Generalized Lie algebras and cocycle twists*

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Abstract

We use the results of Etingof and Gelaki on the classification of (co)triangular Hopf algebras to extend Scheunert's "discoloration" technique to Lie algebras in the category of (co)modules. As an application, we prove a PBW-type theorem for such Lie algebras. We also discuss the relationship between Lie algebras in the category of (co)modules and symmetric braided Lie algebras introduced by Gurevich. Finally, we construct examples of symmetric braided Lie algebras that are essentially different from Lie coloralgebras.

1 Introduction

The notion of a superalgebra is well-known. The idea is to introduce a $\mathbb{Z}_2$-grading on the algebra and replace the usual flip $x \otimes y \mapsto y \otimes x$ in the defining identities of a class of algebras (commutative, Lie, Jordan, etc.) by the map $x \otimes y \mapsto (-1)^{p(x)p(y)} y \otimes x$, where $p(x)$ and $p(y)$ are the "parities" of the elements x and y. A further development of this idea leads to a Lie coloralgebra [14], where the algebra is graded by an abelian group G and the flip in the anticommutativity and Jacobi identities is replaced by the map $t : x \otimes y \mapsto \beta(g, h) y \otimes x$ where x is homogeneous of degree g, y is homogeneous of degree h, and $\beta(g, h)$ is a skew-symmetric bicharacter on G, called the commutation factor (the skew-symmetry is needed to ensure that $t^2 = id$). More generally, one can introduce a braiding operator t on the

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algebra and replace the flip by the braiding. This leads to Lie "S-algebras" of Gurevich [7]. As in [7], we restrict ourselves to the case when the braiding t is *symmetric*, i.e., $t^2 = id$.

Symmetric braidings arise naturally in the category of (co)modules over a (co)triangular Hopf algebra. Lie algebras in the category of comodules over a cotriangular Hopf algebra (H, β) were studied in [1, 2] where they were called (H, β)-Lie algebras. They generalize Lie coloralgebras.

A very useful technique in studying Lie coloralgebras is the so-called "discoloration" introduced by Scheunert [14]. The idea is to change the bracket on a Lie superalgebra by replacing $[x, y]$ for homogeneous x, y with $[x, y]_\sigma = \sigma(g, h)[x, y]$ where σ is a nonzero scalar that depends on the degrees of x and y. If $\sigma(g, h)$ is a 2-*cocycle* on the group G, then the new bracket also satisfies the anticommutativity and Jacobi identities, but with a different commutation factor. It turns out that by choosing a suitable σ one can always make the new bracket satisfy the identities of a Lie superalgebra.

In [2, 3] Scheunert's idea was carried over to (H, β)-Lie algebras for cocommutative (and hence commutative) cotriangular Hopf algebra H. The recent progress in the classification of (co)triangular Hopf algebras [4, 5] allows us in this paper to obtain "discoloration" results for more general cotriangular Hopf algebras.

In Section 2 we recall the basics on symmetric categories. In Section 3 we prove a general result on the behavior of polynomial identities in symmetric categories under a braided monoidal functor (Theorem 3.3) and then specialize to a cocycle twist in the category of (co)modules over a (co)triangular bialgebra (Corollary 3.4). In Section 4 we use the results of [4, 6] and of Section 3 to prove "discoloration" theorems in the categories of modules and comodules (Theorems 4.6 and 4.3). As an application we prove a version of PBW Theorem for (H, β)-Lie algebras (Theorem 4.9). Section 5 is devoted to the discussion of the relationship between braided algebras in the category of (co)modules and "stand-alone" braided (non-associative) algebras such as Lie S-algebras. Given a finite-dimensional braided algebra A, we use the FRT construction to find a cotriangular bialgebra H such that A is an H-comodule algebra and the given braiding on A coincides with the one coming from the category of H-comodules (see Theorem 5.3). We use this construction to show that, under some natural "minimality" conditions on (H, β), the notion of a braided algebra is equivalent to the notion of an algebra in the category of H-comodules (Theorem 5.4). In Section 6 we give explicit examples of (H, β)-Lie algebras that are essentially different from Lie coloralgebras.

Now we fix the notation that will be used throughout the paper. The ground field will be denoted by $\Bbbk$. All vector spaces, algebras, coalgebras, and their tensor products will be taken over $\Bbbk$. The comultiplication on a bialgebra H will be denoted by Δ, counit by ε, and the antipode (if it exists) by S. A right H-comodule structure on a vector space V will be denoted by $\rho : V \to V \otimes H$. We will also use the sigma notation for comultiplication:

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$$

and coaction:

$$\rho(v) = \sum v_{(0)} \otimes v_{(1)}.$$

We refer the reader to [13, 10] for basic facts on bialgebras and Hopf algebras.

2 Symmetric Categories

In this section we briefly recall the basic definitions on symmetric categories and fix the notation. The reader is referred to [12] for details.

By a *monoidal category* we will always mean a strict monoidal $\Bbbk$-linear category. In fact, we will be mostly interested in categories of $\Bbbk$-vector spaces endowed with some additional structure. We will also assume that all *monoidal functors* are $\Bbbk$-linear and preserve the unit objects. Consequently, we will omit parentheses in tensor products. For an object V, we denote its tensor powers by $V^{\otimes n}$, for all $n = 0, 1, 2, \dots$ (where $V^{\otimes 0}$ is the unit object).

A *symmetric category* is a monoidal category $\mathfrak{C}$ with a *symmetric braiding* t, i.e., a natural family of isomorphisms $t_{V,W} : V \otimes W \to W \otimes V$ in $\mathfrak{C}$ satisfying the hexagon and symmetry axioms:

$$(t_{V,W} \otimes id_U) \circ (id_V \otimes t_{U,W}) \circ (t_{U,V} \otimes id_W) = (id_W \otimes t_{U,V}) \circ (t_{U,W} \otimes id_V) \circ (id_U \otimes t_{V,W})$$

and $t_{W,V} \circ t_{V,W} = id_{V \otimes W}$, for all U, V, W in $\mathfrak{C}$.

Then for any V and n, the symmetric group $\mathcal{S}_n$ acts on $V^{\otimes n}$ on the left in the usual way: we let the transpositions $s_i = (i, i+1)$, $i = 1, \dots, n-1$, act by

$$(id_V)^{\otimes (i-1)} \otimes t_{V,V} \otimes (id_V)^{\otimes (n-i-1)}$$

and extend this action to $\mathcal{S}_n$. For $\pi \in \mathcal{S}_n$, denote by $t_{V,n}(\pi)$ the corresponding automorphism of $V^{\otimes n}$. In particular, if $\mathfrak{C}$ is the category of vector spaces with the usual flip $\tau_{V,W} : v \otimes w \mapsto w \otimes v$, then $\tau_{V,n}$ is given by $\tau_{V,n} : v_1 \otimes \cdots \otimes v_n \mapsto v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(n)}$.

Now let $\mathfrak{C}$ and $\mathfrak{C}'$ be symmetric categories and (Φ, φ_2) be a braided monoidal functor from $\mathfrak{C}$ to $\mathfrak{C}'$, i.e., $\Phi : \mathfrak{C} \to \mathfrak{C}'$ is a functor and

$$\varphi_2(V, W) : \Phi(V) \otimes_{\mathfrak{C}'} \Phi(W) \to \Phi(V \otimes_{\mathfrak{C}} W)$$

is a natural family of morphisms in $\mathfrak{C}'$ such that, for all U, V, W in $\mathfrak{C}$, the following diagrams commute:

$$\begin{array}{ccc}
\Phi(U) \otimes_{\mathfrak{C}'} \Phi(V) \otimes_{\mathfrak{C}'} \Phi(W) & \xrightarrow{\varphi_2 \otimes_{\mathfrak{C}'} id} & \Phi(U \otimes_{\mathfrak{C}} V) \otimes_{\mathfrak{C}'} \Phi(W) \\
{\scriptstyle id \otimes_{\mathfrak{C}'} \varphi_2} \big\downarrow & & \big\downarrow {\scriptstyle \varphi_2} \\
\Phi(U) \otimes_{\mathfrak{C}'} \Phi(V \otimes_{\mathfrak{C}} W) & \xrightarrow{\varphi_2} & \Phi(U \otimes_{\mathfrak{C}} V \otimes_{\mathfrak{C}} W)
\end{array} \qquad (1)$$

$$\begin{array}{ccc}
\Phi(V) \otimes_{\mathfrak{C}'} \Phi(W) & \xrightarrow{t_{\mathfrak{C}'}} & \Phi(W) \otimes_{\mathfrak{C}'} \Phi(V) \\
{\scriptstyle \varphi_2} \big\downarrow & & \big\downarrow {\scriptstyle \varphi_2} \\
\Phi(V \otimes_{\mathfrak{C}} W) & \xrightarrow{\Phi(t_{\mathfrak{C}})} & \Phi(W \otimes_{\mathfrak{C}} V)
\end{array} \qquad (2)$$

From (1) it follows that one can unambiguously define the morphisms $\varphi_n(V) : \Phi(V)^{\otimes n} \to \Phi(V^{\otimes n})$, for all V in $\mathfrak{C}$ and $n = 0, 1, 2, \ldots$ (where $\varphi_0(V)$ and $\varphi_1(V)$ are identity morphisms). In its turn, (2) implies that

$$\varphi_n(V) \circ t_{\Phi(V),n}(\pi) = \Phi(t_{V,n}(\pi)) \circ \varphi_n(V) \text{ for all } \pi \in \mathcal{S}_n. \qquad (3)$$

3 Polynomial identities in symmetric categories

Let $\mathfrak{C}$ be a symmetric category. Recall that a *non-associative algebra in* $\mathfrak{C}$ is an object A endowed with a multiplication morphism $\mu_A : A \otimes A \to A$. In the following sections, we will be interested mostly in Lie algebras, but the main result of this section holds for arbitrary non-associative algebras and, in fact, is easier to prove in this generality.

Let $\mathcal{F}$ be the free non-associative algebra over $\Bbbk$ with free generators $x_1, x_2, \ldots$. Let $F = F(x_1, \ldots, x_n) \in \mathcal{F}$. Recall that an algebra A over $\Bbbk$ is said to satisfy the polynomial identity $F = 0$ if $F(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. We want to restate this definition in an element-free way so that it will make sense in any symmetric $\Bbbk$-linear category. For simplicity, assume that F is multilinear in $x_1, \ldots, x_n$ (recall that if $\operatorname{char} \Bbbk = 0$, then any set of polynomial identities is equivalent to a set of multilinear identities).

First we introduce some notation. Let $\mathcal{M}$ be the set of all non-associative monomials in one variable x (i.e., the free magma in one generator) and $\mathcal{M}_n$ the set of elements of $\mathcal{M}$ that have degree n. Then there is a natural one-to-one correspondence between the set of multilinear non-associative monomials in $x_1, \ldots, x_n$ and the set of pairs (u, π) where $u \in \mathcal{M}_n$ and

$\pi \in \mathcal{S}_n$. Namely, given such a pair (u, π), the corresponding monomial $M(u, \pi)$ is $x_{\pi^{-1}(1)} \cdots x_{\pi^{-1}(n)}$ with brackets arranged as in u. So, given a multilinear $F = F(x_1, \dots, x_n) \in \mathcal{F}$, we can write:

$$F = \sum_{u \in \mathcal{M}_n} \sum_{\pi \in \mathcal{S}_n} \lambda_{(u,\pi)} M(u, \pi). \tag{4}$$

Let $\mathfrak{C}$ be a symmetric ($\Bbbk$-linear) category. Let A be an algebra in $\mathfrak{C}$. For $u \in M_n$, define the morphisms $\mu_{A,u} : A^{\otimes n} \to A$ by induction on $n \geq 1$. Set $\mu_{A,x} = id_A$. For $n > 1$, write $u = u_1 u_2$ and set $\mu_{A,u} = \mu_A \circ (\mu_{A,u_1} \otimes \mu_{A,u_2})$. If A is an algebra over $\Bbbk$ in the usual sense, then A satisfies $F = 0$ iff $\sum_{u \in \mathcal{M}_n} \sum_{\pi \in \mathcal{S}_n} \lambda_{(u,\pi)} \mu_{A,u} \circ \tau_{A,n}(\pi) = 0$. The latter condition extends to algebras in $\mathfrak{C}$ in the standard way, i.e., by replacing τ with t:

Definition 3.1. Let A be an algebra in a symmetric category $\mathfrak{C}$ with braiding t and let F be as in (4). We say that A satisfies the *polynomial identity* $F = 0$ if

$$\sum_{u \in \mathcal{M}_n} \sum_{\pi \in \mathcal{S}_n} \lambda_{(u,\pi)} \mu_{A,u} \circ t_{A,n}(\pi) = 0. \tag{5}$$

Definition 3.2. Let $\mathcal{T}$ be a set of (multilinear) polynomial identities. The *variety* of algebras $\mathrm{Var}(\mathfrak{C}, \mathcal{T})$ is the class of all algebras in $\mathfrak{C}$ that satisfy every identity in $\mathcal{T}$. We can view $\mathrm{Var}(\mathfrak{C}, \mathcal{T})$ as a full subcategory of the category $\mathrm{Var}(\mathfrak{C})$ of all algebras in $\mathfrak{C}$.

In particular, one can speak about associative, commutative, Lie, Jordan, nilpotent, etc. algebras in $\mathfrak{C}$. Note that the variety of associative algebras in $\mathfrak{C}$ (or, more generally, any A-homogeneous variety) does not depend on the braiding t.

Now let $\mathfrak{C}'$ be another symmetric category and (Φ, φ_2) a braided monoidal functor from $\mathfrak{C}$ to $\mathfrak{C}'$. If A is an algebra in $\mathfrak{C}$, then $\Phi(A)$ is an algebra in $\mathfrak{C}'$ with multiplication morphism defined by

$$\mu_{\Phi(A)} = \Phi(\mu_A) \circ \varphi_2(A) : \Phi(A) \otimes_{\mathfrak{C}'} \Phi(A) \to \Phi(A).$$

From naturality of φ_2 it follows that if $f : A \to B$ is a morphism of algebras in $\mathfrak{C}$, then $\Phi(f) : \Phi(A) \to \Phi(B)$ is a morphism of algebras in $\mathfrak{C}'$.

Theorem 3.3. *Let $(\Phi, \varphi_2) : \mathfrak{C} \to \mathfrak{C}'$ be a braided monoidal functor. Let A be an algebra in $\mathfrak{C}$. If A satisfies the (multilinear) polynomial identity $F = 0$, then so does the algebra $\Phi(A)$ in $\mathfrak{C}'$. Moreover, if Φ is* strong *(i.e., all $\varphi_2(V, W)$ are isomorphisms), then A satisfies $F = 0$ in $\mathfrak{C}$ iff $\Phi(A)$ satisfies $F = 0$ in $\mathfrak{C}'$.*

Proof. We must verify (5) for the algebra $\Phi(A)$. First, one proves by induction on $n \geq 1$ and using naturality of φ_2 that

$$\mu_{\Phi(A),u} = \Phi(\mu_{A,u}) \circ \varphi_n(A) \text{ for all } u \in \mathcal{M}_n. \tag{6}$$

Then the left-hand side of (5) for $\Phi(A)$ can be rewritten as follows:

$$\begin{aligned}
& \sum_{u\in\mathcal{M}_n} \sum_{\pi\in\mathcal{S}_n} \lambda_{(u,\pi)} \mu_{\Phi(A),u} \circ t_{\Phi(A),n}(\pi) \\
= & \sum_{u\in\mathcal{M}_n} \sum_{\pi\in\mathcal{S}_n} \lambda_{(u,\pi)} \Phi(\mu_{A,u}) \circ \varphi_n(A) \circ t_{\Phi(A),n}(\pi) \\
= & \sum_{u\in\mathcal{M}_n} \sum_{\pi\in\mathcal{S}_n} \lambda_{(u,\pi)} \Phi(\mu_{A,u}) \circ \Phi(t_{A,n}(\pi)) \circ \varphi_n(A) \\
= & \ \Phi\left(\sum_{u\in\mathcal{M}_n} \sum_{\pi\in\mathcal{S}_n} \lambda_{(u,\pi)} \mu_{A,u} \circ t_{A,n}(\pi) \right) \circ \varphi_n(A) = 0,
\end{aligned}$$

where we used (6), (3), and the fact that A satisfies $F = 0$ in $\mathfrak{C}$. ■

We will be mostly interested in the case when $\mathfrak{C} = \mathfrak{M}^H$ where H is a cotriangular bialgebra with universal R-form $\beta : H \otimes H \to \Bbbk$ (see e.g. [10] or [13], but note the left-right difference between axioms in these books — we follow the version of [10]). Recall that the braiding on $\mathfrak{M}^H$ is given by

$$t_{V,W} : v \otimes w \mapsto \sum \beta(v_{(1)}, w_{(1)}) w_{(0)} \otimes v_{(0)}.$$

Suppose (H, β) is a cotriangular bialgebra and $\sigma : H \otimes H \to \Bbbk$ a *right 2-cocycle*, i.e., a convolution-invertible map that satisfies the equations:

$$\sum \sigma(a, b_{(1)} c_{(1)}) \sigma(b_{(2)}, c_{(2)}) = \sum \sigma(a_{(1)} b_{(1)}, c) \sigma(a_{(2)}, b_{(2)})$$

and $\sigma(a, 1) = \sigma(1, a) = \varepsilon(a)$, for all $a, b, c \in H$. It is well-known (see e.g. [10]) that (H_σ, β_σ) is again a cotriangular bialgebra where $H_\sigma = H$ as a coalgebra, the multiplication of H_σ is given by

$$h \cdot_\sigma k = \sum \sigma^{-1}(h_{(1)}, k_{(1)}) h_{(2)} k_{(2)} \sigma(h_{(3)}, k_{(3)}), \tag{7}$$

and

$$\beta_\sigma(h, k) = \sum \sigma^{-1}(k_{(1)}, h_{(1)}) \beta(h_{(2)} k_{(2)}) \sigma(h_{(3)}, k_{(3)}). \tag{8}$$

Also $\Phi = id : \mathfrak{M}^H \to \mathfrak{M}^{H_\sigma}$ and

$$\varphi_2(V, W) : v \otimes w \mapsto \sum \sigma(v_{(1)}, w_{(1)}) v_{(0)} \otimes w_{(0)}$$

define an equivalence of braided monoidal categories $\mathfrak{M}^H$ and $\mathfrak{M}^{H_\sigma}$. If A is an algebra in $\mathfrak{M}^H$ with multiplication $\mu : A \otimes A \to A$, then $\Phi(A) = A$ as an H-comodule and the multiplication of $\Phi(A)$ is given by

$$\mu_\sigma(a \otimes b) = \sum \sigma(a_{(1)}, b_{(1)})\mu(a_{(0)} \otimes b_{(0)}). \tag{9}$$

We denote $\Phi(A)$ by A_σ and call it the *σ-twist* of A.

Dually, let H be a (finite-dimensional) triangular bialgebra with universal R-matrix $R \in H \otimes H$ (see e.g. [10] or [13]). Recall that $J \in H \otimes H$ is a *right twist* for a (finite-dimensional) bialgebra H if J is a right 2-cocycle when viewed as a map $H^* \otimes H^* \to \Bbbk$. The twisted bialgebra H^J is H as an algebra, with comultiplication defined by $\Delta^J(h) = J^{-1}\Delta(h)J$. If (H, R) is a triangular bialgebra, then so is (H^J, R^J), where $R^J = J_{21}^{-1}RJ$ and $J_{21} = \tau(J)$.

Also $\Phi = id : {}_H\mathfrak{M} \to {}_{H^J}\mathfrak{M}$ and

$$\varphi_2(V, W) : v \otimes w \mapsto J \cdot (v \otimes w)$$

define an equivalence of braided monoidal categories ${}_H\mathfrak{M}$ and ${}_{H^J}\mathfrak{M}$. For an algebra A in ${}_H\mathfrak{M}$, we denote $\Phi(A)$ by A_J and call it the *J-twist* of A. The multiplication of A_J is given by

$$\mu_J(a \otimes b) = \mu(J \cdot (a \otimes b)). \tag{10}$$

Specializing Theorem 3.3 to the case $\mathfrak{C} = \mathfrak{M}^H$, resp. $\mathfrak{C} = {}_H\mathfrak{M}$, we obtain the following:

Corollary 3.4. *Let $\mathcal{T}$ be a set of multilinear polynomial identities. Then the functor defined by $A \mapsto A_\sigma$, resp., $A \mapsto A_J$, and $f \mapsto f$ on morphisms, is an equivalence of the categories* $\mathrm{Var}(\mathfrak{M}^H, \mathcal{T})$ *and* $\mathrm{Var}(\mathfrak{M}^{H_\sigma}, \mathcal{T})$, *resp.* $\mathrm{Var}({}_H\mathfrak{M}, \mathcal{T})$ *and* $\mathrm{Var}({}_{H^J}\mathfrak{M}, \mathcal{T})$. ■

4 Twisting Lie algebras in $\mathfrak{M}^H$ and ${}_H\mathfrak{M}$

Let (H, β) be a cotriangular bialgebra. We specialize Definition 3.2 to the variety of Lie algebras in $\mathfrak{M}^H$, i.e., $\mathfrak{C} = \mathfrak{M}^H$, $\mathcal{T} = \{x_1x_2 + x_2x_1, (x_1x_2)x_3 + (x_3x_1)x_2 + (x_2x_3)x_1\}$. As usual for Lie algebras, we write brackets to denote multiplication.

Definition 4.1. Let L be an algebra in $\mathfrak{M}^H$ with $\mu_L(x \otimes y)$ denoted by $[x, y]$. Then L is a Lie algebra in $\mathfrak{M}^H$ if it satisfies the (braided) anticommutativity:

$$[x, y] + \sum \beta(x_{(1)}, y_{(1)})[y_{(0)}, x_{(0)}] = 0, \tag{11}$$

and the (braided) Jacobi identity:

$$
\begin{aligned}
[[x,y],z] \quad &+ \quad \sum \beta(x_{(1)}y_{(1)}, z_{(1)})[[z_{(0)}, x_{(0)}], y_{(0)}] \\
&+ \quad \sum \beta(x_{(1)}, y_{(1)}z_{(1)})[[y_{(0)}, z_{(0)}], x_{(0)}] = 0. \qquad (12)
\end{aligned}
$$

Following [2], we will call these objects (H,β)-*Lie algebras*.

Applying Corollary 3.4 to (H,β)-Lie algebras, we obtain the following corollary (special cases of which already appeared in [2] and [11, Chapter 5]).

Corollary 4.2. *Let L be an (H,β)-Lie algebra. Then L_σ is an (H_σ,β_σ)-Lie algebra. Moreover, L and L_σ have the same H-comodule subalgebras and ideals. L is solvable (resp., nilpotent) iff so is L_σ.*

Now if we can find a suitable 2-cocycle σ, we can simplify (H,β)-Lie algebras by twisting. The well-known result of Scheunert [14] on "discoloration" of Lie coloralgebras is of this form. Indeed, let G be an abelian group and $\beta : G \times G \to \Bbbk^\times$ a skew-symmetric bicharacter. Then $H = \Bbbk G$ is a commutative and cocommutative Hopf algebra, β gives a cotriangular structure on H, and (H,β)-Lie algebras are precisely Lie coloralgebras with commutation factor β: $L = \oplus_{g\in G} L_g$ where $L_g = \{a \in L \,|\, \rho(a) = a \otimes g\}$. It is shown in [14] that there exists a 2-cocycle $\sigma : G \times G \to \Bbbk^\times$ such that β_σ is a "sign bicharacter":

$$
\beta_\sigma(g,h) = \begin{cases} -1 & \text{if } g,h \in G_-, \\ 1 & \text{otherwise;} \end{cases}
$$

where $G_- = G \setminus G_+$ and G_+ is a subgroup of index ≤ 2. It follows that σ twists any Lie coloralgebra L with commutation factor β into a Lie superalgebra $L_\sigma = L_+ \oplus L_-$, where $L_+ = \oplus_{g\in G_+} L_g$ and $L_- = \oplus_{g\in G_-} L_g$. Scheunert's result was generalized to an arbitrary cocommutative (and hence commutative) cotriangular Hopf algebra H over a field of characteristic zero (see [2]) or positive characteristic not equal to 2 (see [3]). Note that for cocommutative H, we have $H = H_\sigma$, so we only need to keep track of β. A further generalization (in characteristic zero) follows from a recent result of Etingof and Gelaki [4] on the structure of cotriangular Hopf algebras. A cotriangular Hopf algebra H is called *pseudoinvolutive* if, for any finite-dimensional subcoalgebra $C \subset H$, we have $\mathrm{tr}(S^2|_C) = \dim C$. In particular, this holds if H is *involutive*, i.e., $S^2 = id$.

Theorem (Etingof–Gelaki)**.** *Let (H, β) be a cotriangular Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic zero. Assume that H is pseudoinvolutive. Then there exists a right 2-cocycle $\sigma : H \otimes H \to \Bbbk$ such that H_σ is commutative, i.e., $H_\sigma = \mathcal{O}(G)$, the algebra of regular functions on the pro-algebraic group $G = \mathrm{Alg}(H_\sigma, \Bbbk)$, and*

$$\beta_\sigma = \frac{1}{2}(\varepsilon \otimes \varepsilon + \varepsilon \otimes c + c \otimes \varepsilon - c \otimes c)$$

for some central element $c \in G$ with $c^2 = 1$.

It is well-known that right $\mathcal{O}(G)$-comodules are in one-to-one correspondence with *algebraic* representations of G. For an $\mathcal{O}(G)$-comodule V, the corresponding G-action is given by

$$g \cdot v = \sum g(v_{(1)})v_{(0)} \text{ for all } g \in G = \mathrm{Alg}(\mathcal{O}(G), \Bbbk) \text{ and } v \in V.$$

The following is a "discoloration" result for (H, β)-Lie algebras.

Theorem 4.3. *Let (H, β) be a pseudoinvolutive cotriangular Hopf algebra, G, σ, c as above. Let L be an (H, β)-Lie algebra. Set $L_0 = \{x \in L \mid c \cdot x = x\}$ and $L_1 = \{x \in L \mid c \cdot x = -x\}$. Then $L \mapsto L_\sigma = L_0 \oplus L_1$ is an equivalence of the category of (H, β)-Lie algebras and the category of Lie superalgebras equipped with an algebraic G-action by automorphisms of graded algebras.*

Proof. By Corollary 4.2, L_σ is an $(\mathcal{O}(G), \beta_\sigma)$-Lie algebra. Thus G acts on L_σ algebraically by automorphisms, which preserve the grading $L_\sigma = L_0 \oplus L_1$ (because c is central). For β_σ as above, (11) takes the form

$$[x, y] + \frac{1}{2}([y, x] + [c \cdot y, x] + [y, c \cdot x] + [c \cdot y, c \cdot x]) = 0,$$

which gives $[x, y] + [y, x] = 0$ for homogeneous x, y that are not both in L_1, and $[x, y] - [y, x] = 0$ for $x, y \in L_1$. Similarly, (12) gives the usual Jacobi identity for superalgebras. ■

In particular, this theorem applies if H is a semisimple finite-dimensional cotriangular Hopf algebra (in this case, G is a finite group). Now we want to consider the general finite-dimensional case. It will be more convenient to state the results in the dual language, so let H be a finite-dimensional triangular Hopf algebra with universal R-matrix R. We will refer to Lie algebras in the category ${}_H\mathfrak{M}$ as (H, R)-*Lie algebras*.

Applying Corollary 3.4 to (H, R)-Lie algebras, we obtain:

Corollary 4.4. *Let L be an (H, R)-Lie algebra. Then L_J is an (H^J, R^J)-Lie algebra. Moreover, L and L_J have the same H-module subalgebras and ideals. L is solvable (resp., nilpotent) iff so is L_J.*

Now we need to recall the definition of a supergroup algebra. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and G a group that acts on $\mathfrak{g}$ by automorphisms of graded algebras. Then the smash product $U(\mathfrak{g})\#\Bbbk G$ admits the structure of a (cocommutative) Hopf superalgebra where the elements of G are group-like (and even), and the elements of $\mathfrak{g}$ are primitive. (By a theorem of Kostant, any cocommutative Hopf superalgebra over an algebraically closed field of characteristic zero has this form.)

The structure of a $U(\mathfrak{g})\#\Bbbk G$-module superalgebra on a non-associative superalgebra $A = A_0 \oplus A_1$ is defined by specifying an action of G on A by automorphisms of graded algebras, an action of $\mathfrak{g}_0$ on A by even derivations, and an action of $\mathfrak{g}_1$ on A by odd derivations in such a way that $g \cdot (x \cdot a) = (g \cdot x) \cdot (g \cdot a)$ for all $g \in G$, $x \in \mathfrak{g}$, and $a \in A$.

In particular, let V be a finite-dimensional vector space and G a finite group that acts on V. Then V can be viewed as a purely odd Lie superalgebra (with zero bracket), whose universal enveloping algebra is $\Lambda(V)$, the exterior algebra of V. Thus $\Lambda(V)\#\Bbbk G$ is a finite-dimensional Hopf superalgebra, which can be considered as the "group algebra of the supergroup $V \rtimes G$". Hopf superalgebras of this form are called *supergroup algebras*.

Definition 4.5. We say that a non-associative superalgebra $A = A_0 \oplus A_1$ is equipped with *an action of the supergroup* $V \rtimes G$ if A is a $\Lambda(V)\#\Bbbk G$-module superalgebra, i.e., G acts on A by automorphisms of graded algebras and V acts on A by pairwise anticommuting odd derivations in such a way that

$$g \cdot (v \cdot a) = (g \cdot v) \cdot (g \cdot a) \text{ for all } g \in G,\ v \in V, \text{ and } a \in A. \tag{13}$$

Now suppose there exists $c \in Z(G)$ such that $c^2 = 1$ and $c \cdot v = -v$ for all $v \in V$. Let $H = \Lambda(V)\#\Bbbk G$ as an algebra, but with comultiplication modified as follows: $\Delta v = v \otimes 1 + c \otimes v$ for $v \in V$ (and still $\Delta g = g \otimes g$ for $g \in G$). Then H is a Hopf algebra in the usual sense (where S is given by $Sg = g^{-1}$ for $g \in G$ and $Sv = vc$ for $v \in V$). Set

$$R_c = \frac{1}{2}(1 \otimes 1 + 1 \otimes c + c \otimes 1 - c \otimes c).$$

Then (H, R_c) is a triangular Hopf algebra, which is called a *modified supergroup algebra*. The following structure theorem of Etingof and Gelaki says that every finite-dimensional cotriangular Hopf algebra can be obtained from a modified supergroup algebra by a twist — see e.g. [6] (where the theorem is stated under the assumption that H satisfies the "Chevalley property", which was later shown [5] to hold for any finite-dimensional H).

Theorem (Etingof–Gelaki)**.** *Let (H,R) be a finite-dimensional triangular Hopf algebra over an algebraically closed field of characteristic zero. Then there exists a twist $J \in H \otimes H$ such that (H^J, R^J) is a modified supergroup algebra.*

Combining this with Corollary 4.4, we obtain the following "discoloration" result for (H,R)-Lie algebras.

Theorem 4.6. *Let (H,R) be a finite-dimensional triangular Hopf algebra, $V \rtimes G$, J, c as above. Let L be an (H,R)-Lie algebra. Set $L_0 = \{x \in L \mid c \cdot x = x\}$ and $L_1 = \{a \in L \mid c \cdot x = -x\}$. Then $L \mapsto L_J = L_0 \oplus L_1$ is an equivalence of the category of (H,R)-Lie algebras and the category of Lie superalgebras equipped with an action of the supergroup $V \rtimes G$.*

Proof. As in the proof of Theorem 4.3, we see that L_J is a Lie superalgebra. Also L_J is an H^J-module algebra and $H^J = \Lambda(V)\#\Bbbk G$ is a modified supergroup algebra. One checks that the structure of H^J-module algebra is equivalent to the action of the supergroup. ∎

As an application of our "discoloration" results, we obtain the following version of PBW Theorem for (H,β)-Lie algebras.

Definition 4.7. Let A be an associative algebra in a symmetric category $\mathfrak{C}$. Then the multiplication morphism

$$[\,,\,] = \mu_A - \mu_A \circ t_{A,A}$$

satisfies the anticommutativity and Jacobi identities in $\mathfrak{C}$, so $(A, [\,,\,])$ is a Lie algebra in $\mathfrak{C}$, denoted by $[A]$.

Definition 4.8. Let L be a Lie algebra in a symmetric category $\mathfrak{C}$. The *universal enveloping algebra* of L is the pair $(U_{\mathfrak{C}}(L), \eta)$ (unique up to an isomorphism) where $U_{\mathfrak{C}}(L)$ is a unital associative algebra in $\mathfrak{C}$ and $\eta : L \to [U_{\mathfrak{C}}(L)]$ is a morphism of algebras such that for any unital associative algebra A in $\mathfrak{C}$ and a morphism of algebras $f : L \to [A]$, there exists a unique morphism of unital algebras $\bar{f} : U(L) \to A$ such that $f = \bar{f} \circ \eta$.

We are interested in the case $\mathfrak{C} = \mathfrak{M}^H$ where (H,β) is a cotriangular bialgebra. If A is an associative H-module algebra, then we write $[\,,\,]_\beta$ for the morphism $\mu_A - \mu_A \circ t_{A,A}$ to emphasize the fact that it depends on β. Explicitly,

$$[a, b]_\beta = ab - \sum \beta(a_{(1)}, b_{(1)}) b_{(0)} a_{(0)} \text{ for all } a, b \in A.$$

For any (H, β)-Lie algebra L, there exists the universal enveloping algebra $(U_\beta(L), \eta)$. Namely, $U_\beta(L)$ is the quotient of the tensor algebra $T(L)$ by the ideal $I_\beta(L)$ generated by the elements $x \otimes y - \sum \beta(x_{(1)}, y_{(1)}) y_{(0)} \otimes x_{(0)} - [x, y]$, for all $x, y \in L$, and η is induced by the embedding $L \to T(L)$. Set

$$U_n = \operatorname{span}\{(\eta(L))^k \,|\, k = 0, 1, \dots, n\}.$$

Then $\{U_n\}$ is a filtration of $U_\beta(L)$.

Theorem 4.9. *Let (H, β) be a cotriangular Hopf algebra over an algebraically closed field of characteristic zero. Assume that H is either pseudoinvolutive or finite-dimensional. Let L be an (H, β)-Lie algebra. Then the associated graded algebra $\mathrm{gr}U_\beta(L)$ is naturally isomorphic to $U_\beta(L^\circ)$ where L° is L as an H-comodule, but with zero bracket. In particular, $\eta : L \to U_\beta(L)$ is injective.*

Proof. We apply Theorem 4.3 if H is pseudoinvolutive and Theorem 4.6 (with H^* instead of H) if H is finite-dimensional.

Now $L \mapsto L_\sigma$ is an equivalence of the category of (H, β)-Lie algebras and the category of (H_σ, β_σ)-Lie algebras. Also $A \mapsto A_\sigma$ is an equivalence of the category of unital associative H-module algebras and the category of unital associative H_σ-algebras. Since $U_\beta(L)$ is defined by a universal property, we have $(U_\beta(L))_\sigma \cong U_{\beta_\sigma}(L_\sigma)$. Moreover, this isomorphism preserves the filtration defined above. If one would like to see a more explicit proof of this claim, one can proceed as follows. For any H-comodule V, write V_σ for the same comodule, but viewed as an H_σ-comodule (recall that H and H_σ have the same comultiplication). Then one checks that the identity map $V_\sigma \to V$ induces an isomorphism of H_σ-comodule algebras $T(V_\sigma) \to (T(V))_\sigma$ that preserves degrees of tensors (in fact, this isomorphism coincides on the component $V^{\otimes n}$ with $\varphi_n(V)$ introduced in Section 3). Finally, one verifies that, for $V = L$, this isomorphism maps the ideal $I_{\beta_\sigma}(L_\sigma)$ onto the ideal $(I_\beta(L))_\sigma$.

Since L_σ is a Lie superalgebra, the usual PBW Theorem tells us that $\mathrm{gr}U_{\beta_\sigma}(L_\sigma)$ is naturally isomorphic to $U_{\beta_\sigma}((L_\sigma)^\circ)$. We know that $U_{\beta_\sigma}(L_\sigma) \cong (U_\beta(L))_\sigma$ as filtered algebras, so $\mathrm{gr}U_{\beta_\sigma}(L_\sigma) \cong \mathrm{gr}(U_\beta(L))_\sigma$. Now L_σ is the same as L when vied as an H-comodule, so $(L_\sigma)^\circ = (L^\circ)_\sigma$. It follows that

$U_{\beta_\sigma}((L_\sigma)^\circ) \cong (U_\beta(L^\circ))_\sigma$. Therefore, $\mathrm{gr}(U_\beta(L))_\sigma$ is naturally isomorphic to $(U_\beta(L^\circ))_\sigma$ as H_σ-comodule algebras. Since $A \mapsto A_\sigma$ is an equivalence of categories, we conclude that $\mathrm{gr}U_\beta(L)$ is naturally isomorphic to $U_\beta(L^\circ)$ as H-comodule algebras. ■

Remark 4.10. V. Kharchenko [9] has recently proved a similar result for symmetric braided Lie algebras (defined in the next section). In particular, his result applies to (H,β)-Lie algebras where (H,β) is an arbitrary cotriangular bialgebra (see the discussion on the relationship of symmetric braided Lie algebras and (H,β)-Lie algebras in the next section).

Remark 4.11. Since our "discoloration" results depend only on the theorem of Etingof–Gelaki and Corollary 3.4, they are not limited to generalized Lie algebras. One can consider, say, (H,β)-Jordan algebras and twist them into Jordan superalgebras.

5 Braided Lie algebras

So far we discussed Lie algebras in a symmetric category, especially in $\mathfrak{M}^H$ or ${}_H\mathfrak{M}$, where H is a cotriangular, resp. triangular, bialgebra. The defining identities of these algebras were obtained from the usual anticommutativity and Jacobi identities by replacing the flip with the symmetric braiding defined in the category. There is another approach to (symmetric) braided Lie algebras that does not involve a category, but starts from a vector space equipped with a symmetric braiding operator. Such braided Lie algebras were introduced by Gurevich [7] under the name "Lie S-algebras" (S was the letter used by Gurevich to denote the braiding).

Definition 5.1. Let L be a vector space, $t : L \otimes L \to L \otimes L$ a symmetric braiding, and $[\,,\,] : L \otimes L \to L$ a linear map. Then $(L, [\,,\,], t)$ is said to be a *braided Lie algebra* if

$$t \circ ([\,,\,] \otimes id) = (id \otimes [\,,\,]) \circ t(123) \text{ (compatibility)}, \tag{14}$$

$$[\,,\,] \circ (id + t) = 0 \text{ (anticommutativity), and} \tag{15}$$

$$[\,,\,] \circ ([\,,\,] \otimes id) \circ (id + t(123) + t(132)) = 0 \text{ (Jacobi)}, \tag{16}$$

where $t(\pi)$ denotes the action of a permutation $\pi \in \mathcal{S}_n$ on $L^{\otimes n}$ induced by the braiding t.

Clearly, if (H,β) is a cotriangular bialgebra and L is an (H,β)-Lie algebra, then L is a braided Lie algebra with $t = t_{L,L}$. However, we lose some information when we forget about the H-comodule structure and keep only the braiding. Namely, a braided Lie algebra $(L, [\,,\,], t)$ may admit different structures of H-comodule algebra over different cotriangular bialgebras H that all lead to the braiding t.

Given an (H, β)-Lie algebra L, there are two obvious reductions that one can make without altering the braiding.

Firstly, recall that, for any bialgebra H and a right H-comodule V, the *coefficient coalgebra* $C_H(V)$ is defined as follows. Let $\rho : V \to V \otimes H$ be the comodule structure map and $\{e_i\}$ a basis of V. Then we can write: $\rho(e_i) = \sum_j e_j \otimes c_i^j$ for some $c_i^j \in H$ (where all but finitely many c_i^j are zero for a fixed i). Then we necessarily have $\Delta c_i^j = \sum_k c_k^j \otimes c_i^k$ and $\varepsilon(c_i^j) = \delta_i^j$, where δ_i^j is the Kronecker delta. It follows that $C_H(V) := \operatorname{span}\{c_i^j\}$ is a subcoalgebra of H (which does not depend on the choice of the basis) and $\rho(V) \subset V \otimes C_H(V)$. Now if L is an (H, β)-Lie algebra and $H_0 = \langle C_H(L) \rangle$ is the subalgebra generated by $C_H(L)$, then H_0 is a subbialgebra and L can be viewed as an $(H_0, \beta|_{H_0})$-Lie algebra.

Secondly, given a cotriangular bialgebra (H, β), let I be the left(=right) kernel of the bilinear form β. Then I is a biideal and β factors through $\bar{\beta} : \bar{H} \otimes \bar{H} \to \Bbbk$, where $\bar{H} = H/I$. Moreover, $(\bar{H}, \bar{\beta})$ is a *minimal* cotriangular bialgebra, i.e., $\bar{\beta}$ is a nondegenerate bilinear form. Now if L is an (H, β)-Lie algebra, then L has a natural structure of an $\bar{H}$-comodule that makes L an $(\bar{H}, \bar{\beta})$-Lie algebra.

Therefore, if we wish to consider an (H, β)-Lie algebra L just as a braided Lie algebra, we can assume without loss of generality that 1) H is generated by $C_H(L)$ and 2) (H, β) is a minimal cotriangular bialgebra.

Conversely, let L be a *finite-dimensional* braided Lie algebra. Then a version of the FRT construction given by Theorem 5.3, below, shows that there exists a cotriangular bialgebra (H, β) such that L is an (H, β)-Lie algebra. Moreover, such a cotriangular bialgebra (H, β) will be unique up to an isomorphism if we require that (H, β) satisfy the above conditions 1) and 2) — see Theorem 5.4, below. Thus the notion of a braided Lie algebra is essentially equivalent to that of an (H, β)-Lie algebra.

Before we proceed, observe that if (H, β) is a cotriangular bialgebra, L is an H-comodule equipped with a bracket $[\,,\,]$, and t is induced by β, i.e.,

$$t(x \otimes y) = \sum \beta(x_{(1)}, y_{(1)}) y_{(0)} \otimes x_{(0)}, \tag{17}$$

then the identity (11) is equivalent to (15) and (12) is equivalent to (16). Therefore, the anticommutativity and Jacobi identitities will play no role in the construction and can be replaced with any set of multilinear polynomial identities.

Definition 5.2. Let $(L, [\,,\,])$ be a non-associative algebra equipped with a symmetric braiding t. We will say that $(L, [\,,\,], t)$ is a *braided algebra* if the compatibility condition (14) holds.

Theorem 5.3. *Let $(L, [\,,\,], t)$ be a finite-dimensional braided algebra. Then there exists a cotriangular bialgebra $H = \mathcal{B}(L, [\,,\,], t)$ with the "universal R-form" β and an H-comodule structure $\rho : L \to L \otimes H$ such that $(L, [\,,\,])$ is an H-comodule algebra and t is induced by β and ρ. If (H', β') is another cotriangular bialgebra and $\rho' : L \to L \otimes H'$ is a comodule structure satisfying these two properties, then there exists a unique homomorphism of cotriangular bialgebras $f : (H, \beta) \to (H', \beta')$ such that $\rho' = (id \otimes f) \circ \rho$.*

Proof. Fix a basis $\{e_i\}$ of L and write

$$t(e_i \otimes e_j) = \sum_{k,l} R_{ij}^{kl}\, e_l \otimes e_k.$$

Let $\mathcal{A}(L, t)$ be the FRT bialgebra associated to the braided vector space (L, t) and β the "universal R-form" of $\mathcal{A}(L, t)$ — see e.g. [10]. Namely, $\mathcal{A}(L, t)$ is the unital associative algebra generated by the symbols u_i^j subject to the relations

$$\sum_{k,l} R_{kl}^{ji} u_m^k u_n^l = \sum_{k,l} u_k^i u_l^j R_{mn}^{lk},$$

with comultiplication and counit defined by

$$\Delta u_i^j = \sum_k u_k^j \otimes u_i^k \text{ and } \varepsilon(u_i^j) = \delta_i^j, \tag{18}$$

and the "universal R-form" β defined by

$$\beta(u_i^k, u_j^l) = R_{ij}^{kl}.$$

Then

$$\rho(e_i) = \sum_j e_j \otimes u_i^j \tag{19}$$

defines an $\mathcal{A}(L, t)$-comodule structure on L such that t is induced by β and ρ. However, L need not be an $\mathcal{A}(L, t)$-*comodule algebra*. We have to impose additional relations on u_i^j to make $[\,,\,] : L \otimes L \to L$ a comodule map.

Let γ_{ij}^k be the structure constants of L relative to the basis $\{e_i\}$, i.e.,

$$[e_i, e_j] = \sum_k \gamma_{ij}^k e_k.$$

Then taking into account (19), we get

$$\rho([e_i, e_j]) = \sum_k \gamma_{ij}^k \rho(e_k) = \sum_{k,l} \gamma_{ij}^k e_l \otimes u_k^l,$$

and

$$\sum [(e_i)_{(0)}, (e_j)_{(0)}] \otimes (e_i)_{(1)}(e_j)_{(1)} = \sum_{s,t} [e_s, e_t] \otimes u_i^s u_j^t = \sum_{s,t,l} \gamma_{st}^l e_l \otimes u_i^s u_j^t.$$

Thus $(L, [\ ,\ ])$ is an $\mathcal{A}(L,t)$-comodule algebra iff $\sum_k \gamma_{ij}^k u_k^l = \sum_{s,t} \gamma_{st}^l u_i^s u_j^t$.

Set $W_{ij}^l = \sum_k \gamma_{ij}^k u_k^l - \sum_{s,t} \gamma_{st}^l u_i^s u_j^t$. We claim that span$\{W_{ij}^l\}$ is a coideal of $\mathcal{A}(L,t)$. Indeed, taking into account (18), we get

$$\begin{aligned}
\Delta(W_{ij}^l) &= \sum_{k,s} \gamma_{ij}^k u_p^l \otimes u_k^p - \sum_{s,t,q,r} \gamma_{st}^l u_q^s u_r^t \otimes u_i^q u_j^r \\
&= \sum_p u_p^l \otimes W_{ij}^p + \sum_{p,q,r} u_p^l \otimes \gamma_{qr}^p u_i^q u_j^r - \sum_{s,t,q,r} \gamma_{st}^l u_q^s u_r^t \otimes u_i^q u_j^r \\
&= \sum_p u_p^l \otimes W_{ij}^p + \sum_{q,r} \left(\sum_p \gamma_{qr}^p u_p^l - \sum_{s,t} \gamma_{st}^l u_q^s u_r^t \right) \otimes u_i^q u_j^r \\
&= \sum_p u_p^l \otimes W_{ij}^p + \sum_{q,r} W_{qr}^l \otimes u_i^q u_j^r,
\end{aligned}$$

and

$$\varepsilon(W_{ij}^l) = \sum_k \gamma_{ij}^k \delta_k^l - \sum_{s,t} \gamma_{st}^l \delta_i^s \delta_j^t = \gamma_{ij}^l - \gamma_{ij}^l = 0.$$

Set $I_{[\ ,\ ]} = (W_{ij}^l)$. Then $I_{[\ ,\ ]}$ is a biideal of $\mathcal{A}(L,t)$. Set $\mathcal{B}(L, [\ ,\ ], t) = \mathcal{A}(L,t)/I_{[\ ,\ ]}$. Then ρ induces a $\mathcal{B}(L, [\ ,\ ], t)$-comodule structure on L. By construction, L is a $\mathcal{B}(L, [\ ,\ ], t)$-comodule algebra.

Now we show that β induces a bilinear form on $\mathcal{B}(L, [\ ,\ ], t)$, making the latter a cotriangular bialgebra. Let I be the left(=right) kernel of the bilinear form β. Taking into account (17), the compatibility condition (14) reads:

$$t([x,y] \otimes z) = \sum z_{(0)} \otimes [x_{(0)}, y_{(0)}] \beta(x_{(1)} y_{(1)}, z_{(1)}) \text{ for all } x, y, z \in L.$$

Substituting $z = e_k$ and using (19), we get

$$\sum_l e_l \otimes \sum [x,y]_{(0)}\beta([x,y]_{(1)}, u_k^l) = \sum_l e_l \otimes \sum [x_{(0)}, y_{(0)}]\beta(x_{(1)}y_{(1)}, u_k^l).$$

Thus for all k, l we have

$$\sum [x,y]_{(0)}\beta([x,y]_{(1)}, u_k^l) = \sum [x_{(0)}, y_{(0)}]\beta(x_{(1)}y_{(1)}, u_k^l).$$

Set $\varphi_k^l(\cdot) = \beta(\cdot, u_k^l)$. Then $\varphi_k^l \in \mathcal{A}(L,t)^\circ$ and the above equation means that, for all k, l,

$$\sum [x,y]_{(0)} \otimes [x,y]_{(1)} - [x_{(0)}, y_{(0)}] \otimes x_{(1)}y_{(1)} \in \mathrm{Ker}(id \otimes \varphi_k^l)$$

Since $\mathcal{A}(L,t)$ is generated by u_k^l as an algebra, we have $\cap_{k,l}\mathrm{Ker}\varphi_k^l = I$. Therefore,

$$\sum [x,y]_{(0)} \otimes [x,y]_{(1)} - [x_{(0)}, y_{(0)}] \otimes x_{(1)}y_{(1)} \in \mathcal{A}(L,t) \otimes I.$$

Substituting $x = e_i$ and $y = e_j$, we get

$$\sum_{k,l} \gamma_{ij}^k e_l \otimes u_k^l - \sum_{p,q,l} \gamma_{pq}^l e_l \otimes u_i^p u_j^q \in \mathcal{A}(L,t) \otimes I,$$

which implies that $W_{ij}^l = \sum_k \gamma_{ij}^k u_k^l - \sum_{p,q} \gamma_{pq}^l u_i^p u_j^q \in I$. Therefore, $I_{[\,,\,]}$ is annihilated by β. Thus β induces a bilinear form on $\mathcal{B}(L,[\,,\,],t) = \mathcal{A}(L,t)/I_{[\,,\,]}$, as desired.

It remains to prove the universal property of $H = \mathcal{B}(L,[\,,\,],t)$. Suppose that (H', β') is another cotriangular bialgebra and $\rho' : L \to L \otimes H'$ is a comodule structure such that L is an H'-comodule algebra and t is induced by β' and ρ'. Write $\rho'(e_i) = \sum_j e_j \otimes c_i^j$ for some $c_i^j \in H'$. By the universal property of $\mathcal{A}(L,t)$ (see e.g. [10]), there exists a unique homomorphism of cotriangular bialgebras $\bar{f} : (\mathcal{A}(L,t), \beta) \to (H', \beta')$ such that $\rho' = (id \otimes \bar{f}) \circ \rho$. This homomorphism is defined by $u_i^j \mapsto c_i^j$. Since $(L, [\,,\,])$ is an H'-comodule algebra, we have

$$\sum_k \gamma_{ij}^k c_k^l = \sum_{p,q} \gamma_{pq}^l c_i^p c_j^q,$$

which implies that $\bar{f}(W_{ij}^l) = 0$. Therefore, $I_{[\,,\,]}$ is annihilated by $\bar{f}$. Thus $\bar{f}$ factors through a homomorphism of cotriangular bialgebras $f : (H, \beta) \to (H', \beta')$. ■

Theorem 5.4. *Let $(L, [\,,\,], t)$ be a finite-dimensional braided algebra. Then there exists a minimal cotriangular bialgebra (H, β) and an H-comodule structure $\rho : L \to L \otimes H$ such that $(L, [\,,\,])$ is an H-comodule algebra, H is generated by $C_H(L)$, and t is induced by β and ρ. If (H', β', ρ') is another triple satisfying these properties, then there exists a unique isomorphism of cotriangular bialgebras $f : (H, \beta) \to (H', \beta')$ such that $\rho' = (id \otimes f) \circ \rho$.*

Proof. Let $\bar{H} = \mathcal{B}(L, [\,,\,], t)$ be the cotriangular bialgebra constructed in Theorem 5.3 and I the kernel of the bilinear form β. Set $H = \bar{H}/I$. Then H is a minimal cotriangular bialgebra. Also $(L, [\,,\,])$ is an H-comodule algebra for the induced comodule structure $\rho : L \to L \otimes H$, H is generated by $C_H(L)$, and t is induced by β and ρ.

Let (H', β', ρ') be another such triple. By the universal property of $\mathcal{B}(L, [\,,\,], t)$, there exists a unique homomorphism of cotriangular bialgebras $\bar{f} : (\bar{H}, \beta) \to (H', \beta')$ such that $\rho' = (id \otimes \bar{f}) \circ \rho$. Since H' is generated by $C_{H'}(L)$, $\bar{f}$ is an epimorphism. Suppose $h \in I$. Then $\beta'(\bar{f}(h), H') = \beta'(\bar{f}(h), \bar{f}(\bar{H})) = \beta(h, \bar{H}) = 0$. Since (H', β') is minimal, $\bar{f}(h) = 0$. Conversely, suppose $\bar{f}(h) = 0$. Then $\beta(h, \bar{H}) = \beta'(\bar{f}(h), H') = 0$. Thus $h \in I$. It follows that $\bar{f}$ factors through an isomorphism of cotriangular bialgebras $f : H \to H'$. ■

An interesting question is when the minimal bialgebra H constructed in Theorem 5.4 is actually a Hopf algebra. In particular, this happens when $\dim H < \infty$, because then the linear map $S : H \to H$ defined by $\beta(Sh, \cdot) = \beta(\cdot, h)$ is easily checked to satisfy the antipode axiom.

Corollary 5.5. *Let $(L, [\,,\,], t)$ be a finite-dimensional braided algebra. Suppose that there exists a finite-dimensional bialgebra $(\tilde{H}, \tilde{\beta})$ and an $\tilde{H}$-comodule structure on L such that $(L, [\,,\,])$ is an $\tilde{H}$-comodule algebra and t is induced by $\tilde{\beta}$. Then the minimal cotriangular bialgebra H constructed in Theorem 5.4 is a finite-dimensional Hopf algebra. It is uniquely characterized by the following properties: (H, β) is minimal, there exists an H-comodule structure $\rho : L \to L \otimes H$ such that $(L, [\,,\,])$ is an H-comodule algebra, H is generated by $C_H(L)$ as a Hopf algebra, and t is induced by β and ρ.*

Proof. Since H is isomorphic to a quotient of a subbialgebra of $\tilde{H}$, $\dim H < \infty$ and thus H is a Hopf algebra. It clearly satisfies the properties listed above.

Let (H', β', ρ') be another triple that satisfies those properties. As in the proof of Theorem 5.4, let $\bar{H} = \mathcal{B}(L, [\,,\,], t)$ be the cotriangular bialgebra constructed in Theorem 5.3, I the kernel of β, and $\bar{f} : \bar{H} \to H'$. Then H' is generated by $\bar{f}(\bar{H})$ as a Hopf algebra. Suppose $h \in I$. Then

$\beta'(\bar{f}(\bar{H}), \bar{f}(h)) = \beta(\bar{H}, h) = 0$ and $\beta'(S\bar{f}(\bar{H}), \bar{f}(h)) = \beta'(\bar{f}(h), \bar{f}(\bar{H})) = \beta(h, \bar{H}) = 0$. Therefore, $\beta'(H', \bar{f}(h)) = 0$ and hence $\bar{f}(h) = 0$ by minimality of (H', β'). It follows that $\bar{f}$ factors through a monomorphism of cotriangular bialgebras $f : H \to H'$. Since both H and H' are Hopf algebras, f is a monomorphism of Hopf algebras. But H' is generated by $f(H)$ as a Hopf algebra, so $f(H) = H'$. ∎

6 Examples

In [1] the authors asked if there exist (H, β)-Lie algebras that are not Lie coloralgebras. We are going to construct examples of (H, β)-Lie algebras L where L is not isomorphic to any Lie coloralgebra *as a braided algebra*. In the entire section we assume that $\Bbbk$ is an algebraically closed field of characteristic zero.

The idea is to reverse the "discoloration" process of Theorem 4.3 or Theorem 4.6. Consider the case when (H, β) is a finite-dimensional cotriangular Hopf algebra. Then H^* is a twist of a modified supergroup algebra. Let $V \rtimes G$ be the supergroup and J a twist of $\Lambda(V)\#\Bbbk G$ such that $H^* = (\Lambda(V)\#\Bbbk G)^J$. If $V \rtimes G$ acts on a Lie superalgebra $L = L_0 \oplus L_1$ (in the sense of Definition 4.5) and $c \in Z(G)$ is such that $c^2 = 1$, $c \cdot v = -v$, for all $v \in V$, and $c \cdot x = x$ for $x \in L_0$ and $c \cdot x = -x$ for $x \in L_1$, then L_J is an (H, β)-Lie algebra where $H = ((\Lambda(V)\#\Bbbk G)^J)^*$ and $\beta = J_{21}^{-1} R_c J$.

Without loss of generality, we may assume that 1) H is generated by $C_L(H)$ as a Hopf algebra and 2) (H, β) is minimal. The first condition holds iff the kernel of the H^*-action on L contains no nonzero Hopf ideals, which is equivalent to saying that $V \rtimes G$ acts faithfully on L. In the terminology of [6], the second condition says that J is a *minimal twist*. Under these conditions, Corollary 5.5 tells us that L_J is isomorphic to a Lie coloralgebra (as a braided algebra) iff $V = 0$ and G is abelian.

First we consider the case of *semisimple* H, i.e., $V = 0$. By [6], the minimal order of a nonabelian G whose group algebra admits a minimal twist is 16. The corresponding semisimple Hopf algebra $(\Bbbk G)^J$ of dimension 16 first appeared in [8], but the minimal twist J was found in [6]. The construction in [6] is as follows.

Take $G = A \rtimes K$ where $A = \langle a \rangle_4$, $K = \langle g \rangle_2 \times \langle h \rangle_2$, and $g \cdot a = a$, $g \cdot a = a^{-1}$. Let $\pi : K \to A$ be a 1-cocycle defined by $\pi(g) = a^2$ and $\pi(h) = a$. Then π is bijective and

$$J = \frac{1}{|A|} \sum_{x \in A, y^* \in \hat{A}} \langle x, y^* \rangle \pi^{-1}(x) \otimes y^*$$

is a minimal twist for $\Bbbk G$. Then $(\Bbbk G)^J$ is a minimal triangular Hopf algebra with $R = J_{21}^{-1}J$ (in this case $c = 1$).

It remains to find a finite-dimensional Lie algebra L on which G can act faithfully by automorphisms. One can easily check that $PSL_2(\Bbbk)$ does not contain a subgroup isomorphic to G, so we cannot take $L = sl_2(\Bbbk)$. We give examples with $L = sl_2(\Bbbk) \times sl_2(\Bbbk)$ and $L = sl_3(\Bbbk)$.

Example 6.1. Take $L = sl_2(\Bbbk) \times sl_2(\Bbbk)$ and let G act on L as follows:

1) g swaps the two $sl_2(\Bbbk)$ components;
2) h acts by $\mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on each component;
3) a acts by $\mathrm{Ad}\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$ on each component, where ω is a primitive 8-th root of unity.

The author used Maple to compute the multiplication table of L_J and the braiding relative to the basis $h_1, e_1, f_1, h_2, e_2, f_2$ where h_j, e_j, f_j is the standard basis in the j-th $sl_2(\Bbbk)$ component, $j = 1, 2$.

Multiplication table of the twisted $sl_2 \times sl_2$

	h_1	e_1	f_1	h_2	e_2	f_2
h_1	0	0	$2f_1$	0	$-2e_2$	0
e_1	$-2e_1$	0	0	0	$-h_2$	0
f_1	$2f_1$	0	h_1	0	0	0
h_2	0	$-2e_1$	0	0	0	$2f_2$
e_2	0	$-h_1$	0	$-2e_2$	0	0
f_2	0	0	0	$2f_2$	0	h_2

Braiding on the twisted $sl_2 \times sl_2$

	h_1	e_1	f_1	h_2	e_2	f_2
h_1	$h_1 \otimes h_1$	$-e_1 \otimes h_2$	$-f_1 \otimes h_1$	$h_2 \otimes h_1$	$-e_2 \otimes h_2$	$-f_2 \otimes h_1$
e_1	$-h_2 \otimes e_1$	$f_1 \otimes f_2$	$e_1 \otimes f_1$	$-h_1 \otimes e_1$	$f_2 \otimes f_2$	$e_2 \otimes f_1$
f_1	$-h_1 \otimes f_1$	$f_2 \otimes e_2$	$e_2 \otimes e_1$	$-h_2 \otimes f_1$	$f_1 \otimes e_2$	$e_1 \otimes e_1$
h_2	$h_1 \otimes h_2$	$-e_1 \otimes h_1$	$-f_1 \otimes h_2$	$h_2 \otimes h_2$	$-e_2 \otimes h_1$	$-f_2 \otimes h_2$
e_2	$h_2 \otimes e_2$	$f_1 \otimes f_1$	$e_1 \otimes f_2$	$-h_1 \otimes e_2$	$f_2 \otimes f_1$	$e_2 \otimes f_2$
f_2	$-h_1 \otimes f_2$	$f_2 \otimes e_1$	$e_2 \otimes e_2$	$-h_2 \otimes f_2$	$f_1 \otimes e_1$	$e_1 \otimes e_2$

Example 6.2. Take $L = sl_3(\Bbbk)$. Then L is generated by $h_1 = E_{11} - E_{22}$, $h_2 = E_{22} - E_{33}$, $e_1 = E_{12}$, $e_2 = E_{23}$, $f_1 = E_{21}$, $f_2 = E_{32}$. Let G act by automorphisms of L in the following way:

1) g swaps e_l and f_l, $h_l \mapsto -h_l$, $l = 1, 2$;

2) h swaps h_1 and h_2, e_1 and e_2, f_1 and f_2;

3) a swaps h_1 and h_2, $e_1 \mapsto e_2$, $e_2 \mapsto -e_1$, $f_1 \mapsto f_2$, $f_2 \mapsto -f_1$.

The author used Maple to compute the multiplication table of L_J relative to the basis $h_+ = h_1 + h_2$, $h_- = h_1 - h_2$, $e = E_{13}$, $f = E_{31}$, $x_1 = -if_1 - f_2$, $x_2 = -ie_1 + e_2$, $x_3 = if_1 - f_2$, $x_4 = ie_1 + e_2$, where $i = \sqrt{-1}$.

Multiplication table of the twisted sl_3

	h_+	h_-	e	f	x_1	x_2	x_3	x_4
h_+	0	0	$2e$	$-2f$	x_1	x_2	$-x_3$	$-x_4$
h_-	0	0	0	0	$-3x_3$	$-3x_4$	$3x_1$	$3x_2$
e	$-2e$	0	0	h_+	0	0	ix_2	$-ix_1$
f	$2f$	0	$-h_+$	0	ix_4	$-ix_3$	0	0
x_1	x_1	$3x_2$	ix_4	0	ih_+	ih_-	0	$-2e$
x_2	$-x_2$	$-3x_1$	0	$-ix_3$	0	$2e$	ih_+	ih_-
x_3	x_3	$3x_4$	ix_2	0	$-ih_-$	$-ih_+$	$-2f$	0
x_4	$-x_4$	$-3x_3$	0	ix_1	$2f$	0	$-ih_-$	$-ih_+$

Remark 6.3. If one forgets about the braiding and views the above examples just as non-associative algebras, then one can ask whether or not it is possible to find a grading by an abelian group and a bicharacter that would make the bracket satisfy the identities of a Lie coloralgebra. I do not know the answer. However, from the multiplication tables one can show that it is not possible to make the above two examples Lie superalgebras: the condition $[x, y] = -[y, x]$ for all $x \in L_0$ and $y \in L$ would force $L_0 = 0$.

Remark 6.4. From the multiplication tables above one can deduce that the twisted $sl_2 \times sl_2$ and the twisted sl_3 are both *simple* non-associative algebras. The cocycle twist does not preserve simplicity.

Now we turn to the case of *non-semisimple* H. We obtain the smallest example by setting $V = \langle x \rangle$, $G = \langle g \rangle_2$, and $c = g$. Then the modified supergroup algebra is the Taft algebra of dimension 4:

$$H_4 = \langle x, g \,|\, g^2 = 1,\, x^2 = 0,\, gx = -xg \rangle \text{ with } \Delta g = g \otimes g,\ \Delta x = x \otimes 1 + g \otimes x.$$

In [6] the following twist is given:

$$J_\lambda = 1 \otimes 1 - \frac{\lambda}{2} gx \otimes g \text{ where } \lambda \in \Bbbk.$$

One can easily check that $(H_4)^{J_\lambda} = H_4$, and

$$(R_g)^{J_\lambda} = R_g - \frac{\lambda}{2}(x \otimes x + gx \otimes x - x \otimes gx + gx \otimes gx)$$

turns H_4 into a minimal triangular Hopf algebra if $\lambda \neq 0$.

It remains to find a Lie superalgebra that admits a faithful action of $V \rtimes G$. Note that it cannot be a Lie algebra, because this would force V to act by zero.

Example 6.5. Take $L = pl_{1,1}(\Bbbk)$. Let g act by parity (since $c = g$) and let x act by $\mathrm{ad}E_{12}$. Then the condition (13) is clearly satisfied and thus L is an H_4-module algebra. It turns out that J_λ does not change the bracket: $[\,,\,]_{J_\lambda} = [\,,\,]$. However, if $\lambda \neq 0$, then the braiding on L_{J_λ} is different from the braiding on L, so L_{J_λ} is not a Lie coloralgebra.

Example 6.6. Take $L = spl_{2,1}(\Bbbk) = \langle h, e, f, z\rangle \oplus \langle E_{13}, E_{23}, E_{31}, E_{32}\rangle$, where h, e, f is the standard basis of $sl_2(\Bbbk)$ in the upper left corner of $spl_{2,1}(\Bbbk)$ and $z = \mathrm{diag}(1, 1, 2)$. Let g act by parity and let x act by $\mathrm{ad}E_{13}$. Then L is an H_4-module algebra. One can check that $[\,,\,]_{J_\lambda}$ coincides with $[\,,\,]$ on all basis elements except the following:

$$\begin{aligned}
&[f, E_{31}]_{J_\lambda} = [E_{31}, f]_{J_\lambda} = -\frac{\lambda}{2}E_{23},\\
&[f, E_{32}]_{J_\lambda} = -E_{31} + \frac{\lambda}{2}E_{13}, \quad [E_{32}, f]_{J_\lambda} = E_{31} + \frac{\lambda}{2}E_{13},\\
&[E_{31}, E_{32}]_{J_\lambda} = -\frac{\lambda}{2}e, \quad [E_{32}, E_{31}]_{J_\lambda} = \frac{\lambda}{2}e.
\end{aligned}$$

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