{R},\{e\},1,e,\kappa,\sigma,1)$ for some κ and σ. Note that $p-q=\text{signature}(\varphi)$ is given by Equation (19), which simplifies in view of the fact $T=\{e\}$. This proves the first assertion. The second assertion follows from Corollary 26. □

Remark 39. This result is valid for $r=1$ as well, and gives another parametrization of gradings for the real forms of type A_1: $\mathfrak{so}_{3,0}(\mathbb{R})\cong\mathfrak{su}(2)\cong\mathfrak{sl}_1(\mathbb{H})$ and $\mathfrak{so}_{2,1}(\mathbb{R})\cong\mathfrak{su}(1,1)\cong\mathfrak{sl}_2(\mathbb{R})$.

4.8. **Series C.** Let $(\mathcal{R},\varphi)=M(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ as defined in Subsection 4.2, where $\mathrm{Arf}(\mu)=-\delta$ if $\Delta_0\in\{\mathbb{R},\mathbb{C}\}$ and $\mathrm{Arf}(\mu)=\delta$ if $\Delta_0=\mathbb{H}$, so that φ is a symplectic involution. The restriction of the G-grading of $\mathcal{R}$ to its Lie subalgebra $\mathrm{Skew}(\mathcal{R},\varphi)$ will be denoted by $\Gamma_C(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$.

Theorem 40. *Let $\mathcal{L}$ be one of the real forms of type C_r for $r\geq 2$, namely, $\mathfrak{sp}_{2r}(\mathbb{R})$ or $\mathfrak{sp}(p,q)$ where $p+q=r$, $p\geq q$. Then any G-grading on $\mathcal{L}$ is isomorphic to $\Gamma_C(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ where $g_0\in\Theta$, $r=\frac{1}{2}|\kappa|\sqrt{|T|\dim\Delta_0}$ and, in the first case, $\delta=-1$, while in the second case, $\delta=1$ and $p-q$ equals the right-hand side of Equation* (19). *Moreover, two such gradings, $\Gamma_C(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ and $\Gamma_C(\Delta_0',T',\mu',g_0',\kappa',\sigma',\delta')$, are isomorphic if and only if $\Delta_0=\Delta_0'$, $T=T'$, $\mu=\mu'$, $g_0=g_0'$, $\delta=\delta'$, and (κ,σ) and (κ',σ') are in the same $G_{[2]}\times\{\pm1\}$-orbit in the set $\mathsf{X}(G,\Delta_0,T,\mu,g_0,\delta)$.*

Proof. If we give the algebra $M_{2r}(\mathbb{R})$ or $M_r(\mathbb{H})$ a G-grading that is compatible with a fixed symplectic involution, then, by Theorem 18, the resulting graded algebra with involution must be isomorphic to some $M(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ as above. If $\delta=-1$ then $\mathrm{Arf}(\mu)=1$ for $\Delta_0\in\{\mathbb{R},\mathbb{C}\}$ and $\mathrm{Arf}(\mu)=-1$ for $\Delta_0=\mathbb{H}$, hence $\mathcal{D}\cong M_\ell(\mathbb{R})$ as an ungraded algebra, where $\ell=\sqrt{|T|\dim\Delta_0}$. This implies that $\mathrm{Skew}(\mathcal{R},\varphi)\cong\mathfrak{sp}_{k\ell}(\mathbb{R})$ where $k=|\kappa|$. Similarly, if $\delta=1$ then $\mathcal{D}\cong M_\ell(\mathbb{H})$ as an ungraded algebra, where $\ell=\frac{1}{2}\sqrt{|T|\dim\Delta_0}$, and this implies $\mathrm{Skew}(\mathcal{R},\varphi)\cong\mathfrak{sp}(p,q)$ where $p+q=k\ell$ and $p-q=\text{signature}(\varphi)$. This proves the first assertion. The second assertion follows from Corollary 26. □

Remark 41. This result is valid for $r=1$ as well, and gives yet another parametrization of gradings for the real forms of type A_1: $\mathfrak{sp}_2(\mathbb{R})=\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{sp}(1,0)=\mathfrak{sl}_1(\mathbb{H})$. We also get another parametrization of gradings for type $B_2=C_2$.

4.9. **Series D.** Let $(\mathcal{R},\varphi)=M(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ as defined in Subsection 4.2, where $\mathrm{Arf}(\mu)=\delta$ if $\Delta_0\in\{\mathbb{R},\mathbb{C}\}$ and $\mathrm{Arf}(\mu)=-\delta$ if $\Delta_0=\mathbb{H}$, so that φ is an orthogonal involution. The restriction of the G-grading of $\mathcal{R}$, with even $|\kappa|\ell$ if $\delta=1$, to its Lie subalgebra $\mathrm{Skew}(\mathcal{R},\varphi)$ will be denoted by $\Gamma_D(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$.

The proof of the next result is completely analogous to Theorem 40.

Theorem 42. *Let $\mathcal{L}$ be one of the real forms of type D_r for $r=3$ or $r\geq 5$, namely, $\mathfrak{u}^*(r)$ or $\mathfrak{so}_{p,q}(\mathbb{R})$ where $p+q=2r$, $p\geq q$. Then any G-grading on $\mathcal{L}$ is isomorphic to $\Gamma_D(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ where $g_0\in\Theta$, $r=\frac{1}{2}|\kappa|\sqrt{|T|\dim\Delta_0}$ and, in the first case, $\delta=-1$, while in the second case, $\delta=1$ and $p-q$ equals the right-hand side of Equation* (19). *Moreover, two such gradings, $\Gamma_D(\Delta_0,T,\mu,g_0,\kappa,\sigma,\delta)$ and $\Gamma_D(\Delta_0',T',\mu',g_0',\kappa',\sigma',\delta')$, are isomorphic if and only if $\Delta_0=\Delta_0'$, $T=T'$,*

$\mu=\mu'$, $g_0=g_0'$, $\delta=\delta'$, and (κ,σ) and (κ',σ') are in the same $G_{[2]}\times\{\pm1\}$-orbit in the set $\mathsf{X}(G,\Delta_0,T,\mu,g_0,\delta)$. $\square$

Remark 43. This theorem gives another parametrization of gradings for type $A_3=D_3$.

5. Appendix: special linear Lie algebras

Type II gradings on $\mathfrak{sl}_n(\mathbb{F})$ (or $\mathfrak{psl}_n(\mathbb{F})$ if $\operatorname{char}\mathbb{F}$ divides n) over an algebraically closed field $\mathbb{F}$, $\operatorname{char}\mathbb{F}\neq 2$, were constructed and classified in [2, 9, 11] by means of refining a grading on the associative algebra $M_n(\mathbb{F})$ with a suitable (degree-preserving) anti-automorphism of this algebra. In this paper, we have classified gradings on $\mathfrak{sl}_n(\Delta)$, $\Delta\in\{\mathbb{R},\mathbb{H}\}$, in terms of gradings on the associative algebra $M_n(\Delta)\times M_n(\Delta)$ with involution of the second kind (Theorem 36). The purpose of this section is to establish a link between these two models.

Let $\mathbb{F}$ be a field that is either algebraically closed with $\operatorname{char}\mathbb{F}\neq 2$ or real closed (and hence $\operatorname{char}\mathbb{F}=0$). Let $\mathcal{R}$ be a central simple associative algebra over $\mathbb{F}$ and let $\tilde{\mathcal{R}}=\mathcal{R}\otimes_{\mathbb{F}}\mathbb{K}$, where $\mathbb{K}=\mathbb{F}\times\mathbb{F}$. Suppose $\tilde{\mathcal{R}}$ is given a G-grading of Type II, that is, its center $\mathbb{K}$ is nontrivially graded: $\mathbb{K}=\mathbb{K}_e\oplus\mathbb{K}_f$ where $\mathbb{K}_e=\mathbb{F}$, $\mathbb{K}_f=\mathbb{F}\zeta$, $\zeta=(1,-1)$, and $f\in G$ is an element of order 2, referred to as the distinguished element.

Assume that there exists a character $\chi:G\to\mathbb{F}^\times$ such that $\chi(f)=-1$. (This is automatic if $\mathbb{F}$ is algebraically closed, but note that χ cannot always be chosen to satisfy $\chi^2=1$.) Fix one such χ and consider its action on $\tilde{\mathcal{R}}$ associated to the G-grading: $\psi(a):=\chi\cdot a=\chi(g)a$ for all $a\in\tilde{\mathcal{R}}_g$ and $g\in G$. Since $\chi\cdot\zeta=-\zeta$, the automorphism ψ interchanges the central idempotents $\varepsilon_1=(1,0)$ and $\varepsilon_2=(0,1)$, and hence we can use the restriction $\psi:\tilde{\mathcal{R}}\varepsilon_1\to\tilde{\mathcal{R}}\varepsilon_2$ to identify these simple components of $\tilde{\mathcal{R}}$ with each other. Thus, we can identify $\tilde{\mathcal{R}}$ with $\mathcal{R}\times\mathcal{R}$ such that $(x,y)\in\mathcal{R}\times\mathcal{R}$ corresponds to $x\varepsilon_1+\psi(y)\varepsilon_2\in\tilde{\mathcal{R}}$, where we regard $\mathcal{R}$ as a subalgebra of $\tilde{\mathcal{R}}$ through the canonical map $x\mapsto x\otimes 1$. Then the action of χ on $\mathcal{R}\times\mathcal{R}$ is given by $\chi\cdot(x,y)=(\psi^2(y),x)$.

Consider the coarsening of the grading on $\tilde{\mathcal{R}}$ associated to the quotient map $G\to\overline{G}:=G/\langle f\rangle$, that is, $\tilde{\mathcal{R}}=\bigoplus_{\bar g\in\overline{G}}\tilde{\mathcal{R}}_{\bar g}$ where $\tilde{\mathcal{R}}_{\bar g}=\tilde{\mathcal{R}}_g\oplus\tilde{\mathcal{R}}_{gf}$ for all $g\in G$. The idempotents ε_1 and ε_2 have degree $\bar e$, hence the simple components of $\tilde{\mathcal{R}}$ are $\overline{G}$-graded and so is its subalgebra $\mathcal{R}$. Moreover, the projections pr_1 and pr_2 of $\mathcal{R}\times\mathcal{R}$ onto $\mathcal{R}$ are homomorphisms of $\overline{G}$-graded algebras. The action of χ can be used to recover the G-grading on $\tilde{\mathcal{R}}$ from its coarsening as follows:

$$\tilde{\mathcal{R}}_g=\{a\in\tilde{\mathcal{R}}_{\bar g}\mid\chi\cdot a=\chi(g)a\}=\{(x,\chi(g)^{-1}x)\mid x\in\mathcal{R}_{\bar g}\}.$$

Since $\mathcal{R}$ is simple, we can identify $\mathcal{R}=\operatorname{End}_{\mathcal{D}}(\mathcal{V})$ as a $\overline{G}$-graded algebra, for some graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$. Let $\tilde{\mathcal{D}}=\mathcal{D}\times\mathcal{D}$ and $\tilde{\mathcal{V}}=\mathcal{V}\times\mathcal{V}$. We can refine the $\overline{G}$-gradings on $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{V}}$ by setting $\tilde{\mathcal{D}}_g:=\{(d,\chi(g)^{-1}d)\mid d\in\mathcal{D}_{\bar g}\}$ and similarly for $\mathcal{V}$. Then $\tilde{\mathcal{V}}$ is a graded right $\tilde{\mathcal{D}}$-module and $\tilde{\mathcal{R}}\cong\operatorname{End}_{\tilde{\mathcal{D}}}(\tilde{\mathcal{V}})$ as a G-graded algebra.

Remark 44. The above is an example of the loop construction, which makes G-graded algebras and modules from $\overline{G}$-graded ones (see [1, 13]).

It is easy to see that $\tilde{\mathcal{D}}$ is a graded-division algebra with $\tilde{\mathcal{D}}_e\cong\mathcal{D}_e$, the support T of $\tilde{\mathcal{D}}$ contains f, and the support of $\mathcal{D}$ is $\overline{T}:=T/\langle f\rangle$. Moreover, the alternating bicharacter β of $\tilde{\mathcal{D}}$ has radical $\langle f\rangle$ and induces the nondegenerate bicharecter $\bar{\beta}$ of $\mathcal{D}$ passing to the quotient modulo $\langle f\rangle$.

Now, any involution of the second kind on $\mathcal{R}\times\mathcal{R}$ has the form $\tilde{\varphi}(x,y) = (\varphi(y),\varphi^{-1}(x))$ where φ is an anti-automorphism of $\mathcal{R}$. Moreover, $\tilde{\varphi}$ is an involution of $\mathcal{R}\times\mathcal{R}$ as a G-graded algebra if and only if φ is an anti-automorphism of $\mathcal{R}$ as a $\overline{G}$-graded algebra and $\varphi^2(x) = \chi^2\cdot x$ for all $x\in\mathcal{R}$. (Note that $\chi^2(f) = 1$, so χ^2 can be regarded as a character of $\overline{G}$ and therefore acts on the $\overline{G}$-graded algebra $\mathcal{R}$; its action is the restriction of ψ^2.) If this is the case, the G-grading on $\tilde{\mathcal{R}}$ restricts to its Lie subalgebra $\mathcal{L} = \mathrm{Skew}(\tilde{\mathcal{R}},\tilde{\varphi})$, which is isomorphic to $\mathcal{R}^{(-)}$ by means of pr_1. Since $\mathcal{L} = \{(x,-\varphi^{-1}(x)) \mid x\in\mathcal{R}\}$, the isomorphism $\mathrm{pr}_1|_{\mathcal{L}}$ maps $\mathcal{L}_g$ onto

$$\mathcal{R}_g = \{x\in\mathcal{R}_{\bar{g}} \mid \varphi(x) = -\chi(g)x\}.$$

Note that $\mathcal{R} = \bigoplus_{g\in G}\mathcal{R}_g$ is a grading of the Lie algebra $\mathcal{R}^{(-)}$ but not of the associative algebra $\mathcal{R}$. It restricts to the Lie subalgebra $[\mathcal{R},\mathcal{R}]$, which is a special linear algebra. If $\mathrm{char}\,\mathbb{F} > 0$ then $\mathcal{R}' = [\mathcal{R},\mathcal{R}]$ may have a 1-dimensional center; passing modulo the center, we obtain a grading on the corresponding projective special linear algebra, which is simple.

If $\mathbb{F}$ is real closed then we know that the existence of $\tilde{\varphi}$ forces T to be an elementary 2-group. The same is true if $\mathbb{F}$ is algebraically closed: because of φ, $\overline{T}$ must be an elementary 2-group (see e.g. [11, Lemma 2.50]) and the same argument as in Subsection 3.5 shows that f cannot be a square in T. Therefore, χ always takes values ±1 on T, and we can write $T = \overline{T}\times\langle f\rangle$ by identifying $\overline{T}$ with the subgroup $\{t\in T \mid \chi(t) = 1\}$ (compare with Proposition 3.25 in [11], where H stands for our T). Hence, $\mathcal{D}$ can be considered G-graded and $\tilde{\mathcal{D}}$ can be identified with $\mathcal{D}\otimes_{\mathbb{F}}\mathbb{K}$ as a G-graded algebra. If $\mathbb{F}$ is algebraically closed, then $\mathcal{D}_e = \mathbb{F}$ and the isomorphism class of $\mathcal{D}$ is determined by $(\overline{T},\bar{\beta})$. Using the same approach as in Section 4, we can obtain the following analog of Theorem 36 for algebraically closed fields.

Let $\tilde{\mathcal{R}} = \mathrm{End}_{\tilde{\mathcal{D}}}(\tilde{\mathcal{V}})$ where $\tilde{\mathcal{D}}$ is determined by (T,β) and $\tilde{\mathcal{V}}$ is determined by a multiplicity function $\kappa : G/T\to\mathbb{Z}_{\geq 0}$ (with finite support). For each (T,β), we fix a quadratic form η on T with polar form β such that $\eta(f) = -1$. This gives us an involution $\tilde{\varphi}_0$ of the second kind on $\tilde{\mathcal{D}}$. For a given $g_0\in G$, if κ is admissible, that is, $\kappa(g_0^{-1}x^{-1}) = \kappa(x)$ for all $x\in G/T$, and $\kappa(x)\equiv 0 \pmod 2$ if $x\in (G/T)_{g_0}$ and $\eta(\tau(x)) = -1$ (compare with Definition 16), then we have a nondegenerate hermitian form $\tilde{B}$ of degree g_0 on $\tilde{\mathcal{V}}$, and all such hermitian forms are isomorphic (no signature functions in the algebraically closed case!). The form $\tilde{B}$ gives us an involution on $\tilde{\mathcal{R}}$, and the G-grading of $\mathcal{R}$ induces an outer G-grading on the quotient of the Lie algebra $\mathrm{Skew}(\mathcal{R},\varphi)'$ modulo its center. Denote this grading by $\Gamma^{(\mathrm{II})}_{\mathfrak{psl}}(T,\beta,\eta,g_0,\kappa)$.

Theorem 45. *Let $\mathbb{F}$ be an algebraically closed field,* $\mathrm{char}\,\mathbb{F}\neq 2$. *Let $\mathcal{L}$ be the simple Lie algebra of type A_r for $r\geq 2$, namely, $\mathfrak{psl}_{r+1}(\mathbb{F})$. If* $\mathrm{char}\,\mathbb{F} = 3$, *assume that $r\geq 3$. Then any outer G-grading on $\mathcal{L}$ is isomorphic to $\Gamma^{(\mathrm{II})}_{\mathfrak{psl}}(T,\beta,\eta,g_0,\kappa)$ where $r = |\kappa|\sqrt{\frac{1}{2}|T|}-1$. Moreover, $\Gamma^{(\mathrm{II})}_{\mathfrak{psl}}(T,\beta,\eta,g_0,\kappa)$ and $\Gamma^{(\mathrm{II})}_{\mathfrak{psl}}(T',\beta',\eta',g_0',\kappa')$ are isomorphic if and only if $T = T'$, $\beta = \beta'$ (hence $\eta = \eta'$), and (g_0,κ) and (g_0',κ') are in the same G-orbit, where $g\cdot(g_0,\kappa) = (g^{-2}g_0, g\cdot\kappa)$ and $(g\cdot\kappa)(x) = \kappa(g^{-1}x)$ for all $x\in G/T$.* $\square$

To compare this result with Theorem 3.53 in [11], we note that in that work the parameters of the grading are $(H, h, \beta, \kappa, \gamma, \mu_0, \bar{g}_0)$, where H stands for our T, h for f, β for $\bar{\beta}$, and (κ, γ) for κ (γ refers to the support of the multiplicity function and κ to the multiplicities of the elements of the support). Clearly, f and $\bar{\beta}$ carry the same information as β. As to the parameters $\mu_0 \in \mathbb{F}^\times$ and $\bar{g}_0 \in \overline{G}$, they determine the φ_0-sesquilinear form $\mathcal{V} \times \mathcal{V} \to \mathcal{D}$ that gives the anti-automorphism φ on $\mathcal{R}$. In [11], the involution φ_0 on $\mathcal{D}$ was obtained by fixing a standard matrix realization of $\mathcal{D}$ and setting $\varphi_0(X) := X^T$. (These are the involutions that correspond to quadratic forms $\bar{\eta}$ with polar form $\bar{\beta}$ and $\mathrm{Arf}(\bar{\eta}) = 1$.) Let ι be the involution of $\mathbb{K} = \mathbb{F} \times \mathbb{F}$ that interchanges the two components (in other words, $\iota(\zeta) = -\zeta$). Then we can take $\varphi_0 \otimes \iota$ on $\mathcal{D} \otimes_{\mathbb{F}} \mathbb{K}$ as our $\tilde{\varphi}_0$. (The corresponding η is related to $\bar{\eta}$ as follows: $\eta(t) = -\eta(tf) = \bar{\eta}(t)$ for all $t \in \overline{T}$.) Any hermitian form $\tilde{B}$ must restrict to zero on each of the components $\mathcal{V} \times \{0\}$ and $\{0\} \times \mathcal{V}$ of $\tilde{\mathcal{V}}$, so it is determined by its values $\tilde{B}((0, v), (w, 0))$ for all $v, w \in \mathcal{V}$. These values lie in $\mathcal{D} \times \{0\}$ and hence we obtain a φ_0-sesquilinear form B on $\mathcal{V}$ by setting

$$(B(v, w), 0) := \tilde{B}((0, v), (w, 0)).$$

It is straightforward to verify that B is nondegenerate, has degree $\bar{g}_0 = g_0\langle f\rangle$ with respect to the $\overline{G}$-grading, determines φ by Equation (3), and has the following weak form of hermitian property: $\varphi_0(B(w, v)) = \chi^{-1}(g_0) B(\chi^{-2} \cdot v, w)$ for all $v, w \in \mathcal{V}$. It follows that the parameter μ_0 in [11] is given by $\mu_0 = \chi^{-1}(g_0)$. Since $\chi(f) = -1$, the element g_0 is determined by $\bar{g}_0$ and μ_0.

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Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, A1C5S7, Canada

E-mail address: bahturin@mun.ca

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, A1C5S7, Canada

E-mail address: mikhail@mun.ca

Departamento de Matemáticas e Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address: adrian.rodrigo.escudero@gmail.com