COUNTING FINE GRADINGS ON MATRIX ALGEBRAS AND ON CLASSICAL SIMPLE LIE ALGEBRAS

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Abstract. Known classification results allow us to find the number of (equivalence classes of) fine gradings on matrix algebras and on classical simple Lie algebras over an algebraically closed field $\mathbb{F}$ (assuming char $\mathbb{F} \neq 2$ in the Lie case). The computation is easy for matrix algebras and especially for simple Lie algebras of type B_r (the answer is just $r+1$), but involves counting orbits of certain finite groups in the case of Series A, C and D. For $X \in \{A, C, D\}$, we determine the exact number of fine gradings, $N_X(r)$, on the simple Lie algebras of type X_r with $r \leq 100$ as well as the asymptotic behaviour of the average, $\hat{N}_X(r)$, for large r. In particular, we prove that there exist positive constants b and c such that $\exp(br^{2/3}) \leq \hat{N}_X(r) \leq \exp(cr^{2/3})$. The analogous average for matrix algebras $M_n(\mathbb{F})$ is proved to be $a \ln n + O(1)$ where a is an explicit constant depending on char $\mathbb{F}$.

1. Introduction

Let $\mathcal{A}$ be an algebra (not necessarily associative) over a field $\mathbb{F}$ and let G be a semigroup (written multiplicatively).

Definition 1. A *G-grading* on $\mathcal{A}$ is a vector space decomposition

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

such that

$$\mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{gh} \quad \text{for all} \quad g, h \in G.$$

If such a decomposition is fixed, $\mathcal{A}$ is referred to as a *G-graded algebra*. The *support* of Γ is the set $\mathrm{Supp}\,\Gamma := \{g \in G \mid \mathcal{A}_g \neq 0\}$.

The reader may consult the recent monograph [EK13] for background on gradings. In particular, there is more than one natural equivalence relation on graded algebras, depending on whether or not it is desirable to fix G. In this paper we will use the following version, where G is not fixed.

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Definition 2. Let $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{B} = \bigoplus_{h\in H} \mathcal{B}_h$ be two graded algebras, with supports S and T, respectively. We say that the graded algebras $\mathcal{A}$ and $\mathcal{B}$ (or the gradings Γ and Γ') are *equivalent* if there exists an isomorphism of algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a bijection $\alpha\colon S \to T$ such that $\varphi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$ for all $s \in S$.

It is known that if Γ is a grading on a simple Lie algebra by any semigroup, then $\operatorname{Supp}\Gamma$ generates an abelian group (see e.g. [Koc09] or [EK13, Proposition 1.12]). From now on, we will assume that all gradings are by *abelian groups*, which will be written additively. The cyclic group $\mathbb{Z}/m\mathbb{Z}$ will be denoted by $\mathbb{Z}_m$. We will also assume that the ground field $\mathbb{F}$ is *algebraically closed*.

The so-called fine gradings on an algebra (defined below) are of special importance since they reveal the structure of the algebra and its automorphism group (if $\operatorname{char}\mathbb{F} = 0$, then the fine gradings on a finite-dimensional algebra $\mathcal{A}$ correspond to maximal quasitori in the automorphism group of $\mathcal{A}$).

Definition 3. Let $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}'_h$ be two gradings on the same algebra, with supports S and T, respectively. We will say that Γ' is a *refinement* of Γ (or Γ is a *coarsening* of Γ') if for any $t \in T$ there exists (unique) $s \in S$ such that $\mathcal{A}'_t \subset \mathcal{A}_s$. If, moreover, $\mathcal{A}'_t \neq \mathcal{A}_s$ for at least one $t \in T$, then the refinement is said to be *proper*. Finally, Γ is said to be *fine* if it does not admit any proper refinements (in the class of gradings by abelian groups).

Gradings have recently been classified for many interesting algebras (see e.g. [EK13] and references therein). In particular, a classification of fine gradings up to equivalence is known for matrix algebras over an algebraically closed field $\mathbb{F}$ of arbitrary characteristic [HPP98a, BSZ01, BZ03] and for classical simple Lie algebras except D_4 in characteristic different from 2 [Eld10, EK12c]. Type D_4 is different from all other members of Series D due to the phenomenon of triality. In [Eld10], fine gradings on the simple Lie algebra of type D_4 are classified in characteristic 0; there are 17 equivalence classes. As to the exceptional simple Lie algebras, fine gradings are classified for type G_2 ($\operatorname{char}\mathbb{F} \neq 2, 3$) in [DM06, EK12a], for type F_4 ($\operatorname{char}\mathbb{F} \neq 2$) in [DM09, EK12a], and for type E_6 ($\operatorname{char}\mathbb{F} = 0$) in [DV]. The number of equivalence classes is, respectively, 2, 4 (only 3 in the case $\operatorname{char}\mathbb{F} = 3$), and 14.

In the present paper, we are interested in the number of (equivalence classes of) fine gradings for matrix algebras and for classical simple Lie algebras of Series A, C and D. It is easy to see that there are 2 fine gradings on $\mathfrak{sl}_2(\mathbb{F})$ ($\operatorname{char}\mathbb{F} \neq 2$). The number of fine gradings on a few other members of these series over $\mathbb{C}$ (and on their real forms) have been found in [HPP98b, PPS01, PPS02, Svo08] using the description of maximal quasitori ("MAD subroups") in [HPP98a]. The more recent classification results, as stated in [EK12c] and [EK13], reduce the problem to counting orbits of certain finite groups, so the number of fine gradings can be computed, in principle, for any member of these series over an algebraically closed field of characteristic different from 2. Note that there is no work to be done for Series B because there are exactly $r+1$ gradings on the simple Lie algebra of type B_r (see [Eld10] or [EK13, §3.4]). We count the orbits using Burnside–Cauchy–Frobenius Lemma and the computer algebra system GAP (see [GAP]) to obtain the exact number of fine gradings for simple Lie algebras of types A_r, C_r and D_r up to $r = 100$ (see Tables 4, 6 and 8, respectively). The number of fine gradings on $M_n(\mathbb{F})$ is easily computed since it is expressed in terms of the partition function and the multiplicities of the prime factors of n. We state the answer for n up to 100 for completeness (see Table 1). The behaviour of the average number of fine gradings on $M_j(\mathbb{F})$ with $j \le n$ as $n \to \infty$ (Theorem 2) is derived from the known asymptotics of the number of abelian groups of order $\le n$ [ES35]. We also establish the asymptotic behaviour of the average number of fine gradings for simple Lie algebras of Series A (Theorem 5), C (Theorem 7) and D (Theorem 9); this number exhibits *intermediate growth*: faster than any polynomial but slower than any exponential. The proof of these results is based on the asymptotic analysis of certain binomial coefficients (Section 6).

2. Matrix Algebras

Fine gradings on $M_n(\mathbb{C})$ were classified in [HPP98a] in terms of the corresponding "MAD subgroups" of $\mathrm{PGL_n}(\mathbb{C})$. The approach in [BSZ01] was to look directly at the structure of the graded algebra, which allowed a generalization to $M_n(\mathbb{F})$ over any algebraically closed field [BZ03]. We state the classification using the notation of [EK12b, Corollary 2.6] and [EK13, §2.3]. We do not explain this notation here, as our present concern is the number of gradings and not their explicit form. The subscript M stands for matrices; later we will use subscripts A, C and D for the corresponding series of classical Lie algebras.

Theorem 1 ([HPP98a, BZ03])**.** *Let Γ be a fine abelian group grading on the matrix algebra $M_n(\mathbb{F})$ over an algebraically closed field $\mathbb{F}$. Then Γ is equivalent to some $\Gamma_M(T,k)$ where T is a finite abelian group of the form $\mathbb{Z}_{\ell_1}^2 \times \cdots \times \mathbb{Z}_{\ell_r}^2$ (i.e., a Cartesian square),* $\operatorname{char}\mathbb{F}$ *does not divide $|T|$, and $k\ell_1\cdots\ell_r = n$. Two gradings $\Gamma_M(T_1,k_1)$ and $\Gamma_M(T_2,k_2)$ are equivalent if and only if $T_1 \cong T_2$ and $k_1 = k_2$.* $\square$

It follows that the number of fine gradings on $M_n(\mathbb{F})$ is given by

$$N_M(n) = \sum_{\ell \,|\, n} N_{ab}(\ell), \tag{1}$$

where $N_{ab}(\ell)$ is the number of (isomorphism classes of) abelian groups of order ℓ (Online Encyclopaedia of Integer Sequences A000688) and, if $\operatorname{char}\mathbb{F} = p$, the summation is restricted to ℓ that are not divisible by p.

2.1. **Counting gradings.** Factoring $\ell = p_1^{m_1}\cdots p_s^{m_s}$ in equation (1), where $p_i \ne \operatorname{char}\mathbb{F}$ are distinct primes, we obtain $N_{ab}(\ell) = P(m_1)\cdots P(m_s)$ where $P(m)$ denotes the number of partitions of a non-negative integer m (with the convention $P(0) = 1$). Hence, if $n = p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ and $\operatorname{char}\mathbb{F} = 0$ or if $n = p_1^{\alpha_1}\cdots p_s^{\alpha_s}p^{\alpha}$ and $\operatorname{char}\mathbb{F} = p$ then equation (1) can be rewritten as

$$N_M(n) = \prod_{i=1}^{s}\sum_{j=0}^{\alpha_i} P(j). \tag{2}$$

Table 1 displays the numbers $N_M(n)$ for $n \le 100$ in the case $\operatorname{char}\mathbb{F} = 0$, which were calculated using equation (2).

2.2. **Asymptotic behaviour.** The function $N_M(n)$ behaves irregularly, so we introduce the following:

$$\hat{N}_M(n) = \frac{1}{n}\sum_{j=1}^{n} N_M(j), \tag{3}$$

n	$N_M(n)$	n	$N_M(n)$	n	$N_M(n)$	n	$N_M(n)$	n	$N_M(n)$
1	1	21	4	41	2	61	2	81	12
2	2	22	4	42	8	62	4	82	4
3	2	23	2	43	2	63	8	83	2
4	4	24	14	44	8	64	30	84	16
5	2	25	4	45	8	65	4	85	4
6	4	26	4	46	4	66	8	86	4
7	2	27	7	47	2	67	2	87	4
8	7	28	8	48	24	68	8	88	14
9	4	29	2	49	4	69	4	89	2
10	4	30	8	50	8	70	8	90	16
11	2	31	2	51	4	71	2	91	4
12	8	32	19	52	8	72	28	92	8
13	2	33	4	53	2	73	2	93	4
14	4	34	4	54	14	74	4	94	4
15	4	35	4	55	4	75	8	95	4
16	12	36	16	56	14	76	8	96	38
17	2	37	2	57	4	77	4	97	2
18	8	38	4	58	4	78	8	98	8
19	2	39	4	59	2	79	2	99	8
20	8	40	14	60	16	80	24	100	16

TABLE 1. Number of fine gradings, $N_M(n)$, on the matrix algebra $M_n(\mathbb{F})$, where char $\mathbb{F} = 0$.

which is the average number of fine gradings on the algebras $M_j(\mathbb{F})$ with $j \le n$.

Theorem 2. *Let $\mathbb{F}$ be an algebraically closed field of characteristic c. The following asymptotic formula holds:*

$$\hat{N}_M(n) = a_c \ln n + O(1). \tag{4}$$

In the case $c = 0$, the constant is

$$a_0 = \prod_{m=2}^{\infty} \zeta(m) \approx 2.2948566; \tag{5}$$

here $\zeta(\cdot)$ is the Riemann zeta function.

In the case $c > 0$, the constant is

$$a_c = a_0 \prod_{m=2}^{\infty} (1 - c^{-m}). \tag{6}$$

The infinite product appearing in the formula for a_c is a particular case of q-*Pochhammer symbol* (with $q = 1/c$) known in the theory of elliptic functions, theory of partitions, and elsewhere. In standard notation, it is abbreviated as

$$\prod_{m=2}^{\infty} (1 - c^{-m}) = (c^{-2}; c^{-1})_{\infty}.$$

The numerical values of $(c^{-2}; c^{-1})_{\infty}$ for prime $c \le 13$ and the corresponding values of the constants a_c are given in Table 2.

c	2	3	5	7	11	13
$(c^{-2};c^{-1})_\infty$	0.577576	0.840189	0.950416	0.976261	0.990916	0.993593
a_c	1.325455	1.928114	2.181068	2.240380	2.274010	2.280153

TABLE 2. Numerical values of $(c^{-2};c^{-1})_\infty$ and a_c.

Proof. (a) Case char $\mathbb{F}=0$. We refer to the classical result [ES35] on the average number of abelian groups of order j with $j \le n$:

$$\frac{1}{n}\sum_{j=1}^{n} N_{ab}(j) = a_0 + O(n^{-1/2}). \tag{7}$$

For simplicity denote the sum on the left side of (7) by $F(n)$ and also set $F(0)=0$ by definition. Then

$$n\hat{N}_M(n) = \sum_{j\le n}\sum_{\ell\,|\,j} N_{ab}(\ell) = \sum_{j=1}^{n} N_{ab}(j)\left\lfloor \frac{n}{j}\right\rfloor$$

$$= \sum_{j=1}^{n}(F(j)-F(j-1))\frac{n}{j} - \sum_{j=1}^{n} N_{ab}(j)\left(\frac{n}{j} - \left\lfloor\frac{n}{j}\right\rfloor\right).$$

The second sum has positive terms and is majorized by $\sum_{j=1}^{n} N_{ab}(j) = O(n)$. Now,

$$\sum_{j=1}^{n}(F(j)-F(j-1))\frac{n}{j} = F(n) + n\sum_{j=1}^{n-1}\frac{F(j)}{j(j+1)};$$

here $F(n)=O(n)$, the sum

$$\sum_{j=1}^{n-1}\frac{F(j)-a_0 j}{j(j+1)}$$

is $O(1)$ since its terms are $O(j^{-3/2})$ by (7), while

$$\sum_{j=1}^{n}\frac{a_0 j}{j(j+1)} = a_0\ln n + O(1).$$

This completes the proof for char $\mathbb{F}=0$.

(b) Case $c = \operatorname{char}\mathbb{F} > 0$. The summatory function $n\hat{N}_M(n)$ in this case is

$$n\hat{N}_M(n) = \sum_{j\le n}\sum_{\ell\,|\,j,\; c\nmid \ell} N_{ab}(\ell) = \sum_{\ell=1}^{n} f(\ell)\left\lfloor\frac{n}{\ell}\right\rfloor,$$

where we set

$$f(n) = \begin{cases} N_{ab}(n) & \text{if } c\nmid n,\\ 0 & \text{if } c\,|\,n.\end{cases}$$

Similarly to (a), the formula (4) with constant (6) will follow from the asymptotics of the summatory function $F(n)=\sum_{j=1}^{n} f(j)$

$$F(n) = a_c n + O(n^{1/2}). \tag{8}$$

A proof of (8) is a slight variation of the proof of (7) given in [ES35]. The argument goes through with only one change: the prime c is not participating in any products. This results in the constant A_1 of [ES35], which equals our a_0, being replaced by

$$a_c = \prod_{m=2}^{\infty} \prod_{p \neq c} (1 + p^{-m} + p^{-2m} + \cdots),$$

where the inner product runs over all primes $p \neq c$. Thus

$$a_c = \prod_{m=2}^{\infty} \zeta(m)(1 - c^{-m}) = a_0 \prod_{m=2}^{\infty} (1 - c^{-m}).$$

□

Remark 1. The asymptotic formula (7) has been significantly refined by many later authors, see e.g. [Liu91, HB89]. There is little doubt that formula (4) can be refined similarly, but this endeavour is beyond the scope of the current paper.

3. Lie Algebras of Series A

The classification of fine gradings on all simple Lie algebras of Series A was established in [Eld10] for the case char $\mathbb{F} = 0$. We state the result in purely combinatorial terms, as it appears in [EK12c] and [EK13, §3.3]. There are two types of gradings, which we distinguish using superscripts (I) and (II). The subscript A refers to the series of Lie algebras. We do not introduce the gradings explicitly because we are only interested in their number. A *multiset* in a set X is a function $X \to \mathbb{Z}_{\geq 0}$ that assigns to each point its multiplicity. If a group G acts on X, then it also acts on the multisets in X. The relevant group here is $\mathrm{ASp}_{2m}(2)$, the semidirect product $\mathbb{Z}_2^{2m} \rtimes \mathrm{Sp}_{2m}(2)$ of the symplectic group $\mathrm{Sp}_{2m}(2)$ and the vector group $\mathbb{Z}_2^{2m}$. Each element $(t, A) \in \mathbb{Z}_2^{2m} \rtimes \mathrm{Sp}_{2m}(2)$ acts on $\mathbb{Z}_2^{2m}$ in the natural way: $x \mapsto Ax + t$.

Theorem 3 ([Eld10, EK12c])**.** *Let $\mathbb{F}$ be an algebraically closed field,* char $\mathbb{F} \neq 2$. *Let $n \geq 3$ if* char $\mathbb{F} \neq 3$ *and $n \geq 4$ if* char $\mathbb{F} = 3$. *Then any fine grading on $\mathfrak{psl}_n(\mathbb{F})$ is equivalent to one of the following:*

- *$\Gamma_A^{(\mathrm{I})}(T, k)$ where T is as in Theorem 1, k is a positive integer, $k\sqrt{|T|} = n$, and $k \geq 3$ if T is an elementary 2-group;*
- *$\Gamma_A^{(\mathrm{II})}(T, q, s, \tau)$ where T is an elementary 2-group of even rank, q and s are non-negative integers, $(q + 2s)\sqrt{|T|} = n$, $\tau = (t_1, \ldots, t_q)$ is a q-tuple of elements of T, and $t_1 \neq t_2$ if $q = 2$ and $s = 0$.*

Gradings belonging to different types listed above are not equivalent. Within each type, we have the following:

- *$\Gamma_A^{(\mathrm{I})}(T_1, k_1)$ and $\Gamma_A^{(\mathrm{I})}(T_2, k_2)$ are equivalent if and only if $T_1 \cong T_2$ and $k_1 = k_2$;*
- *$\Gamma_A^{(\mathrm{II})}(T_1, q_1, s_1, \tau_1)$ and $\Gamma_A^{(\mathrm{II})}(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2m}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the natural action of* $\mathrm{ASp}_{2m}(2)$, *where $\Sigma(\tau)$ is the multiset for which the multiplicity of any point t is the number of times t appears among the components of the q-tuple τ.* □

Since, for Series A, the rank r is related to the matrix size n as $n = r + 1$, Theorem 3 implies that the number of fine gradings of Type I on the simple Lie algebra A_r ($r \geq 2$) is given by

$$N_A^{(\mathrm{I})}(r) = \begin{cases} N_M(r+1) - 2 & \text{if } r+1 \text{ is a power of 2,} \\ N_M(r+1) & \text{otherwise,} \end{cases} \tag{9}$$

where $N_M(n)$ is the number of fine gradings on $M_n(\mathbb{F})$, which is discussed in the previous section.

In order to calculate the number of fine gradings of Type II, we need to determine the number of orbits, $N(m,q)$, of $\mathrm{ASp}_{2m}(2)$ on multisets of size q in $T = \mathbb{Z}_2^{2m}$. Clearly, $N(m,q)$ does not exceed the total number of such multisets. Note that any multiset of size q determines a partition of the integer q by looking at the (nonzero) multiplicities and forgetting to which points in T they belong. Clearly, any permutation of T leaves invariant the set of multisets belonging to a fixed partition. Hence, we obtain bounds:

$$P_{2^{2m}}(q) \leq N(m,q) \leq \binom{q + 2^{2m} - 1}{q}, \tag{10}$$

where $P_M(q)$ is the number of partitions of q into at most M positive parts. Note that $\mathrm{ASp}_2(2)$ is the full group of permutations on $\mathbb{Z}_2^2$, hence the lower bound is achieved if $m = 1$. It is also achieved if $q \leq 2$ because $\mathrm{Sp}_{2m}(2)$ acts irreducibly and hence $\mathrm{ASp}_{2m}(2)$ acts 2-transitively on $\mathbb{Z}_2^{2m}$.

Another lower bound for $N(m,q)$, which is better than that in (10) for any fixed $m > 1$ and sufficiently large q, comes from the obvious fact that the size of a G-orbit cannot exceed the size of G. Thus

$$\frac{1}{|G_m|}\binom{q + 2^{2m} - 1}{q} \leq N(m,q), \tag{11}$$

where $G_m = \mathrm{ASp}_{2m}(2)$. This bound is indeed better as $q \to \infty$, because $\binom{q+M-1}{q} \sim \frac{q^{M-1}}{(M-1)!}$, G_m is not the full group of permutations if $m > 1$, and it is known that $P_M(q) \sim \frac{1}{M!}\frac{q^{M-1}}{(M-1)!}$. The upper bound in (10) and the lower bound (11) will be used to obtain asymptotic results in Section 6.

3.1. **Counting orbits.** We start with a few general remarks. It is customary to write partitions as decreasing sequences of positive integers: $\kappa = (k_1, \ldots, k_\ell)$, where $k_1 \geq \ldots \geq k_\ell$ and $\ell = \ell(q)$ is called the *length* of κ. The notation $\kappa \vdash q$ means $\sum_j k_j = q$. We can also write $\kappa = (q_1^{(\ell_1)}, \ldots, q_s^{(\ell_s)})$ where $q_1 > \ldots > q_s$ and the superscript (ℓ_j) indicates the number of times the value q_j is repeated; $\sum_j \ell_j = \ell$. For example, $(4,4,4,3,1)$ can be written as $(4^{(3)}, 3^{(1)}, 1^{(1)})$ or just $(4^{(3)}, 3, 1)$. When working with partitions of length $\leq M$ for a fixed M, it is sometimes convenient to append zeros at the end so the total number of parts is formally M. For example, with $M = 7$, the partition $(4^{(3)}, 3, 1)$ may be written as $(4^{(3)}, 3, 1, 0^{(2)})$.

Let T be a set of M elements. As pointed out above, any multiset of size q in T determines a partition $\kappa \vdash q$ of length $\leq M$, and the set of all multisets belonging to a fixed $\kappa \vdash q$ is invariant under any group G acting on T. For $T = \mathbb{Z}_2^{2m}$ and $G = G_m$, let $N(m,\kappa)$ be the number of orbits in this set. Thus

$$N(m,q) = \sum_{\kappa \vdash q,\, \ell(\kappa) \leq 2^{2m}} N(m,\kappa). \tag{12}$$

Similarly, a partition $\kappa = (q_1^{(\ell_1)}, \dots, q_s^{(\ell_s)})$ of length M determines a partition of M given by sorting the sequence $(\ell_1, \dots, \ell_s)$. If κ has length $\ell < M$, we write $\kappa = (q_1^{(\ell_1)}, \dots, q_s^{(\ell_s)}, 0^{(M-\ell)})$ and sort the sequence $(\ell_1, \dots, \ell_s, M - \ell)$. In other words, we regard κ as a multiset of size M in $\mathbb{Z}_{\geq 0}$ and assign to it a partition of M as was done before to multisets in T. For example, with $M = 7$, the partition $(4, 4, 4, 3, 1)$ gives $(3, 2, 1, 1)$.

If $\kappa = (k_1, \dots, k_M)$ and $\kappa' = (k'_1, \dots, k'_M)$ are partitions of length $\leq M$ that determine the same partition of M, then there is a bijection $\varphi\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $k'_j = \varphi(k_j)$ for all j. If T is a set of M elements, then post-composition with any such φ defines a bijection from the multisets in T belonging to κ to those belonging to κ'. Clearly, this bijection is G-equivariant for any group G acting on T. In particular, we have $N(m, \kappa) = N(m, \kappa')$.

The above remarks show that, for a fixed m, the structure of G_m-orbits on multisets of all sizes in $T = \mathbb{Z}_2^{2m}$ is determined by the orbits belonging to a finite number of partitions. However, this number is very large: $P(M) \sim \frac{1}{4\sqrt{3}M} \exp\left(\pi\sqrt{\frac{2M}{3}}\right)$, $M = 2^{2m}$.

For small values of m and q, one can find the orbits on multisets of size q using a standard function of GAP. (To speed up the calculation, one can work separately with multisets belonging to each partition $\kappa \vdash q$.) If we do not need representatives of the orbits (say, in order to construct explicitly all fine gradings of Type II on a simple Lie algebra of Series A) and just want to count their number then we can use the following well-known fact.

Lemma 4 (Burnside–Cauchy–Frobenius)**.** *Let G be a finite group acting on a finite set X. Then the number of orbits equals the average number of fixed points:*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g),$$

where $\mathrm{Fix}(g)$ *is the number of points in X fixed by g.* □

An important addition to this lemma is the observation that if g and g' are conjugate, then $\mathrm{Fix}(g) = \mathrm{Fix}(g')$, so

$$|X/G| = \frac{1}{|G|} \sum_i c_i \mathrm{Fix}(g_i), \tag{13}$$

where the summation is over the conjugacy classes of G, with g_i and c_i being a representative and the size of the i-th class. The number of conjugacy classes in G_m is small compared to $|G_m|$.

We also used the following observation to make the calculation of the number of fixed points more efficient. Let g be a permutation on a set T of size M and let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash M$ be the cycle structure of g, i.e., the λ_j are the cycle lenghs of g (including the trivial cycles). Let κ be a partition of length $\leq M$ and let $\mu = (\mu_1, \dots, \mu_s)$ be the corresponding partition of M. Then the number of fixed points of g among the multisets in T belonging to κ is equal to the number of functions $\varphi\colon \{1, \dots, \ell\} \to \{1, \dots, s\}$ such that $\sum_{j\,:\,\varphi(j)=k} \lambda_j = \mu_k$ for all $k = 1, \dots, s$. We denote this number by $f(\lambda, \mu)$. Informally, $f(\lambda, \mu)$ is the number of ways to fit ℓ pigeons of volumes given by λ into s holes of capacities given by μ so that each hole is filled to its full capacity. For example, for $\lambda = (1^{(M)})$ and $\mu = (M)$ we have $f(\lambda, \mu) = 1$; for $\lambda = (M)$ and $\mu = (1^{(M)})$ we have $f(\lambda, \mu) = 0$; for $\lambda = (3, 3, 2)$ and

$\mu = (5,3)$ we have $f(\lambda,\mu) = 2$. (Pigeons are distinguishable even if they have the same volume; holes are distinguishable even if they have the same capacity.)

Hence equation (13) can be rewritten as

$$N(m,\kappa) = \frac{1}{|G_m|} \sum_{\lambda \vdash 2^{2m}} c(\lambda) f(\lambda,\mu), \tag{14}$$

where $c(\lambda)$ is the sum of c_i over all i such that g_i has cycle structure λ. Finally, let $P_\mu(q)$ be the number of partitions $\kappa \vdash q$ of length $\leq M$ such that the corresponding partition of M is μ. Then, in view of (14), we can rewrite (12) as follows:

$$N(m,q) = \frac{1}{|G_m|} \sum_{\lambda,\mu \vdash 2^{2m}} c(\lambda) f(\lambda,\mu) P_\mu(q). \tag{15}$$

Note that the number of partitions λ that actualy occur in the sum is at most the number of conjugacy classes of G_m; the number of partitions μ that actually occur is bounded by $P_{2^{2m}}(q) \leq P(q)$.

In GAP, we defined G_m by converting $\mathrm{Sp}_{2m}(2)$ to a permutation group on $M = 2^{2m}$ points and adding one generator for the translation by a nonzero vector. Then we obtained representatives and sizes of conjugacy classes using a standard function and found the numbers $N(m,q)$ using (15). Some of these numbers are shown in Table 3.

Remark 2. Alternatively, one can use a reformulation of (15) in terms of generating functions afforded by Pólya's Theorem. Namely, the generating function $g_m(t) = 1 + \sum_{q=1}^{\infty} N(m,q)t^q$ is given by

$$g_m(t) = Z_m\Big(\frac{1}{1-t}, \frac{1}{1-t^2}, \frac{1}{1-t^3}, \ldots\Big)$$

where $Z_m(x_1, x_2, x_3, \ldots)$ is the *cycle index* of G_m, i.e., the sum of the terms

$$\frac{c(\lambda)}{|G_m|} x_1^{k_1} \cdots x_M^{k_M}$$

where k_i is the number of parts of $\lambda \vdash M$ that are equal to i. The numbers $N(m,q)$ can be obtained using a computer algebra system to expand $g_m(t)$ into a power series at $t = 0$.

q	1	2	3	4	5	6	7	8	9	10	11	12
$N(1,q)$	1	2	3	5	6	9	11	15	18	23	27	34
$N(2,q)$	1	2	4	9	17	38	74	158	318	657	1304	2612
$N(3,q)$	1	2	4	10	22	67	202	755	3082	14493	72584	379501
$N(4,q)$	1	2	4	10	23	75	265	1352	9432	98773	1398351	23613147
$N(5,q)$	1	2	4	10	23	76	275	1495	12196	183053	5075226	226160064

TABLE 3. Number of orbits, $N(m,q)$, of $\mathrm{ASp}_{2m}(2)$ on the set of multisets of size q in $T = \mathbb{Z}_2^{2m}$.

3.2. **Counting gradings.** The number of fine gradings of Type I, $N_A^{(\mathrm{I})}(r)$, is given by equation (9), see also Table 1. As to Type II gradings, we will use the following notation. For $n = 2^\alpha k$ where k is odd, set

$$f_0(n) = \sum_{m=0}^{\alpha} \sum_{s=0}^{\lfloor 2^{\alpha-m-1}k \rfloor} N(m, 2^{\alpha-m}k - 2s), \tag{16}$$

with the convention $N(0, q) = 1$ for all q. Then Theorem 3 implies that the number of fine gradings of Type II, $N_A^{(\mathrm{II})}(r)$, is given by

$$N_A^{(\mathrm{II})}(r) = \begin{cases} f_0(r+1) - 1 & \text{if } r+1 \text{ is a power of } 2, \\ f_0(r+1) & \text{otherwise.} \end{cases} \tag{17}$$

The total number of fine gradings on A_r is, of course, $N_A(r) = N_A^{(\mathrm{I})}(r) + N_A^{(\mathrm{II})}(r)$. We calculated these numbers for $r \le 100$ assuming char $\mathbb{F} = 0$. The results are stated in Table 4. Note that if r is even then $N_A^{(\mathrm{II})}(r) = \frac{r}{2} + 1$.

Remark 3. If char $\mathbb{F} = p > 0$ ($p \neq 2$), then the numbers $N_A^{(\mathrm{I})}(r)$ must be adjusted to exclude the prime factor p when it divides $r + 1$. The case A_2 requires special treatment if char $\mathbb{F} = 3$; the number of fine gradings turns out to be 2 instead of 3.

3.3. **Asymptotic behaviour.** The function $N_A(r)$ behaves irregularly, so we use averaging similar to (3). We record for future reference:

$$\hat{N}_X(r) = \frac{1}{r} \sum_{j \le r} N_X(j) \quad \text{where} \quad X \in \{A, C, D\}. \tag{18}$$

This is the average number of fine gradings on the simple Lie algebras of type X_j with $j \le r$. (The summation can start with $j = 1$ if $X = A$ or C and with $j = 3$ if $X = D$, but such details are irrelevant for the asymptotics.)

Theorem 5. *Let $\mathbb{F}$ be an algebraically closed field,* char $\mathbb{F} \neq 2$*. Let $\hat{N}_A(r)$ be defined by* (18)*. The following asymptotic formula holds:*

$$\ln \hat{N}_A(r) = b(r+1) r^{2/3} + O((\ln r)^2),$$

where $b(\cdot)$ is the bounded continuous function defined by (35)*. Specifically, the maximum and minimum values of $b(\cdot)$ are the constants $b_0 \approx 1.581080$ and $b_1 \approx 1.512173$ defined in Lemma* 10 *and Equation* (31)*.*

Proof. The number of fine gradings of Type II is given by (17), and Theorem 11 implies that the logarithm of the average number is $b(r+1)r^{2/3} + O((\ln r)^2)$. The number of fine gradings of Type I is given by (9), and Theorem 2 implies that the average number is asymptotically negligible compared to Type II. The result follows. □

4. Lie Algebras of Series C

For Series C and D, the classification of fine gradings (see e.g. [EK13, §3.5, 3.6]) requires a certain action of $\mathrm{Sp}_{2m}(2)$ on $T = \mathbb{Z}_2^{2m}$, which is different from the natural action (cf. [DM96, p.245]). The group $\mathrm{Sp}_{2m}(2) = \mathrm{Sp}(T)$ is defined as the group of isometries of a nondegenerate alternating bilinear form on T, say, the following:

$$(x, y) = \sum_{i=1}^{m} x_i y_{2m+1-i} - \sum_{i=1}^{m} x_{2m+1-i} y_i = \sum_{i=1}^{2m} x_i y_{2m+1-i} \quad \text{for all } x, y \in \mathbb{Z}_2^{2m}. \tag{19}$$

r	(I)	(II)	$N_A(r)$	r	(I)	(II)	$N_A(r)$	r	(I)	(II)	$N_A(r)$
2	2	2	4	35	16	717	733	68	4	35	39
3	2	6	8	36	2	19	21	69	8	2451	2459
4	2	3	5	37	4	345	349	70	2	36	38
5	4	8	12	38	4	20	24	71	28	160306	160334
6	2	4	6	39	14	1305	1319	72	2	37	39
7	5	16	21	40	2	21	23	73	4	2964	2968
8	4	5	9	41	8	467	475	74	8	38	46
9	4	16	20	42	2	22	24	75	8	275919	275927
10	2	6	8	43	8	2269	2277	76	4	39	43
11	8	29	37	44	8	23	31	77	8	3554	3562
12	2	7	9	45	4	619	623	78	2	40	42
13	4	29	33	46	2	24	26	79	24	494159	494183
14	4	8	12	47	24	4284	4308	80	12	41	53
15	10	56	66	48	4	25	29	81	4	4228	4232
16	2	9	11	49	8	806	814	82	2	42	44
17	8	49	57	50	4	26	30	83	16	816756	816772
18	2	10	12	51	8	7700	7708	84	4	43	47
19	8	88	96	52	2	27	29	85	4	4994	4998
20	4	11	15	53	14	1033	1047	86	4	44	48
21	4	78	82	54	4	28	32	87	14	1450304	1450318
22	2	12	14	55	14	14592	14606	88	2	45	47
23	14	157	171	56	4	29	33	89	16	5860	5876
24	4	13	17	57	4	1305	1309	90	4	46	50
25	4	119	123	58	2	30	32	91	8	2276709	2276717
26	7	14	21	59	16	26426	26442	92	4	47	51
27	8	247	255	60	2	31	33	93	4	6834	6838
28	2	15	17	61	4	1628	1632	94	4	48	52
29	8	175	183	62	8	32	40	95	38	4116511	4116549
30	2	16	18	63	28	49420	49448	96	2	49	51
31	17	441	458	64	4	33	37	97	8	7925	7933
32	4	17	21	65	8	2008	2016	98	8	50	58
33	4	249	253	66	2	34	36	99	16	5997150	5997166
34	4	18	22	67	8	87728	87736	100	2	51	53

TABLE 4. Number of fine gradings, $N_A(r)$, on the simple Lie algebra of type A_r assuming char $\mathbb{F} = 0$.

(This is the form used by GAP to define symplectic groups.) Now consider the quadratic form

$$Q(x) = \sum_{i=1}^{m} x_i x_{2m+1-i} \quad \text{for all } x \in \mathbb{Z}_2^{2m}. \tag{20}$$

Clearly, the bilinear form (19) is the polar of Q, i.e., $(x,y) = Q(x+y) - Q(x) - Q(y)$ for all x, y. But since we are now in characteristic 2, Q cannot be expressed in terms of the bilinear form and, consequently, an element $A \in \mathrm{Sp}(T)$ does not necessarily preserve Q. One verifies that the mapping $x \mapsto Q(A^{-1}x) + Q(x)$ is linear (a peculiarity of the field of order 2), so there exists unique $t_A \in T$ such that

$$(t_A, x) = Q(A^{-1}x) + Q(x) \quad \text{for all } x \in T.$$

It follows that $Q(Ax + t_A) = Q(x) + Q(t_A)$ for all x. Let

$$T_+ = \{x \in T \mid Q(x) = 0\} \quad \text{and} \quad T_- = \{x \in T \mid Q(x) = 1\}.$$

One verifies that $|T_\pm| = 2^{m-1}(2^m \pm 1)$. Since the mapping $x \mapsto Ax + t_A$ is bijective and $|T_+| \neq |T_-|$, it cannot swap T_+ and T_-, hence $Q(t_A) = 0$ and $Q(Ax + t_A) = Q(x)$ for all x.

Definition 4. For $x \in T$ and $A \in \mathrm{Sp}(T)$, define $A \cdot x = Ax + t_A$. One verifies that the mapping $\mathrm{Sp}(T) \to \mathrm{ASp}(T) = T \rtimes \mathrm{Sp}(T)$, $A \mapsto (t_A, A)$, is a homomorphism, so $\cdot$ is an action of $\mathrm{Sp}(T)$ on T, which we call the *twisted action* to distinguish from the natural one.

By construction, the twisted action of $\mathrm{Sp}(T)$ preserves Q, i.e., T_+ and T_- are invariant subsets. One can show that the twisted action is 2-transitive on each of T_+ and T_-.

Theorem 6 ([Eld10, EK12c]). *Let $\mathbb{F}$ be an algebraically closed field,* char $\mathbb{F} \neq 2$. *Let $n \geq 4$ be even. Then any fine grading on $\mathfrak{sp}_n(\mathbb{F})$ is equivalent to $\Gamma_C(T, q, s, \tau)$ where T is an elementary 2-group of even rank, q and s are non-negative integers, $(q + 2s)\sqrt{|T|} = n$, $\tau = (t_1, \ldots, t_q)$ is a q-tuple of elements of T_-, and $t_1 \neq t_2$ if $q = 2$ and $s = 0$. Moreover, $\Gamma_C(T_1, q_1, s_1, \tau_1)$ and $\Gamma_C(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2m}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of* $\mathrm{Sp}_{2m}(2)$ *as in Definition* 4. □

Hence, to calculate the number of fine gradings on simple Lie algebras of Series C, we need to determine the number of orbits, $N_-(m, q)$, of the twisted action of $\mathrm{Sp}_{2m}(2)$ on multisets of size q in $T_- \subset \mathbb{Z}_2^{2m}$. Similarly, we will need $N_+(m, q)$ for Series D. By the same argument as for (10), we obtain the following bounds:

$$P_{2^{m-1}(2^m \pm 1)}(q) \leq N_\pm(m, q) \leq \binom{q + 2^{m-1}(2^m \pm 1) - 1}{q}, \tag{21}$$

where $P_k(q)$ is the number of partitions of q into at most k positive parts. Note that if $m = 1$ then we have $|T_+| = 3$ and $|T_-| = 1$, so $\mathrm{Sp}_2(2)$ acts as the full group of permutations on T_+ and on T_-, hence the lower bound is achieved in this case. It is also achieved if $q \leq 2$ because of 2-transitivity.

An alternative lower bound, similar to (11), is the following:

$$\frac{1}{|G_m|}\binom{q + 2^{m-1}(2^m \pm 1) - 1}{q} \leq N_\pm(m, q), \tag{22}$$

where $G_m = \mathrm{Sp}_{2m}(2)$. This will be used in Section 6 to obtain asymptotic results.

4.1. **Counting orbits.** Using the same method as in the previous section, we can compute the numbers $N_-(m, q)$. Some of them are displayed in Table 5. We note that if $m = 2$ then $|T_-| = 6$, so $\mathrm{Sp}_4(2)$ acts as the full group of permutations on T_- (but not on T_+). Hence the lower bound (21) for $N_-(2, q)$ is achieved.

q	1	2	3	4	5	6	7	8	9	10	11	12
$N_-(1,q)$	1	1	1	1	1	1	1	1	1	1	1	1
$N_-(2,q)$	1	2	3	5	7	11	14	20	26	35	44	58
$N_-(3,q)$	1	2	4	8	16	37	80	186	444	1091	2711	6857
$N_-(4,q)$	1	2	4	9	20	57	172	660	3093	18413	131556	1059916
$N_-(5,q)$	1	2	4	9	21	63	210	986	6773	77279	1432570	36967692
$N_-(6,q)$	1	2	4	9	21	64	217	1058	7898	110027	3156144	172638169

TABLE 5. Number of orbits, $N_-(m,q)$, of $\mathrm{Sp}_{2m}(2)$ on the set of multisets of size q in $T_- \subset \mathbb{Z}_2^{2m}$.

4.2. **Counting gradings.** We will use the following notation. For $n = 2^\alpha k$ where k is odd, set

$$f_\pm(n) = \sum_{m=0}^{\alpha} \sum_{s=0}^{\lfloor 2^{\alpha-m-1}k \rfloor} N_\pm(m, 2^{\alpha-m}k - 2s), \tag{23}$$

with the convention $N_+(0,q) = 1$ for all q and $N_-(0,q) = \delta_{0,q}$ (Kronecker delta). Since, for Series C, the rank r is related to the matrix size n as $n = 2r$, Theorem 6 implies that the number of fine gradings, $N_C(r)$, on the simple Lie algebra C_r ($r \geq 2$) is given by

$$N_C(r) = \begin{cases} f_-(2r) - 1 & \text{if } r \text{ is a power of 2,} \\ f_-(2r) & \text{otherwise.} \end{cases} \tag{24}$$

We calculated these numbers for $r \leq 100$. The results are stated in Table 6, where we included the case $C_1 = A_1$ for completeness. Note that if r is odd, then (23) involves only $m = 0$ and $m = 1$, hence $N_C(r) = \lfloor r/2 \rfloor + 2$.

4.3. **Asymptotic behaviour.** The following is an immediate consequence of (24) and Theorem 13:

Theorem 7. *Let $\mathbb{F}$ be an algebraically closed field,* $\mathrm{char}\,\mathbb{F} \neq 2$. *Let $\hat{N}_C(r)$ be defined by* (18). *The following asymptotic formula holds:*

$$\ln \hat{N}_C(r) = 2^{1/3} b_-(2r) r^{2/3} + O((\ln r)^2),$$

where $b_-(\cdot)$ is the bounded continuous function defined by (46) *and described in Lemma* 15. *In particular, $b_-(t) = b(t) + O(t^{-1/3})$ where $b(\cdot)$ is the function in Theorem* 5. □

5. Lie Algebras of Series D

This case is very similar to Series C, which was described in the previous section.

Theorem 8 ([Eld10, EK12c])**.** *Let $\mathbb{F}$ be an algebraically closed field,* $\mathrm{char}\,\mathbb{F} \neq 2$. *Let $n \geq 6$ be even. Assume $n \neq 8$. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_D(T,q,s,\tau)$ where T is an elementary 2-group of even rank, q and s are non-negative integers, $(q+2s)\sqrt{|T|} = n$, $\tau = (t_1, \ldots, t_q)$ is a q-tuple of elements of T_+, and $t_1 \neq t_2$ if $q = 2$ and $s = 0$. Moreover, $\Gamma_D(T_1,q_1,s_1,\tau_1)$ and $\Gamma_D(T_2,q_2,s_2,\tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2m}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of $\mathrm{Sp}_{2m}(2)$ as in Definition* 4. □

r	$N_C(r)$	r	$N_C(r)$	r	$N_C(r)$	r	$N_C(r)$	r	$N_C(r)$
1	2	21	12	41	22	61	32	81	42
2	3	22	108	42	1028	62	5332	82	19346
3	3	23	13	43	23	63	33	83	43
4	7	24	199	44	4510	64	323502	84	21899478
5	4	25	14	45	24	65	34	85	44
6	9	26	181	46	1484	66	7063	86	24283
7	5	27	15	47	25	67	35	87	45
8	17	28	339	48	10044	68	774947	88	48274977
9	6	29	16	49	26	69	36	89	46
10	18	30	293	50	2098	70	9237	90	30227
11	7	31	17	51	27	71	37	91	47
12	32	32	625	52	23038	72	1838997	92	103789470
13	8	33	18	53	28	73	38	93	48
14	34	34	458	54	2911	74	11941	94	37333
15	9	35	19	55	29	75	39	95	49
16	63	36	1122	56	55266	76	4274302	96	220645585
17	10	37	20	57	30	77	40	97	50
18	62	38	695	58	3970	78	15274	98	45777
19	11	39	21	59	31	79	41	99	51
20	107	40	2211	60	133241	80	9788777	100	456000882

TABLE 6. Number of fine gradings, $N_C(r)$, on the simple Lie algebra of type C_r assuming char $\mathbb{F} \neq 2$.

As mentioned in the introduction, the Lie algebra $\mathfrak{so}_8(\mathbb{F})$ (type D_4) requires special treatment.

5.1. **Counting orbits.** We can compute the numbers $N_+(m,q)$ in the same way as $N_-(m,q)$ (see Table 5). Some of the $N_+(m,q)$ are displayed in Table 7.

q	1	2	3	4	5	6	7	8	9	10	11	12
$N_+(1,q)$	1	2	3	4	5	7	8	10	12	14	16	19
$N_+(2,q)$	1	2	4	8	14	27	46	82	140	237	386	630
$N_+(3,q)$	1	2	4	9	20	53	138	408	1265	4161	13999	47628
$N_+(4,q)$	1	2	4	9	21	63	204	882	4945	36909	337821	3428167
$N_+(5,q)$	1	2	4	9	21	64	217	1048	7594	95775	2061395	62537928
$N_+(6,q)$	1	2	4	9	21	64	218	1067	8012	113097	3362409	198208405

TABLE 7. Number of orbits, $N_+(m,q)$, of $\mathrm{Sp}_{2m}(2)$ on the set of multisets of size q in $T_+ \subset \mathbb{Z}_2^{2m}$.

5.2. **Counting gradings.** Since, for Series D, the rank r is related to the matrix size n as $n = 2r$, Theorem 8 implies that the number of fine gradings, $N_D(r)$, on the simple Lie algebra D_r ($r = 3$ or $r \geq 5$) is given by

$$N_D(r) = \begin{cases} f_+(2r) - 1 & \text{if } r \text{ is a power of 2,} \\ f_+(2r) & \text{otherwise,} \end{cases} \tag{25}$$

where f_+ is defined by (23). We calculated these numbers for $r \le 100$. The results are stated in Table 8. For completeness, we included the case D_4 from [Eld10] (where it is assumed that char $\mathbb{F} = 0$), for which the number of fine gradings is 17 instead of 15 given by the above formula. Note that if r is odd, then (23) involves only $m = 0$ and $m = 1$, hence $N_D(r) = \sum_{s=0}^{\lfloor r/2 \rfloor} P_3(1 + 2s) + r + 1 = \sum_{s=0}^{\lfloor r/2 \rfloor} \operatorname{int} \frac{(s+2)^2}{3} + r + 1$, where $\operatorname{int} x$ denotes the integer nearest to x.

r	$N_D(r)$	r	$N_D(r)$	r	$N_D(r)$	r	$N_D(r)$	r	$N_D(r)$
		21	236	41	1302	61	3868	81	8601
		22	858	42	44335	62	944552	82	10125234
3	8	23	294	43	1480	63	4233	83	9219
4	17	24	1387	44	78115	64	7055100	84	1097811150
5	15	25	361	45	1674	65	4620	85	9866
6	26	26	1987	46	87671	66	1588770	86	15332525
7	25	27	438	47	1884	67	5030	87	10543
8	47	28	3186	48	173939	68	19667958	88	2848498443
9	39	29	525	49	2111	69	5464	89	11250
10	68	30	4538	50	166968	70	2606954	90	22842458
11	57	31	623	51	2356	71	5922	91	11988
12	113	32	7292	52	402982	72	54994767	92	7213746853
13	80	33	733	53	2619	73	6405	93	12758
14	161	34	10069	54	307013	74	4181709	94	33520718
15	109	35	855	55	2901	75	6914	95	13560
16	263	36	16255	56	991330	76	152123321	96	17847717516
17	144	37	990	57	3203	77	7449	97	14395
18	372	38	21550	58	546543	78	6569548	98	48505808
19	186	39	1139	59	3525	79	8011	99	15264
20	595	40	35756	60	2586241	80	413256061	100	43141937237

TABLE 8. Number of fine gradings, $N_D(r)$, on the simple Lie algebra of type D_r assuming char $\mathbb{F} \neq 2$ (char $\mathbb{F} = 0$ for $r = 4$).

5.3. **Asymptotic behaviour.** The following is an immediate consequence of (25) and Theorem 13:

Theorem 9. *Let $\mathbb{F}$ be an algebraically closed field,* char $\mathbb{F} \neq 2$. *Let $\hat{N}_D(r)$ be defined by* (18). *The following asymptotic formula holds:*

$$\ln \hat{N}_D(r) = 2^{1/3} b_+(2r) r^{2/3} + O((\ln r)^2),$$

where $b_+(\cdot)$ is the bounded continuous function defined by (46) *and described in Lemma* 15. *In particular, $b_+(t) = b(t) + O(t^{-1/3})$ where $b(\cdot)$ is the function in Theorem* 5. $\square$

6. Asymptotics of the number of fine gradings for Series A, C and D

The asymptotic formulas stated in Theorems 5, 7 and 9 follow from similar (rough) estimates for the functions $f_0(n)$ and $f_\pm(n)$ defined by (16) and (23), respectively. In the case of $f_\pm(n)$, only even values of n are relevant. Define

$$\hat{f}_0(n) = \frac{1}{n}\sum_{j=1}^{n} f_0(j)$$

and

$$\hat{f}_\pm(n) = \frac{1}{n/2}\sum_{2j\le n} f_\pm(2j).$$

Asymptotic analysis of the functions $\hat{f}_0$ and $\hat{f}_\pm$ will be essentially based on solution of the constrained optimization problem

$$\begin{cases} u(x,y) := (x+y)\ln(x+y) - x\ln x - y\ln y \ \to \max, \\ x>0,\ y>0,\ x^2y=1, \end{cases} \tag{26}$$

as well as on the analysis of similar but more delicate problems (slightly different in the three cases) with certain integrality constraints imposed on the arguments of the function u. The latter problems will be dealt with in the course of the proof of Theorems 11 and 13. At present, let us introduce functions and constants needed to state Theorem 11.

The problem (26) is equivalent to maximizing the function

$$v(x) := u(x, x^{-2}),\ x>0. \tag{27}$$

The critical point equation $v'(x) = 0$ can be transformed to a convenient short form, see (29) below, by writing $v = xw(x^3)$, where

$$w(z) = z^{-1}\ln(1+z) + \ln(1+z^{-1}).$$

Some properties of the function u and the solution of problem (26) are summarized in the next lemma for reference.

Lemma 10. *(a) The function $u(x,y)$ increases in both arguments and is homogeneous of degree 1, i.e., $u(tx,ty) = tu(x,y)$.*

(b) The solution of problem (26) *is*

$$\begin{gathered} x_0 = {z_0}^{1/3} \approx 0.575891, \quad y_0 = {z_0}^{-2/3} \approx 3.015227, \\ b_0 := u(x_0,y_0) = v(x_0) \approx 1.581080, \end{gathered} \tag{28}$$

where z_0 is the unique positive root of the transcendental equation

$$z\ln(1+z^{-1}) = 2\ln(1+z). \tag{29}$$

□

The second collection of constants and functions pertains to the first "more delicate" optimization problem mentioned above.

Consider the transcendental equation involving the function (27),

$$v(x/2) = v(x). \tag{30}$$

It has a unique positive root $x_1 \approx 0.800203$. Define also

$$b_1 := v(x_1/2) = v(x_1) \approx 1.512173. \tag{31}$$

Due to (30), the function

$$\tilde{v}(x) := \max\{v(x), v(x/2)\} \ = \begin{cases} v(x), & \text{if } x \le x_1 \\ v(x/2), & \text{if } x > x_1 \end{cases} \tag{32}$$

is continuous in the interval $[x_0, 2x_0]$. We will use this fact in the last set of preliminaries, which follows.

For $t \geq 1$, let $\phi(t)$ be the multiplicative excess of t over the greatest whole power of 2 below t, i.e.,

$$\phi(t) = \frac{t}{2^{\lfloor \log_2 t \rfloor}}. \tag{33}$$

Clearly, $1 \leq \phi(t) < 2$ and $\phi(2t) = \phi(t)$.

Next, for $t \geq x_0^3$, where x_0 is defined in (28), let

$$\lambda(t) = \phi(x_0^{-1} t^{1/3}), \tag{34}$$

and define

$$b(t) = \tilde{v}(x_0 \lambda(t)). \tag{35}$$

The function $b(t)$ is continuous, logarithmically periodic in the sense that $b(8t) = b(t)$, and has bounds

$$\min b(t) = b_1 \approx 1.512173, \qquad \max b(t) = b_0 \approx 1.581080. \tag{36}$$

The upper bound is attained when $x_0^{-1} t^{1/3} = 2^m$ with integer m. The lower bound is attained when $x_1^{-1} t^{1/3} = 2^m$ with integer m. The function is monotone and smooth between these maximum and minimum points.

We are now prepared to state the theorem describing the asymptotic behaviour of $\hat{f}_0(n)$.

Theorem 11. *There exists a constant $C > 0$ such that*

$$-\frac{2}{9 \ln 2} (\ln n)^2 - C \ln n \leq \ln \hat{f}_0(n) - b(n)\, n^{2/3} \leq C \ln n,$$

where the function $b(\cdot)$ is defined in (35)*. It is continuous, has property $b(8t) = b(t)$, and its lower and upper bounds are given in* (36)*.*

Proof. Let $f^*(n) = \max_{1 \leq j \leq n} f_0(j)$. Clearly,

$$\frac{1}{n} f^*(n) \leq \hat{f}_0(n) \leq f^*(n),$$

so $\ln \hat{f}_0(n) = \ln f^*(n) + O(\ln n)$. Therefore, it suffices to prove the desired estimate for $f^*(n)$ instead of $\hat{f}_0(n)$. (The letter C will denote a constant that may have different values in different formulas.)

Observe that in the sum (16) defining $f_0(n)$ the number of summands is $O(n \ln n)$. Hence, repeating the above argument, we see that

$$\ln f_0(n) = \ln N^* + O(\ln n),$$

where N^* is the largest summand. Now, the summands have the form $N(m, q)$, which are the numbers of orbits as described in Section 3. From (10) and (11) we obtain the inequalities:

$$\frac{B(m, q)}{|G_m|} \leq N(m, q) \leq B(m, q), \tag{37}$$

where $B(m, q)$ stands for the binomial coefficient $\binom{q + 2^{2m} - 1}{q}$.

The required upper bound for N^*, and hence for $f^*(n)$, follows from the inequality

$$\max_{2^m q\le n} \ln B(m,q) \le b(n)\, n^{2/3} + C\ln n, \tag{38}$$

which will be proved in Lemma 12.

To obtain the desired lower bound for N^*, it suffices to show that for each n there exist m^* and q^* with $2^{m^*}q^* \le n$ such that the following two inequalities hold with C independent of n:

$$\ln |G_{m^*}| \le \frac{2}{9\ln 2}(\ln n)^2 + C\ln n, \tag{39}$$

and

$$\ln B(m^*,q^*) \ge b(n)\, n^{2/3} - C\ln n. \tag{40}$$

The inequality (40) will be proved in Lemma 12, with m^* satisfying the estimate $2^{m^*} \le Cn^{1/3}$, i.e.,

$$m^* \le \frac{1}{3}\log_2 n + O(1).$$

We claim that this implies (39). Indeed, it is well known that

$$|\mathrm{Sp}_{2m}(2)| = 2^{m^2}\prod_{i=1}^{m}(2^{2i}-1),$$

hence we have

$$|G_m| = |\mathbb{Z}_2^{2m}|\cdot|\mathrm{Sp}_{2m}(2)| \le 2^{2m}\cdot 2^{m^2+m(m+1)} = 2^{2m^2+3m}.$$

Therefore,

$$\ln|G_{m^*}| \le 2(m^*)^2\ln 2 + O(m^*) \le 2\ln 2\left(\frac{\log_2 n}{3}\right)^2 + O(\ln n),$$

as claimed. □

Lemma 12. *Let* $B(m,q) = \binom{q+2^{2m}-1}{q}$*. For* $t>1$*, let*

$$B^*(t) = \max\{B(m,q) \mid m,q\in\mathbb{Z}_{\ge 0}, 2^m q\le t\}.$$

Then $\ln B^*(t) = b(t)\,t^{2/3} + O(\ln t)$*. Moreover, there exist* q^* *and* m^* *such that* $\ln B(m^*,q^*) = b(t)\,t^{2/3} + O(\ln t)$ *and* $q^* \le Ct^{2/3}$*,* $2^{m^*}\le Ct^{1/3}$*.*

Proof. Set $M := 2^{2m}$. Let us discard the trivial case $q=0$, which clearly does not provide maximum. So $\ln M$ and $\ln q$ are defined and nonnegative.

The condition $q\sqrt{M}\le t$ implies $\ln q\le \ln t$ and $\ln M \le 2\ln t$. Since $|\ln\binom{q+M-1}{q} - \ln\binom{q+M}{q}| = |\ln(q+M)-\ln M| \le C\ln t$, we may replace $B(m,q)$ by $\binom{q+M}{q}$.

By Stirling's formula,

$$\ln\binom{q+M}{q} = u(q,M) + \frac{1}{2}\ln\frac{q+M}{qM} + O(1),$$

where the function $u(\cdot,\cdot)$ is defined in (26). Again, due to the estimates $\ln q\le\ln t$ and $\ln M\le 2\ln t$, we obtain

$$\ln\binom{q+M}{q} = u(q,M) + O(\ln t),$$

so we come to the optimization problem

$$u(q, M) \to \max, \quad q \cdot \sqrt{M} \leq t,$$

with a strong additional restriction: $q \in \mathbb{Z}_{>0}$ and $M = 2^{2m}$ where $m \in \mathbb{Z}_{\geq 0}$. Without this restriction, as Lemma 10 tells us, the maximum would be equal to $b_0 t^{2/3}$ and attained at $q_0 = x_0 t^{2/3}$, $M_0 = y_0 t^{2/3}$. The idea is to get our q^* and M^* close to these values.

Suppose first that M is fixed and our only freedom is a choice of q, which is a nonnegative integer. To maximize $u(q, M)$ under the constraint $qM^{1/2} \leq t$, we should choose the largest possible q, i.e., $q = \lfloor tM^{-1/2} \rfloor$. With this value of q we have $u(q, M) = u(tM^{-1/2}, M) + O(\ln t)$. Therefore, the integrality condition on q can be ignored.

Now we write $q = xt^{2/3}$, where $q > 0$ is no longer assumed to be integer, and $M = x^{-2} t^{2/3}$. Thus we arrive at the following simplified optimization problem:

$$\left\{ \begin{array}{l} v(x) = u(x, x^{-2}) \to \max, \\ \\ x > 0, \ x^{-1} t^{1/3} = 2^m, \ m \in \mathbb{Z}_{\geq 0}. \end{array} \right. \tag{41}$$

Let $\mu = \log_2(x_0^{-1} t^{1/3})$. Since the function $v(x)$ is increasing in $(0, x_0)$ and decreasing in (x_0, ∞), the optimal value of m is either $\lfloor \mu \rfloor$ or $\lceil \mu \rceil$. Looking at (34), we see that, in the case $m = \lfloor \mu \rfloor$,

$$2^m = \frac{x_0^{-1} t^{1/3}}{\lambda(t)}, \quad x = t^{1/3} 2^{-m} = x_0 \lambda(t),$$

while in the case $m = \lceil \mu \rceil$ ($\mu \notin \mathbb{Z}$),

$$2^{m-1} = \frac{x_0^{-1} t^{1/3}}{\lambda(t)}, \quad x = t^{1/3} 2^{-m} = \frac{1}{2} x_0 \lambda(t).$$

Denote for a moment $x(t) = x_0 \lambda(t)$. Then we have $x = x(t)$ in the first case and $x = x(t)/2$ in the second case. Recalling the definition (31) of b_1, we see that the inequality $v(x(t)) \geq v(x(t)/2)$ holds if and only if $x(t) \leq x_1$. In view of (32) and (35), we conclude that the solution of the optimization problem (41) is exactly $b(t)$. Hence

$$\max\{u(q, 2^{2m}) \mid q > 0, m \in \mathbb{Z}_{\geq 0}, \ q \cdot 2^m \leq t\} = b(t)\, t^{2/3}.$$

The claimed asymptotics of $\ln B^*(t)$ follows. Finally, note that the values $m^* = \lfloor \mu \rfloor$ or $\lceil \mu \rceil$ (chosen as explained above) and $q^* = \lfloor x(t) t^{2/3} \rfloor$ or $\lfloor \frac{x(t)}{2} t^{2/3} \rfloor$ (respectively) satisfy the required conditions. $\square$

To state Theorem 13 we need two more function, $b_\pm(t)$, which will play the role of $b(t)$ in Theorem 11. We begin with substitutes for the function $v(x)$ defined by (27), which are

$$v_\tau^\pm(x) = u(x, x^{-2} \pm \tau x^{-1}). \tag{42}$$

Here $x, \tau > 0$; in the case of v_τ^- we also assume that $x^{-2} - \tau x^{-1} > 0$.

By the Implicit Function Theorem (IFT), for sufficiently small τ the transcendental equation

$$\frac{d}{dx} v_\tau^\pm(x) = 0 \tag{43}$$

has a unique root near the root x_0 of the equation $v'(x) = 0$. Denote that root $x_{0,\tau}^\pm$. For sufficiently small τ, the function $v^\pm(\tau)$ attains its maximum at $x_{0,\tau}^\pm$.

Similarly to (32), (34) and (35), we define

$$\tilde{v}_\tau^\pm(x) = \max\{v_\tau^\pm(x),\, v_\tau^\pm(x/2)\}, \tag{44}$$

$$\lambda_\tau^\pm(t) = \phi\left(\frac{t^{1/3}}{x_{0,\tau}^\pm}\right), \tag{45}$$

$$b_\pm(t) = \tilde{v}_\tau^\pm\left(x_{0,\tau}^\pm \cdot \lambda_\tau^\pm(t)\right)\Big|_{\tau=(2t)^{-1/3}}, \tag{46}$$

where $\phi(\cdot)$ is defined in (33). (Note that $b_\pm(t)$ are defined for sufficiently large t.)

The behaviour of the functions $b_\pm(t)$ with small τ is similar to that of $b(t)$. In particular, they are positive, bounded, and separated from zero (see Lemma 15, below, for more precise information).

Theorem 13. *There exists a constant $C > 0$ such that*

$$-\frac{2}{9\ln 2}(\ln n)^2 - C\ln n \le \ln \hat{f}_\pm(n) - 2^{-1/3} b_\pm(n)\, n^{2/3} \le C \ln n, \tag{47}$$

where the functions $b_\pm(\cdot)$ are defined in (46).

Proof. We follow the proof of Theorem 11 with minor modifications, so we only describe the changes that need to be made. The cases of $f_+(n)$ and $f_-(n)$ are completely analogous; their only difference is the sign $+$ or $-$ in various formulas.

In the estimate (37), the group is now $G_m = \mathrm{Sp}_2(m)$ and $B(m,q)$ stands for the binomial coefficient $\binom{q+2^{m-1}(2^m\pm1)-1}{q}$. Since $|\mathrm{Sp}_2(m)| < |\mathrm{ASp}_2(m)|$, the inequality (39) is proved as before. It remains to apply the next lemma, which is an adaptation of Lemma 12. $\square$

Lemma 14. *Let $B(m,q) = \binom{q+2^{m-1}(2^m\pm1)-1}{q}$. For $t > 1$, let*

$$B^*(t) = \max\{B(m,q) \mid m, q \in \mathbb{Z}_{\ge 0}, 2^m q \le t\}.$$

Then $\ln B^*(t) = 2^{-1/3} b_\pm(t)\, t^{2/3} + O(\ln t)$. Moreover, there exist q^* and m^* such that $\ln B(m^*, q^*) = 2^{-1/3} b_\pm(t)\, t^{2/3} + O(\ln t)$ and $q^* \le Ct^{2/3}$, $2^{m^*} \le Ct^{1/3}$.

Proof. As in the proof of Lemma 12, maximization of $B^*(t)$, to the accuracy of $O(\ln t)$, reduces to maximization of $u(q, M)$ under the constraints

$$2M = 2^m(2^m \pm 1), \quad q \cdot 2^m = t, \quad m \in \mathbb{Z}_{\ge 0}; \tag{48}$$

the condition $q \in \mathbb{Z}$ is dropped here.

Letting $q = 2^{-1/3} x \cdot t^{2/3}$, we have $2^m = (2t)^{1/3} x^{-1}$,

$$M = 2^{-1/3} x^{-2} t^{2/3} (1 \pm (2t)^{-1/3} x),$$

and

$$u(q, M) = 2^{-1/3} t^{2/3}\, v_\tau^\pm(x)\Big|_{\tau=(2t)^{-1/3}}.$$

The rest of the proof repeats that of Lemma 12. $\square$

The functions $b_\pm(t)$ used in Theorem 13 can be computed by (46) and preceding formulas. However, understanding of their qualitative behaviour is obscured by the involvement of the parameter τ. It is desirable to have simpler even if approximate expressions for $b_\pm(t)$ in terms of functions of just one variable. This is the purpose of Lemma 15. Informally it says that for large t, on every interval between t and $8t$ with deleted subinterval of size $O(t^{-1/3})$ the functions $b_\pm(t)$ have two-term asymptotics (51), while if the exceptional subintervals are not deleted, then there is a uniform but less precise approximation (49).

If we replace $b_\pm(t)$ in Theorem 13 by their asymptotics (51), the error terms $O(t^{-2/3})$ give rise to the error of order $O(1)$ in the middle part of the inequality (47), which can be discarded at the expense of a possible increase of the constant C.

The exceptional intervals are $O(t^{-1/3})$-neighbourhoods of values of t corresponding to the switching point of maximum in (32). In those intervals, $b_\pm(t) = b_1 + O(t^{-1/3})$ with b_1 as in (31) and (36). For the corresponding values of n in (47), our simplified estimate (49) yields a coarser asymptotics,

$$\hat{f}_\pm(n) = 2^{-1/3} b_1 n^{2/3} + O(n^{1/3}).$$

Lemma 15. *(i) The functions $b_\pm(t)$ are continuous and bounded. Moreover,*

$$b_\pm(t) = b(t) + O(t^{-1/3}), \tag{49}$$

where $b(t)$ is defined by (35).

(ii) Recall x_0 defined in Lemma 10 and x_1 defined by (30). *There exists $C > 0$ such that for all sufficiently large t satisfying*

$$\left|\phi\left(\frac{t^{1/3}}{x_0}\right) - \frac{x_1}{x_0}\right| > Ct^{-1/3} \tag{50}$$

we have

$$b_\pm(t) = b(t) \pm t^{-1/3} b^{(1)}(t) + O(t^{-2/3}), \tag{51}$$

where $b^{(1)}(t)$ is a bounded function given by the formula of a type similar to (35)*:*

$$b^{(1)}(t) = \frac{\ln(1 + x^3(t))}{x(t)}, \qquad x(t) = \begin{cases} x_0\lambda(t) & \text{if } x_0\lambda(t) < x_1, \\ (x_0/2)\lambda(t) & \text{if } x_0\lambda(t) > x_1. \end{cases} \tag{52}$$

In particular, $b^{(1)}(8t) = b^{(1)}(t)$.

Proof. Let us analyse the function $b_+(\cdot)$, say. To lighten notation, we will write v_τ instead of v_τ^+, etc., when referring to the functions (42)–(45). Whenever the behaviour of a function with subscript τ with respect to t is discussed, it will be assumed that $\tau = (2t)^{-1/3}$ unless stated otherwise.

(i) First, note the asymptotics of the critical point $x_{0,\tau}$ of the function $v_\tau(x)$: $x_{0,\tau} = x_0 + O(\tau)$ (by IFT).

The argument of the function $\tilde{v}_\tau$ in (46) lies in $[x_{0,\tau}, 2x_{0,\tau}] \subset I_0 := [x_0/4, 4x_0]$ for sufficiently large t. Regarding $v_\tau(x)$ as a small perturbation of the function $v(x) = u(x, x^{-2})$, we have the approximation

$$v_\tau(x) = v(x) + O(\tau) \quad \text{as } \tau \to 0$$

with uniform remainder term for $x \in I_0$. The same approximation holds true with x replaced by $x/2$ and consequently for $\tilde{v}_\tau(x)$ and $\tilde{v}(x)$ instead of, respectively, $v_\tau(t)$ and $v(x)$, cf. (32) and (44). Thus we obtain the asymptotics (49).

The function $b_+(t)$ is certainly continuous at all points of continuity of $\lambda_\tau(t)$. In order to check its continuity for all values of t, it remains to consider the points where $\lambda_\tau(t)$ has jump discontinuities. If $t = \theta$ is such a point, then there are two limit values of $x_{0,\tau}\lambda_\tau(t)$ as $t \to \theta$, namely, $x_{0,\tau}$ and $2x_{0,\tau}$. Hence, the limit values of $b_+(t)$ are $\tilde{v}_\tau(x_{0,\tau})$ and $\tilde{v}_\tau(2x_{0,\tau})$. But these are equal by definition (44) of $\tilde{v}_\tau$.

(ii) We introduce two auxiliary functions of the variables t and τ, temporarily considered as independent:

$$\beta_k(t,\tau) = v_\tau\left(\frac{x_{0,\tau}}{k}\cdot\phi\left(\frac{t^{1/3}}{x_{0,\tau}}\right)\right), \quad k\in\{1,2\}. \tag{53}$$

Then

$$b_+(t) = \max_k \beta_k(t,(2t)^{-1/3}).$$

Let us analyse more carefully which of the two values of k provides the maximum and where the switch between $k=1$ and $k=2$ occurs.

By IFT, for sufficiently small values of τ the equation $v_\tau(x/2) = v_\tau(x)$ has a unique root $x_{1,\tau}$ in the interval $[x_0, 2x_0]$, and $x_{1,\tau} = x_1 + O(\tau)$. (We omit a verification of applicability of IFT, which amounts to numerical evaluation with guaranteed maximum error.)

In the interval $x\in[x_{0,\tau}/2, 2x_{0,\tau}]$ the function $v_\tau(x)$ has maximum at $x_{0,\tau} = x_0+O(\tau)$; it is increasing for $x < x_{0,\tau}$ and decreasing for $x > x_{0,\tau}$. Hence, to obtain $b_+(t)$, we must substitute $\tau = (2t)^{-1/3}$ and take $k=1$ if $\phi(t^{1/3}/x_{0,\tau}) \le x_{1,\tau}/x_{0,\tau}$ and $k=2$ otherwise.

We claim that the condition

$$(-1)^k\left(\phi\left(\frac{t^{1/3}}{x_0}\right) - \frac{x_1}{x_0}\right) > C\tau$$

with sufficiently large C implies

$$(-1)^k\left(\phi\left(\frac{t^{1/3}}{x_{0,\tau}}\right) - \frac{x_{1,\tau}}{x_{0,\tau}}\right) > 0.$$

It is clearly so if $\phi(x_0^{-1}t^{1/3})$ is close to the extreme values 1 or 2, so assume that $\phi(x_0^{-1}t^{1/3}) \in (1+\varepsilon, 2-\varepsilon)$ with some $\varepsilon > 0$. To be specific, suppose $(-1)^k = 1$ and write the implication to prove in the abridged form $(\phi_0 - \xi_0 > C\tau) \overset{?}{\Rightarrow} (\phi_\tau - \xi_\tau > 0)$. By a standard argument with triangle inequality, it suffices to show that $|\phi_0 - \phi_\tau| < C\tau/2$ and $|\xi_0 - \xi_\tau| < C\tau/2$. The latter inequality with appropriate C is equivalent to the obvious estimate $(x_{1,\tau}/x_{0,\tau}) - (x_1/x_0) = O(\tau)$. As for the former one, note that $\phi(\cdot)$ is linear homogeneous on its intervals of continuity, so

$$\frac{\phi_0}{\phi_\tau} = \frac{x_{0,\tau}}{x_0} = 1 + O(\tau).$$

The estimate $\phi_0 - \phi_\tau = O(\tau)$ follows.

The rest is simple: once k is determined by the value of $\phi(x_0^{-1}t^{1/3})$, we write

$$\beta_k(t,\tau) = \beta_k(t,0) + \left.\frac{\partial\beta_k}{\partial\tau}\right|_{\tau=0}\tau + O(\tau^2).$$

Here $\beta_k(t,0) = b(t)$ and the second term has the form $b^{(1)}(t)\tau$. Since $\partial_\tau\beta(t,0)$ depends on t only through the function $\phi(x_0^{-1}t^{1/3})$ in the argument of $v'(\cdot)$, it is clear that $b^{(1)}(t)$ is bounded and $b^{(1)}(t) = b^{(1)}(8t)$.

It remains to complete calculation of $b^{(1)}(t)$. It will follow that the result in the case of $b_-(t)$ differs from that in the case of $b_+(t)$ just in sign.

Observe that the argument of the function v_τ in (53) is locally constant, hence $(\partial/\partial\tau)\beta_k(t,\tau) = (\partial v_\tau/\partial\tau)(\dots)$. Looking at the definition (42) of the functions $v_\tau^\pm$

and recalling the definition of u in (26), we find

$$\frac{\partial v_\tau^\pm}{\partial \tau}\bigg|_{\tau=0} = \pm x^{-1}\,\frac{\partial u(x,y)}{\partial y}\bigg|_{y=x^{-2}} = \pm x^{-1}\ln(x^3+1).$$

Taking $k=1$ or 2 according to the sign of $x_{0,\tau}\phi(t^{1/3}/x_{0,\tau})-x_1$, we come to the formula (52). $\square$

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