ALMOST FINE GRADINGS ON ALGEBRAS AND CLASSIFICATION OF GRADINGS UP TO ISOMORPHISM

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Abstract. We consider the problem of classifying gradings by groups on a finite-dimensional algebra $\mathcal{A}$ (with any number of multilinear operations) over an algebraically closed field. We introduce a class of gradings, which we call almost fine, such that every G-grading on $\mathcal{A}$ is obtained from an almost fine grading on $\mathcal{A}$ in an essentially unique way, which is not the case with fine gradings. For abelian groups, we give a method of obtaining all almost fine gradings if fine gradings are known. We apply these ideas to the case of semisimple Lie algebras in characteristic 0: to any abelian group grading with nonzero identity component, we attach a (possibly nonreduced) root system Φ and, in the simple case, construct an adapted Φ-grading.

1. Introduction

Let G be a group and let $\mathcal{A}$ be an algebra with any number of multilinear operations. $\mathcal{A}$ is said to be a *G-graded algebra* if there is a fixed *G-grading* on $\mathcal{A}$, i.e., a direct sum decomposition of its underlying vector space, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, such that, for any operation φ defined on $\mathcal{A}$, we have $\varphi(\mathcal{A}_{g_1}, \dots, \mathcal{A}_{g_n}) \subset \mathcal{A}_{g_1\cdots g_n}$ for all $g_1, \dots, g_n \in G$, where n is the number of arguments taken by φ. The subspaces $\mathcal{A}_g$ are called *homogeneous components*. For any nonzero element $a \in \mathcal{A}_g$, we will say that a is *homogeneous of degree* g and write $\deg a = g$. (The zero vector is also considered homogeneous, but its degree is undefined.)

For a fixed group G, the class of G-graded vector spaces is a category in which morphisms are the linear maps that preserve degree. In particular, we can speak of isomorphism of G-graded algebras. Two G-gradings, Γ and Γ', on the same algebra $\mathcal{A}$ are said to be *isomorphic* if there exists an isomorphism of G-graded algebras $(\mathcal{A}, \Gamma) \to (\mathcal{A}, \Gamma')$ or, in other words, there exists an automorphism of the algebra $\mathcal{A}$ that maps each component of Γ onto the component of Γ' of the same degree. If the automorphism maps each component of Γ onto a component of Γ', but not necessarily of the same degree, then Γ and Γ' are said to be *equivalent*; in this setting the group G need not be fixed. A grading is said to be *fine* if it has no proper refinement (see Section 2).

Given an algebra $\mathcal{A}$, we may wish to classify all G-gradings on $\mathcal{A}$ up to isomorphism or fine gradings on $\mathcal{A}$ up to equivalence. These two problems received much attention in the last two decades, especially for simple algebras in many varieties: associative, associative with involution, Lie, Jordan, alternative, various triple systems, and so on (see, e.g., [EK13] and the references therein, also

2020 *Mathematics Subject Classification.* Primary 17A01; Secondary 17A36, 17B70, 16W50, 20G07.

Key words and phrases. Grading; classification; automorphism group; Lie algebra; root grading.

A.E. is supported by grant PID2021-123461NB-C21, funded by MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe", and by grant E22_23R (Gobierno de Aragón, Grupo de investigación "Álgebra y Geometría").

M.K. is supported by Discovery Grant 2018-04883 of the Natural Sciences and Engineering Research Council (NSERC) of Canada.

[Ara17, BKR18, AC21, DET21, EKR22]). The second problem may be easier to solve, and the answer sheds some light on the first problem in the following way.

Any group homomorphism $\alpha : G \to H$ gives a functor from G-graded vector spaces to H-graded ones: for $V = \bigoplus_{g\in G} V_g$, we define the H-graded vector space ${}^{\alpha}V$ to be the same space V but equipped with the H-grading $V = \bigoplus_{h\in H} V'_h$ where $V'_h := \bigoplus_{g\in\alpha^{-1}(h)} V_g$. (This functor is the identity map on morphisms.) If the G-grading on V is denoted by Γ, the corresponding H-grading on V will be called *induced by* α and denoted ${}^{\alpha}\Gamma$. Note that the homogeneous elements of degree g with respect to Γ become homogeneous of degree $\alpha(g)$ with respect to ${}^{\alpha}\Gamma$.

For a finite-dimensional algebra $\mathcal{A}$, any grading is a coarsening of a fine grading. Hence, if $\{\Gamma_i\}_{i\in I}$ is a set of representatives of the equivalence classes of fine gradings on $\mathcal{A}$ and U_i is the universal group of Γ_i (again, see Section 2), then any G-grading Γ on $\mathcal{A}$ is isomorphic to the induced grading ${}^{\alpha}\Gamma_i$ for some $i \in I$ and a group homomorphism $\alpha : U_i \to G$. However, both i and α are usually far from unique, so we do not easily obtain a classification of G-gradings up to isomorphism.

The purpose of this paper is to introduce a class of gradings in such a way that any G-grading is induced from one of them in an essentially unique way (see Theorem 4.3), and at the same time this class is not too far removed from fine gradings (see Proposition 5.1 and Theorem 5.4). We call this class *almost fine gradings*.

After reviewing preliminaries on gradings and algebraic groups in Section 2, we introduce almost fine gradings on a finite-dimensional algebra $\mathcal{A}$ in Section 3. Section 4 is dedicated to the question how to classify all G-gradings on $\mathcal{A}$ up to isomorphism if we know almost fine gradings on $\mathcal{A}$ up to equivalence. In Section 5, we discuss, in the case of gradings by abelian groups, how to obtain almost fine gradings if we know fine gradings. Finally, in Section 6, we apply these ideas to the case of abelian group gradings on semisimple Lie algebras that have nontrivial identity component: to any such grading Γ on $\mathcal{L}$, we attach a (possibly nonreduced) root system Φ (Theorem 6.1) and, in the case of simple $\mathcal{L}$, construct a Φ-grading on $\mathcal{L}$ adapted to Γ (Theorem 6.4).

Except in Section 2, we assume that the ground field $\mathbb{F}$ is *algebraically closed*. The characteristic is arbitrary unless stated otherwise.

2. Preliminaries on gradings

In this section we will briefly review some general facts and terminology concerning gradings on algebras, most of which go back to J. Patera and H. Zassenhaus [PZ89]. We will also introduce notation that is used throughout the paper. The reader is referred to Chapter 1 of the monograph [EK13] for more details, and to [Hum75, Wat79] for the background on (linear) algebraic groups and (affine) group schemes.

2.1. **Gradings and their universal groups.** There is a more general concept of a grading on an algebra $\mathcal{A}$, namely, a set S of nonzero subspaces of $\mathcal{A}$, which we write as $\Gamma = \{\mathcal{A}_s\}_{s\in S}$ for convenience, such that $\mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ and, for any n-ary operation φ defined on $\mathcal{A}$ and any $s_1, \ldots, s_n \in S$, there exists $s \in S$ such that $\varphi(\mathcal{A}_{s_1}, \ldots, \mathcal{A}_{s_n}) \subset \mathcal{A}_s$. Any G-grading on $\mathcal{A}$ becomes a grading in this sense if we take S to be its *support*: $S = \{g \in G \mid \mathcal{A}_g \neq 0\}$.

For a given grading Γ on an algebra $\mathcal{A}$ as above, there may or may not exist a realization of Γ over a group G, by which we mean an injective map $\iota : S \to G$ such that assigning the nonzero elements of $\mathcal{A}_s$ degree $\iota(s) \in G$, for all $s \in S$, and taking the homogeneous components of degree in $G \smallsetminus \iota(S)$ to be zero defines a G-grading on $\mathcal{A}$. If such realizations exist, there is a universal one among them.

Indeed, let $U = U(\Gamma)$ be the group generated by the set S subject to all relations of the form $s_1 \cdots s_n = s$ whenever $0 \neq \varphi(\mathcal{A}_{s_1}, \ldots, \mathcal{A}_{s_n}) \subset \mathcal{A}_s$ for an n-ary operation φ on $\mathcal{A}$. Then Γ has a realization over some group if and only if the canonical map $\iota_0 : S \to U$ is injective. If this is the case, then we will say that Γ is a *group grading*. Then (U, ι_0) is universal among all realizations (G, ι) of Γ in the sense that there exists a unique group homomorphism $\alpha : U \to G$ such that $\alpha\iota_0 = \iota$. We will call U the *universal group* of Γ.

Many gradings (for example, all group gradings on simple Lie algebras) can be realized over an abelian group. We will call them *abelian group gradings*. Let $U_{\mathrm{ab}} = U_{\mathrm{ab}}(\Gamma)$ be the abelianization of $U(\Gamma)$, i.e., the *abelian* group generated by S subject to the relations above. Then Γ has a realization over some abelian group if and only if the canonical map $\iota_0 : S \to U_{\mathrm{ab}}$ is injective, and in this case $(U_{\mathrm{ab}}, \iota_0)$ is the universal one among such realizations. We will call U_{ab} the *universal abelian group* of Γ.

2.2. **Equivalence and automorphisms of gradings.** An *equivalence of graded algebras* from $\mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ to $\mathcal{B} = \bigoplus_{t\in T} \mathcal{B}_t$ is an algebra isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that, for any $s \in S$, we have $\psi(\mathcal{A}_s) = \mathcal{B}_t$ for some $t \in T$. Since we assume that all $\mathcal{A}_s$ are nonzero, ψ defines a bijection $\gamma : S \to T$ such that $\psi(\mathcal{A}_s) = \mathcal{B}_{\gamma(s)}$ for all $s \in S$. If these are group gradings and we realize them over their universal groups, U and U', then any equivalence $\psi : \mathcal{A} \to \mathcal{B}$ leads to an isomorphism: the bijection of the supports $\gamma : S \to T$ determined by ψ extends to a unique isomorphism of the universal groups, which we also denote by γ, so that $\psi : {}^{\gamma}\mathcal{A} \to \mathcal{B}$ is an isomorphism of U'-graded algebras.

Two gradings, Γ and Γ', on the same algebra $\mathcal{A}$ are said to be *equivalent* if there exists an equivalence $(\mathcal{A}, \Gamma) \to (\mathcal{A}, \Gamma')$ or, in other words, there exists an automorphism of the algebra $\mathcal{A}$ that maps the set of nonzero components of Γ to that of Γ'. In particular, we can consider the group $\mathrm{Aut}(\Gamma)$ of all equivalences from the graded algebra $(\mathcal{A}, \Gamma)$ to itself. Applying the above property of universal groups, we see that the permutation of the support of Γ defined by any element of $\mathrm{Aut}(\Gamma)$ extends to a unique automorphism of the universal group $U = U(\Gamma)$. This gives us a group homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(U)$, whose kernel is denoted $\mathrm{Stab}(\Gamma)$ and consists of all degree-preserving automorphisms, i.e., isomorphisms from the graded algebra $(\mathcal{A}, \Gamma)$ to itself. The image of this homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(U)$ is known as the *Weyl group* of the grading Γ:

$$W(\Gamma) := \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma) \hookrightarrow \mathrm{Aut}(U(\Gamma)).$$

If we deal with abelian group gradings, the universal abelian groups can be used, and we can regard $W(\Gamma)$ as a subgroup of $\mathrm{Aut}(U_{\mathrm{ab}}(\Gamma))$.

2.3. **Fine gradings.** A grading $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ is said to be a *refinement* of a grading $\Gamma' : \mathcal{A} = \bigoplus_{t\in T} \mathcal{A}'_t$ (or Γ' a *coarsening* of Γ) if, for any $s \in S$, there exists $t \in T$ such that $\mathcal{A}_s \subset \mathcal{A}'_t$. If the inclusion is proper for at least one $s \in S$, the refinement (or coarsening) is called *proper*.

For example, if Γ is a grading by a group G (so $S \subset G$) and $\alpha : G \to H$ is a group homomorphism, then ${}^{\alpha}\Gamma$ is a coarsening of Γ, which is proper if and only if $\alpha|_S$ is not injective. If G is the universal group of Γ, then any coarsening Γ' that is a grading by a group H necessarily has the form ${}^{\alpha}\Gamma$ for a unique group homomorphism $\alpha : G \to H$.

A group grading (respectively, abelian group grading) is said to be *fine* if it does not have a proper refinement that is itself a group (respectively, abelian group) grading. Note that the concept of fine grading is relative to the class that we consider. For example, there is a $\mathbb{Z}^n$-grading defined on the matrix algebra $M_n(\mathbb{F})$ by declaring the degree of the matrix unit E_{ij} to be $\varepsilon_i - \varepsilon_j$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis of $\mathbb{Z}^n$. This grading is fine in the class of group gradings, but has a refinement whose components are the one-dimensional subspaces spanned by E_{ij}. This latter cannot be realized over a group (but can be realized over a semigroup).

2.4. **Gradings and actions.** Given a G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$, any group homomorphism $\chi : G \to \mathbb{F}^\times$, where $\mathbb{F}^\times$ denotes the multiplicative group of $\mathbb{F}$, acts as an automorphism of $\mathcal{A}$ as follows: $\chi \cdot a = \chi(g)a$ for all $a \in \mathcal{A}_g$ and $g \in G$, which is then extended to the whole $\mathcal{A}$ by linearity. Note that this is actually an automorphism of $\mathcal{A}$ as a graded algebra, as it leaves each component $\mathcal{A}_g$ invariant — in fact, acts on it as the scalar operator $\chi(g)\,\mathrm{id}_{\mathcal{A}_g}$. Thus Γ defines a group homomorphism η_Γ from the group of (multiplicative) characters $\widehat{G} := \mathrm{Hom}(G, \mathbb{F}^\times)$ to the automorphism group $\mathrm{Aut}(\mathcal{A})$, which is particularly useful if G is abelian and $\mathbb{F}$ is algebraically closed and of characteristic 0, because then $\widehat{G}$ separates points of G and, therefore, the grading Γ can be recovered as a simultaneous eigenspace decomposition with respect to these automorphisms:

$$\mathcal{A}_g = \{a \in \mathcal{A} \mid \chi \cdot a = \chi(g)a \text{ for all } \chi \in \widehat{G}\}. \tag{2.1}$$

For example, the above $\mathbb{Z}^n$-grading on $M_n(\mathbb{F})$ corresponds to the homomorphism from the (algebraic) torus $(\mathbb{F}^\times)^n$ to $\mathrm{Aut}(M_n(\mathbb{F}))$ that sends $(\lambda_1, \dots, \lambda_n)$ to the inner automorphism $\mathrm{Int}\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

If $\mathcal{A}$ is finite-dimensional and $\mathbb{F}$ is algebraically closed, then $\mathrm{Aut}(\mathcal{A})$ is an algebraic group. If G is a finitely generated abelian group, then $\widehat{G}$ is a *diagonalizable* algebraic group, isomorphic to the direct product of a torus and a finite abelian group whose order is not divisible by $\mathrm{char}\,\mathbb{F}$; such groups are often called *quasitori* (especially in characteristic 0). For any G-grading on $\mathcal{A}$, η_Γ is a homomorphism of algebraic groups. Conversely, the image of any homomorphism of algebraic groups $\eta : \widehat{G} \to \mathrm{Aut}(\mathcal{A})$ consists of commuting diagonalizable operators and, therefore, defines a simultaneous eigenspace decomposition of $\mathcal{A}$ indexed by the homomorpisms of algebraic groups $\widehat{G} \to \mathbb{F}^\times$, which are canonically identified with the elements of G if $\mathrm{char}\,\mathbb{F} = 0$ or $\mathrm{char}\,\mathbb{F} = p$ and G has no p-torsion. Thus the subspaces $\mathcal{A}_g$ defined by (2.1), with respect to the $\widehat{G}$-action $\chi \cdot a = \eta(\chi)a$, form a G-grading Γ on the algebra $\mathcal{A}$, and $\eta = \eta_\Gamma$.

If $\mathbb{F}$ is not necessarily algebraically closed or $\mathrm{char}\,\mathbb{F} \neq 0$, one can recover the above one-to-one correspondence by using group schemes over $\mathbb{F}$, namely, the automorphism group scheme $\mathbf{Aut}(\mathcal{A})$, defined by $\mathbf{Aut}(\mathcal{A})(\mathcal{R}) := \mathrm{Aut}_{\mathcal{R}}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{R})$ for any commutative associative unital $\mathbb{F}$-algebra $\mathcal{R}$, and the Cartier dual G^D of an abelian group G, defined by $G^D(\mathcal{R}) := \mathrm{Hom}(G, \mathcal{R}^\times)$. Then a G-grading Γ on $\mathcal{A}$ corresponds to the homomorphism of group schemes $\eta_\Gamma : G^D \to \mathbf{Aut}(\mathcal{A})$ defined by

$$(\eta_\Gamma)_{\mathcal{R}}(\chi) : a \otimes r \mapsto a \otimes \chi(g)r \text{ for all } \chi \in \mathrm{Hom}(G, \mathcal{R}^\times),\ a \in \mathcal{A}_g,\ g \in G,\ r \in \mathcal{R}. \tag{2.2}$$

The homomorphism $\eta_\Gamma : \widehat{G} \to \mathrm{Aut}(\mathcal{A})$ in the previous paragraph is obtained by applying this one to $\mathbb{F}$-points, i.e., taking $\mathcal{R} = \mathbb{F}$.

The image of the homomorphism $\eta_\Gamma : G^D \to \mathbf{Aut}(\mathcal{A})$ is contained in the following diagonalizable subgroupscheme $\mathbf{Diag}(\Gamma)$ of $\mathbf{Aut}(\mathcal{A})$, which can be defined for any grading $\mathcal{A} = \bigoplus_{s\in S}\mathcal{A}_s$:

$$\mathbf{Diag}(\Gamma)(\mathcal{R}) := \{\psi \in \mathrm{Aut}_{\mathcal{R}}(\mathcal{A} \otimes \mathcal{R}) \mid \psi|_{\mathcal{A}_s \otimes \mathcal{R}} \in \mathcal{R}^\times \,\mathrm{id}_{\mathcal{A}_s \otimes \mathcal{R}} \text{ for all } s \in S\}. \tag{2.3}$$

If Γ is an abelian group grading and we take $G = U_{\mathrm{ab}}(\Gamma)$ in (2.2), then it follows from the defining relations of $U_{\mathrm{ab}}(\Gamma)$ that $\eta_\Gamma : U_{\mathrm{ab}}(\Gamma)^D \to \mathbf{Diag}(\Gamma)$ is an isomorphism. In particular, the group of $\mathbb{F}$-points $\mathrm{Diag}(\Gamma)$ is isomorphic to the group of characters of $U_{\mathrm{ab}}(\Gamma)$. Moreover, the automorphism group scheme $\mathbf{Stab}(\Gamma) := \mathbf{Aut}(\mathcal{A}, \Gamma)$ coincides with the centralizer of $\mathbf{Diag}(\Gamma)$ in $\mathbf{Aut}(\mathcal{A})$.

Since we are going to assume that $\mathbb{F}$ is algebraically closed, the group schemes that are *algebraic* and *smooth* can be identified with algebraic groups, by assigning to such a group scheme its group of $\mathbb{F}$-points. For a finite-dimensional algebra $\mathcal{A}$, $\mathbf{Aut}(\mathcal{A})$ is algebraic and, for an abelian group G, G^D is algebraic if and only if G is finitely generated. The smoothness condition is automatic if $\operatorname{char}\mathbb{F} = 0$, but not so if $\operatorname{char}\mathbb{F} = p$. In particular, G^D is smooth if and only if G has no p-torsion, and $\mathbf{Aut}(\mathcal{A})$ is smooth if and only if the tangent Lie algebra of the algebraic group $\operatorname{Aut}(\mathcal{A})$ coincides with $\operatorname{Der}(\mathcal{A})$, which is the tangent Lie algebra of the group scheme $\mathbf{Aut}(\mathcal{A})$ (in general, the former is contained in the latter). In any case, G^D is a diagonalizable group scheme, and the centralizers of diagonalizable subgroupschemes in smooth group schemes are known to be smooth ([SGA3, Exp. XI, 2.4], cf. [Hum75, §18.4]). Hence, if Γ is an abelian group grading on $\mathcal{A}$ and $\mathbf{Aut}(\mathcal{A})$ is smooth, then so is $\mathbf{Stab}(\Gamma)$.

Proposition 2.1. *Let $\mathcal{A}$ be a finite-dimensional algebra over an algebraically closed field $\mathbb{F}$ such that $\mathbf{Aut}(\mathcal{A})$ is smooth. If $\operatorname{char}\mathbb{F} = p$, then $U_{\mathrm{ab}}(\Gamma)$ has no p-torsion for any fine abelian group grading Γ on $\mathcal{A}$.*

Proof. This is equivalent to the statement that any maximal diagonalizable subgroupscheme $\mathbf{Q}$ of $\mathbf{Aut}(\mathcal{A})$ is smooth. We have $\mathbf{Q} \simeq G^D$ for some finitely generated abelian group G. We can write G as the direct product of a finite p-group, a finite group of order coprime to p, and a free abelian group. Consider the corresponding decomposition $\mathbf{Q} = \mathbf{Q}_0 \times \mathbf{Q}_1 \times \mathbf{T}$. Let $\mathbf{C}$ be the centralizer of $\mathbf{Q}$ in $\mathbf{Aut}(\mathcal{A})$. Then $\mathbf{Q} \subset \mathbf{C}$, $\mathbf{C}$ is smooth, and $\mathbf{T}$ is a maximal torus in $\mathbf{C}$. Indeed, if $\mathbf{T} \subset \mathbf{T}' \subset \mathbf{C}$ for some torus $\mathbf{T}'$, then $\mathbf{Q} \subset \mathbf{QT}'$ and $\mathbf{QT}'$ is diagonalizable (as a homomorphic image of $\mathbf{Q} \times \mathbf{T}'$), so we get $\mathbf{T}' \subset \mathbf{Q}$ by maximality of $\mathbf{Q}$, but then $\mathbf{T} = \mathbf{T}'$ since $\mathbf{T}$ is a maximal torus in $\mathbf{Q}$. Consider the connected component $\mathbf{C}^\circ$ (see e.g. [Wat79, §6.7]). It is smooth and contains $\mathbf{T}$ as its maximal torus. Since $\mathbf{T}$ is central in $\mathbf{C}^\circ$, $\mathbf{C}^\circ$ must be nilpotent (for example, apply [Hum75, §21.4] to the groups of $\mathbb{F}$-points) and, therefore, $\mathbf{C}^\circ = \mathbf{T} \times \mathbf{U}$ where $\mathbf{U}$ is unipotent (see e.g. [Wat79, §10.4]). Now, $\mathbf{Q}_0$ is connected, so it is contained in $\mathbf{C}^\circ$. But its projection to $\mathbf{U}$ must be trivial, since $\mathbf{U}$ does not have nontrivial diagonalizable subgroupschemes. Therefore, $\mathbf{Q}_0 \subset \mathbf{T}$, which forces $\mathbf{Q}_0 = \mathbf{1}$. □

Corollary 2.2. *Any fine abelian group grading on $\mathcal{A}$ is obtained as the eigenspace decomposition with respect to a unique maximal diagonalizable subgroup of* $\operatorname{Aut}(\mathcal{A})$*, namely,* $\operatorname{Diag}(\Gamma)$.

Thus, if $\mathbf{Aut}(\mathcal{A})$ is smooth, then we have a one-to-one correspondence between the equivalence classes of fine abelian group gradings on $\mathcal{A}$ and the conjugacy classes of maximal diagonalizable subgroups of $\operatorname{Aut}(\mathcal{A})$.

3. Definition and construction of almost fine gradings

Let $\mathcal{A}$ be a finite-dimensional algebra over an algebraically closed field $\mathbb{F}$. Let $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ be a grading on $\mathcal{A}$ with nonzero homogeneous components $\mathcal{A}_s$, universal group U and universal abelian group U_{ab} (see Subsection 2.1).

3.1. Toral rank.

It is well known that, over an algebraically closed field, all maximal tori in an algebraic group are conjugate. In particular, they have the same dimension, which is known as the (reductive) rank of the algebraic group. Let r be the rank of the automorphism group $\operatorname{Aut}(\mathcal{A})$.

Definition 3.1. The rank of the algebraic group $\operatorname{Stab}(\Gamma)$ of the automorphisms of the graded algebra $(\mathcal{A}, \Gamma)$ will be called the *toral rank* of Γ and denoted $\operatorname{tor.rank}(\Gamma)$.

Since $\mathrm{Stab}(\Gamma) \subset \mathrm{Aut}(\mathcal{A})$, we have $0 \le \mathrm{tor.rank}(\Gamma) \le r$. A maximal torus of $\mathrm{Aut}(\mathcal{A})$ gives a $\mathbb{Z}^r$-grading on $\mathcal{A}$, called its *Cartan grading* (for example, the root space decomposition of a semisimple complex Lie algebra), whose toral rank is equal to r, since $\mathrm{Stab}(\Gamma)$ contains this maximal torus (see Subsection 2.4).

If Γ' is a coarsening of Γ (see Subsection 2.3), then $\mathrm{Stab}(\Gamma) \subset \mathrm{Stab}(\Gamma')$, so $\mathrm{tor.rank}(\Gamma) \le \mathrm{tor.rank}(\Gamma')$. In particular, any coarsening of the Cartan grading has toral rank r. Those among the coarsenings that are themselves abelian group gradings are known as *toral gradings*.

3.2. **Almost fine gradings.** For any grading Γ, $\mathrm{Stab}(\Gamma)$ contains the quasitorus

$$\mathrm{Diag}(\Gamma) := \{\psi \in \mathrm{Aut}(\mathcal{A}) \mid \psi|_{\mathcal{A}_s} \in \mathbb{F}^\times \mathrm{id}_{\mathcal{A}_s} \text{ for all } s \in S\}$$

in its center. This quasitorus is isomorphic to the group of characters of the finitely generated abelian group $U_{\mathrm{ab}} = U_{\mathrm{ab}}(\Gamma)$, so its dimension is equal to the (free) rank of U_{ab}, i.e., the rank of the free abelian group $U_{\mathrm{ab}}/t(U_{\mathrm{ab}})$, where $t(U_{\mathrm{ab}})$ denotes the torsion subgroup of U_{ab}. Indeed, for any finitely generated abelian group A, the closed subgroup of $\widehat{A}$ consisting of all characters of A that kill $t(A)$ can be identified with the group of characters of $A/t(A)$, so it is a torus of dimension $\mathrm{rank}(A)$. The quotient of $\widehat{A}$ by this subgroup can be identified with the group of characters of $t(A)$, so it is a finite abelian group (whose order is not divisible by $\mathrm{char}\,\mathbb{F}$). Therefore, the connected component of the identity $\mathrm{Diag}(\Gamma)^\circ$ is isomorphic to the group of characters of $U_{\mathrm{ab}}/t(U_{\mathrm{ab}})$, a torus of dimension $\mathrm{rank}(U_{\mathrm{ab}})$. In particular,

$$\mathrm{rank}(U_{\mathrm{ab}}(\Gamma)) \le \mathrm{tor.rank}(\Gamma).$$

Definition 3.2. A grading Γ on $\mathcal{A}$ is *almost fine* if $\mathrm{rank}(U_{\mathrm{ab}}(\Gamma)) = \mathrm{tor.rank}(\Gamma)$ or, in other words, $\mathrm{Diag}(\Gamma)^\circ$ is a maximal torus in $\mathrm{Stab}(\Gamma)$.

For example, if Γ has toral rank 0, then Γ is almost fine and $U_{\mathrm{ab}}(\Gamma)$ is finite.

Remark 3.3. If Γ is almost fine, then $\mathrm{Stab}(\Gamma)^\circ$ is the direct product of the torus $\mathrm{Diag}(\Gamma)^\circ$ and a connected unipotent group (the unipotent radical).

Proof. Since $\mathrm{Diag}(\Gamma)^\circ$ is central in $\mathrm{Stab}(\Gamma)^\circ$ and is a maximal torus by hypothesis, the connected algebraic group $\mathrm{Stab}(\Gamma)^\circ$ is nilpotent (see e.g. [Hum75, §21.4]) and, hence, the direct product of its (unique) maximal torus and unipotent radical (see e.g. [Hum75, §19.2]). □

Remark 3.4. If Γ is almost fine, then any refinement of Γ is almost fine and has the same toral rank.

Proof. If Γ' is a refinement of Γ, then we have

$$\mathrm{Diag}(\Gamma) \subset \mathrm{Diag}(\Gamma') \subset \mathrm{Stab}(\Gamma') \subset \mathrm{Stab}(\Gamma).$$

By hypothesis, $\mathrm{Diag}(\Gamma)^\circ$ is a maximal torus in $\mathrm{Stab}(\Gamma)$ and, hence, in $\mathrm{Stab}(\Gamma')$. But $\mathrm{Diag}(\Gamma')^\circ$ is a torus and contains $\mathrm{Diag}(\Gamma)^\circ$, so $\mathrm{Diag}(\Gamma')^\circ = \mathrm{Diag}(\Gamma)^\circ$ by maximality. The result follows. □

In the next subsection, we will see that any grading Γ admits an almost fine refinement and, moreover, if Γ is a group (respectively, abelian group) grading, then so is this refinement. Therefore, any fine grading (in the class of all gradings, group gradings, or abelian group gradings) is almost fine.

In characteristic 0, almost fine gradings can be characterized in terms of derivations of $\mathcal{A}$. Let $\mathcal{D} = \mathrm{Der}(\mathcal{A})$, which is the Lie algebra of the algebraic group $\mathrm{Aut}(\mathcal{A})$, and let

$$\mathcal{D}_e = \{\delta \in \mathrm{Der}(\mathcal{A}) \mid \delta(\mathcal{A}_s) \subset \mathcal{A}_s \text{ for all } s \in S\}, \tag{3.1}$$

which is the Lie algebra of $\mathrm{Stab}(\Gamma)$. Note that, if Γ can be realized as a G-grading for a group G, then the associative algebra $\mathrm{End}(\mathcal{A})$ has an induced G-grading with the following components:

$$\mathrm{End}(\mathcal{A})_g := \{f \in \mathrm{End}(\mathcal{A}) \mid f(\mathcal{A}_h) \subset \mathcal{A}_{gh} \text{ for all } h \in G\},$$

and $\mathcal{D}_e = \mathcal{D} \cap \mathrm{End}(\mathcal{A})_e$, where e denotes the identity element of G. Moreover, if G is abelian, then $\mathcal{D}$ has an induced G-grading: $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ where $\mathcal{D}_g = \mathcal{D} \cap \mathrm{End}(\mathcal{A})_g$.

Proposition 3.5. *Assume* $\mathrm{char}\,\mathbb{F} = 0$. *A grading* $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ *is almost fine if and only if, for any element* $\delta \in \mathcal{D}_e$, *each of the restrictions* $\delta|_{\mathcal{A}_s}$, $s \in S$, *has a unique eigenvalue.*

Proof. If Γ is almost fine, then the decomposition $\mathrm{Stab}(\Gamma)^\circ = \mathrm{Diag}(\Gamma)^\circ \times R_u$ of Remark 3.3 gives $\mathcal{D}_e = \mathcal{M} \oplus \mathcal{N}$ where any element of $\mathcal{M}$ acts as a scalar on each $\mathcal{A}_s$ and any element of $\mathcal{N}$ is nilpotent. It follows that any element of $\mathcal{D}_e$ has a unique eigenvalue on each $\mathcal{A}_s$. Conversely, suppose $\mathrm{Diag}(\Gamma)^\circ \subset T \subset \mathrm{Stab}(\Gamma)$ where T is a torus. Then every element of the Lie algebra of T is semisimple, so it must act as a scalar on each $\mathcal{A}_s$ by hypothesis. This implies $T = \mathrm{Diag}(\Gamma)^\circ$, proving the maximality of $\mathrm{Diag}(\Gamma)^\circ$. □

There is a simpler characterization in the case of abelian group gradings if we assume that $\mathrm{Aut}(\mathcal{A})$ is a reductive algebraic group, by which we mean that its unipotent radical is trivial, but do not assume connectedness (contrary to the convention in [Hum75]). It turns out that this characterization holds in prime characteristic as well if we assume that the group scheme $\mathbf{Aut}(\mathcal{A})$ is reductive, by which we mean that it is smooth and its group of $\mathbb{F}$-points, $\mathrm{Aut}(\mathcal{A})$, is reductive. The following result is probably known, but we could not find a reference:

Lemma 3.6. *Let G be a reductive algebraic group over an algebraically closed field. Then the centralizer* $\mathrm{Cent}_G(Q)$ *of any diagonalizable subgroup* $Q \subset G$ *is reductive.*

Proof. Without loss of generality, we assume that Q is Zariski closed, so it is the product of a torus and a finite abelian group whose order is not divisible by $\mathrm{char}\,\mathbb{F}$. Since $\mathrm{Cent}_{G^\circ}(Q)$ has finite index in $\mathrm{Cent}_G(Q)$, it suffices to prove that $\mathrm{Cent}_{G^\circ}(Q)$ is reductive. Let H and Z be, respectively, the derived group $[G^\circ, G^\circ]$ and the connected component of the center $Z(G^\circ)^\circ$. Then Z is a torus and the radical of G°, $H \cap Z$ is finite (see e.g. [Hum75, §19.5]), H is connected semisimple, and $G^\circ = HZ$ (see e.g. [Hum75, §27.5]). We claim that it suffices to prove that $\mathrm{Cent}_H(Q)$ is reductive. Indeed, let $C_H = \mathrm{Cent}_H(Q)$ and $C_Z = \mathrm{Cent}_Z(Q)$. Then

$$C_H C_Z \subset \mathrm{Cent}_{G^\circ}(Q) \subset \tilde{C}_H \tilde{C}_Z,$$

where $\tilde{C}_H := \{h \in H \mid [h, q] \in H \cap Z \ \forall q \in Q\}$ and similarly for $\tilde{C}_Z$. Since $H \cap Z$ is finite, we have $\tilde{C}_H^\circ \subset C_H$ and $\tilde{C}_Z^\circ \subset C_Z$, so $C_H C_Z$ has finite index in $\tilde{C}_H \tilde{C}_Z$, and the claim follows.

Replacing G by H and Q by its image in the automorphism group of H (see e.g. [Hum75, §27.4]), we arrive at the following setting: G is a connected semisimple algebraic group, Q is a closed diagonalizable subgroup of $\mathrm{Aut}(G)$, and we have to prove that the group of fixed points G^Q is reductive. Now, Q defines a grading on the Lie algebra $\mathfrak{g}$ of G, and the Lie algebra $\mathfrak{g}_0$ of G^Q is the identity component of this grading. If $\mathrm{char}\,\mathbb{F} = 0$, then a standard argument shows that the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{g}_0$ is nondegenerate and, hence, G^Q is reductive (see e.g. [GOV94, Prop. 6.2 of Chap. 1, Prop. 3.6 of Chap. 3], cf. [EK13, Lemma 6.9]). Unfortunately, this approach does not work if $\mathrm{char}\,\mathbb{F} = p$, so we will make some further reductions.

First, we may suppose that Q is finite (of order not divisible by $\operatorname{char}\mathbb{F}$), because Q° is a torus and the centralizers of tori in reductive algebraic groups are reductive (see e.g. [Hum75, §26.2]). By induction on $|Q|$, we may further suppose that Q is cyclic: $Q = \langle s\rangle$.

Second, we may assume that G is of adjoint type (or, alternatively, simply connected), because a Q-equivariant isogeny $f : G \to H$ restricts to an isogeny $f^{-1}(H^Q) \to H^Q$ and G^Q is a subgroup of finite index in $f^{-1}(H^Q)$, by the same argument as in the first paragraph. So, we may suppose that G is the direct product of its simple factors G_i, $i \in I$. Partitioning I into Q-orbits, we see that G^Q is the direct product of groups of the form $H_J = (\prod_{i\in J} G_i)^Q$ where $J \subset I$ and Q cyclically permutes the elements of J. Let $m = |J|$ and fix $i \in J$. Then the projection $G \to G_i$ restricts to a bijective homomorphism (hence, isogeny) $H_J \to G_i^{\langle s^m\rangle}$. Therefore, we may also suppose that G is simple.

Third, let s^n be a generator of $Q \cap \operatorname{Int}(G)$. Then s^n acts as the conjugation by a semisimple element of G, and centralizers of semisimple elements in connected semisimple algebraic groups are reductive (see e.g. [Hum95, §2.2]). Replacing G with $G^{\langle s^n\rangle}$, we may suppose that $Q \cap \operatorname{Int}(G) = 1$.

This leaves only the following types of G and outer automorphisms s of order k to consider: (i) A_r, $k = 2$; (ii) D_r, $k = 2$; (iii) D_4, $k = 3$; (iv) E_6, $k = 2$ (in all cases $\operatorname{char}\mathbb{F} \neq k$). One can use models for G to classify all such automorphisms s up to conjugation in $\operatorname{Aut}(G)$: in case (i), they are classified by involutions of $M_{r+1}(\mathbb{F})$; in case (ii), by involutive improper isometries in dimension $2r$; in case (iii), by symmetric composition algebras of dimension 8 (see e.g. [KMRT98, §36]); in case (iv), by involutions of the Albert Jordan pair or, equivalently, by exceptional simple Jordan triple systems of dimension 27 (see [Loo75]). The classifications in (i) and (ii) are well known, the one in (iii) is due to A. Elduque and H.C. Myung [EM93], and the one in (iv) is due to O. Loos [Loo71]. In all cases the group of fixed points is semisimple. □

Corollary 3.7. *Assume that* $\mathbf{Aut}(\mathcal{A})$ *is reductive and let* Γ *be a fine abelian group grading on* $\mathcal{A}$*. If* $\operatorname{char}\mathbb{F} = 0$*, then* $\operatorname{Stab}(\Gamma) = \operatorname{Diag}(\Gamma)$*. If* $\operatorname{char}\mathbb{F} = p$*, then the index* $[\operatorname{Stab}(\Gamma) : \operatorname{Diag}(\Gamma)]$ *is a power of* p*.*

Proof. Let $Q = \operatorname{Diag}(\Gamma)$. Then, by Corollary 2.2, Γ is the eigenspace decomposition of $\mathcal{A}$ with respect to Q, so $\operatorname{Stab}(\Gamma)$ is the centralizer of Q in $\operatorname{Aut}(\mathcal{A})$, which is reductive by Lemma 3.6. By Remark 3.3, we then get $\operatorname{Stab}(\Gamma)^\circ = Q^\circ$. Hence, for any $s \in \operatorname{Stab}(\Gamma)$, we have $s^n \in Q$ for some $n > 0$. If $\operatorname{char}\mathbb{F} = 0$, it follows that s is semisimple and, therefore, $\langle Q, s\rangle$ is diagonalizable, which forces $s \in Q$ by maximality of Q. If $\operatorname{char}\mathbb{F} = p$, choose n minimal possible and write $n = mp^k$ where $p \nmid m$. Applying the above argument to s^{p^k}, we see that $s^{p^k} \in Q$. □

Proposition 3.8. *Assume that* $\mathbf{Aut}(\mathcal{A})$ *is reductive. Then, for an abelian group grading* Γ *on* $\mathcal{A}$*, the following conditions are equivalent:*

(i) Γ *is almost fine;*
(ii) $\operatorname{Diag}(\Gamma)^\circ = \operatorname{Stab}(\Gamma)^\circ$*;*
(iii) $\operatorname{rank}(U_{\mathrm{ab}}(\Gamma)) = \dim \mathcal{D}_e$ *where* $\mathcal{D}_e \subset \operatorname{Der}(\mathcal{A})$ *is defined by* (3.1).

If these conditions hold, $\dim \mathcal{D}_e = \operatorname{tor.rank}(\Gamma)$ *and the elements of* $\mathcal{D}_e$ *act as scalars on each component of* Γ*.*

Proof. We will see later (Corollary 5.2) that if Γ is almost fine, then $U_{\mathrm{ab}}(\Gamma)$ has no p-torsion in the case $\operatorname{char}\mathbb{F} = p$. Then the argument in the proof of Corollary 3.7 shows that (i) ⇒ (ii). The converse is trivial.

We always have $\operatorname{rank}(U_{\mathrm{ab}}(\Gamma)) = \dim \operatorname{Diag}(\Gamma)$ and $\dim \operatorname{Stab}(\Gamma) \leq \dim \mathcal{D}_e$, since $\mathcal{D}_e$ is the Lie algebra of $\mathbf{Stab}(\Gamma)$. But we assume that $\mathbf{Aut}(\mathcal{A})$ is smooth, so

$\mathbf{Stab}(\Gamma)$ is smooth, too, and this means $\dim \mathrm{Stab}(\Gamma) = \dim \mathcal{D}_e$. It is now clear that (ii) $\Leftrightarrow$ (iii). $\square$

In particular, if $\mathrm{char}\,\mathbb{F} = 0$ and Γ is an abelian group grading on a semisimple Lie algebra $\mathcal{L}$, then $\mathrm{ad} : \mathcal{L} \to \mathcal{D} = \mathrm{Der}(\mathcal{L})$ is an isomorphism of graded algebras (for any realization of Γ over an abelian group), so $\mathcal{L}_e \simeq \mathcal{D}_e$. Thus, Γ is almost fine if and only if the quasitorus $Q = \mathrm{Diag}(\Gamma)$ satisfies $\dim Q = \dim \mathcal{L}_e$, which is *condition (*)* of Jun Yu [Yu14, Yu16], who studied such quasitori in the automorphism groups of simple complex Lie algebras. At the extreme values of toral rank for these almost fine gradings, we have the Cartan grading for which $\mathcal{L}_e$ is a Cartan subalgebra of $\mathcal{L}$ and *special gradings* of Wim Hesselink [Hes82] for which $\mathcal{L}_e = 0$.

We note that, in general, if a grading Γ on $\mathcal{A}$ satisfies $\mathcal{D}_e = 0$ then Γ is almost fine of toral rank 0.

Example 3.9. Let $\mathbb{H}$ be the split quaternion algebra over $\mathbb{F}$, $\mathrm{char}\,\mathbb{F} \neq 2$, with basis $\{\hat{1}, \hat{\imath}, \hat{\jmath}, \hat{k}\}$ and multiplication defined by $\hat{\imath}^2 = \hat{\jmath}^2 = 1$ and $\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k}$. We have a grading on $\mathbb{H}$ by the Klein group $\mathbb{Z}_2^2$: $\deg \hat{1} = (\bar{0}, \bar{0})$, $\deg \hat{\imath} = (\bar{1}, \bar{0})$, $\deg \hat{\jmath} = (\bar{0}, \bar{1})$, and $\deg \hat{k} = (\bar{1}, \bar{1})$, so we can define a $\mathbb{Z}_2^3$-grading on $M_2(\mathbb{H}) \simeq M_2(\mathbb{F}) \otimes \mathbb{H}$ by setting $\deg(E_{ij} \otimes d) = (\bar{i} - \bar{j}, \deg d) \in \mathbb{Z}_2 \times \mathbb{Z}_2^2$ for any nonzero homogeneous $d \in \mathbb{H}$.

Denote by bar the standard involution of $\mathbb{H}$, which maps $\hat{1} \mapsto \hat{1}$, $\hat{\imath} \mapsto -\hat{\imath}$, $\hat{\jmath} \mapsto -\hat{\jmath}$, $\hat{k} \mapsto -\hat{k}$. The corresponding involution $*$ on $M_2(\mathbb{H})$, $E_{ij} \otimes d \mapsto E_{ji} \otimes \bar{d}$, preserves degrees, so the Lie algebra of skew elements

$$\mathcal{L} = \{X \in M_2(\mathbb{H}) \mid X^* = -X\}$$

becomes $\mathbb{Z}_2^3$-graded. This is a simple Lie algebra of type B_2, which is isomorphic to the algebra of derivations of either itself or the associative algebra with involution $M_2(\mathbb{H})$. Since $\mathcal{L}_e = 0$, we have almost fine gradings of toral rank 0 on $\mathcal{L}$ and $M_2(\mathbb{H})$. However, these gradings are not fine, because they can be refined to $\mathbb{Z}_2^4$-gradings. Indeed, transporting the $\mathbb{Z}_2^2$-grading via the isomorphism $\mathbb{H} \to M_2(\mathbb{F})$ defined by

$$\hat{\imath} \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \ \hat{\jmath} \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

we obtain a $\mathbb{Z}_2^2$-grading on $M_2(\mathbb{F})$, which is a refinement of the original $\mathbb{Z}_2$-grading $\deg E_{ij} = \bar{i} - \bar{j}$. Consequently, we obtain a $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$-grading on $M_2(\mathbb{H}) \simeq M_2(\mathbb{F}) \otimes \mathbb{H}$, which is a refinement of the original $\mathbb{Z}_2 \times \mathbb{Z}_2^2$-grading and is still preserved by the involution $*$ (cf. [EK13, Theorem 3.30 and Remark 6.60]).

3.3. **Canonical almost fine refinement.** Given a grading Γ on $\mathcal{A}$, pick a maximal torus T in $\mathrm{Stab}(\Gamma)$. Then the eigenspace decomposition of each homogeneous component $\mathcal{A}_s$ with respect to the action of T yields a refinement of Γ:

$$\mathcal{A} = \bigoplus_{(s,\lambda) \in S \times \mathfrak{X}(T)} \mathcal{A}_{(s,\lambda)} \quad \text{with } \mathcal{A}_{(s,\lambda)} := \{a \in \mathcal{A}_s \mid \tau(a) = \lambda(\tau) a \ \forall \tau \in T\}, \tag{3.2}$$

where $\mathfrak{X}(T)$ denotes the group of characters of T, i.e., the algebraic group homomorphisms from T to the multiplicative group. We will denote this refinement by Γ_T^*. Clearly, if Γ is a G-grading for some group G, then Γ_T^* is a $G \times \mathfrak{X}(T)$-grading.

Lemma 3.10. *If T and T' are maximal tori of $\mathrm{Stab}(\Gamma)$, then the gradings Γ_T^* and $\Gamma_{T'}^*$ are equivalent.*

Proof. Since we are assuming $\mathbb{F}$ algebraically closed, there exists $\varphi \in \mathrm{Stab}(\Gamma)$ such that $\varphi T \varphi^{-1} = T'$ or, in other words, $T' = (\mathrm{Int}\,\varphi)(T)$, where $\mathrm{Int}\,\varphi$ is the inner automorphism determined by φ. Thus we get an isomorphism $\hat{\varphi} : \mathfrak{X}(T') \to \mathfrak{X}(T)$ sending $\lambda' \mapsto \lambda' \circ \mathrm{Int}\,\varphi$, and it follows from the definition that $\varphi\big(\mathcal{A}_{(s,\lambda)}\big) = \mathcal{A}'_{(s,\hat{\varphi}^{-1}(\lambda))}$ for all $s \in S$, $\lambda \in \mathfrak{X}(T)$. $\square$

Lemma 3.11. $\mathrm{Diag}(\Gamma_T^*)^\circ = T$ *is a maximal torus in* $\mathrm{Stab}(\Gamma_T^*)$. *In particular,* Γ_T^* *is an almost fine grading and* $\mathrm{tor.\,rank}(\Gamma_T^*) = \mathrm{tor.\,rank}(\Gamma)$.

Proof. By definition, every element $\tau \in T$ acts as the scalar $\lambda(\tau)$ on $\mathcal{A}_{(s,\lambda)}$, so we have $T \subset \mathrm{Diag}(\Gamma_T^*)^\circ \subset \mathrm{Stab}(\Gamma_T^*)$. Since Γ_T^* is a refinement of Γ, we also have $\mathrm{Stab}(\Gamma_T^*) \subset \mathrm{Stab}(\Gamma)$. By the maximality of the torus T in $\mathrm{Stab}(\Gamma)$, we conclude that $T = \mathrm{Diag}(\Gamma_T^*)^\circ$ is a maximal torus in $\mathrm{Stab}(\Gamma_T^*)$. □

Corollary 3.12. *For any grading* Γ, $\mathrm{tor.\,rank}(\Gamma)$ *is the maximum of* $\dim \mathrm{Diag}(\Gamma')$ *over all refinements* Γ' *of* Γ. *If* Γ *is a group (respectively, abelian group) grading, then this maximum is attained among group (respectively, abelian group) gradings.*

Proof. If Γ' is a refinement of Γ, then $\mathrm{Diag}(\Gamma') \subset \mathrm{Stab}(\Gamma') \subset \mathrm{Stab}(\Gamma)$. Since $\mathrm{Diag}(\Gamma')^\circ$ is a torus, we get $\dim \mathrm{Diag}(\Gamma') \leq \mathrm{tor.\,rank}(\Gamma)$. The result now follows by Lemma 3.11. □

The last two lemmas justify the following terminology:

Definition 3.13. For any maximal torus $T \subset \mathrm{Stab}(\Gamma)$, the refinement Γ_T^* will be called the *canonical almost fine refinement of* Γ.

The following is an abstract characterization of Γ_T^* among refinements of Γ.

Proposition 3.14. *The following are equivalent for a refinement* Γ' *of* Γ*:*

(i) Γ' *is almost fine and* $\mathrm{tor.\,rank}(\Gamma') = \mathrm{tor.\,rank}(\Gamma)$*;*
(ii) Γ' *is a refinement of* Γ_T^* *for some* T.

Proof. If Γ' is a refinement of Γ_T^*, then Γ' satisfies (i) by Lemma 3.11 and Remark 3.4. Conversely, if Γ' satisfies (i), then $T := \mathrm{Diag}(\Gamma')^\circ$ is a maximal torus in $\mathrm{Stab}(\Gamma')$ and, hence, in $\mathrm{Stab}(\Gamma)$, since $\mathrm{Stab}(\Gamma)$ and its subgroup $\mathrm{Stab}(\Gamma')$ have the same rank by hypothesis. But the elements of T act as scalars on each component of Γ', so Γ' must be a refinement of Γ_T^*. □

4. Classification of group gradings up to isomorphism

We will now show how a classification of almost fine group gradings on $\mathcal{A}$ up to equivalence can be used to obtain, for any group G, a classification of G-gradings on $\mathcal{A}$ up to isomorphism.

Let Γ be a G-grading on $\mathcal{A}$. As discussed in the introduction, Γ can be obtained from a fine group grading Δ by a homomorphism $\alpha : U(\Delta) \to G$, but neither Δ nor α is unique. To remedy the situation, we restrict the class of homomorphisms α that we are going to use, and this forces us to extend the class of gradings from which we will take Δ by allowing Δ to be almost fine.

Definition 4.1. Let Δ be an almost fine group grading on $\mathcal{A}$, $U = U(\Delta)$, $U_{\mathrm{ab}} = U_{\mathrm{ab}}(\Delta)$, and let $\pi_\Delta : U \to U_{\mathrm{ab}}/t(U_{\mathrm{ab}})$ be the composition of the natural homomorphisms $U \to U_{\mathrm{ab}} \to U_{\mathrm{ab}}/t(U_{\mathrm{ab}})$. A group homomorphism $\alpha : U \to G$ is said to be *admissible* if the restriction of the homomorphism $(\alpha, \pi_\Delta) : U \to G \times U_{\mathrm{ab}}/t(U_{\mathrm{ab}})$ to the support of Δ is injective.

In the abelian case, i.e., if G is an abelian group and Δ is an abelian group grading, the restriction of the natural homomorphism $\pi_{\mathrm{ab}} : U \to U_{\mathrm{ab}}$ to the support of Δ is injective and any homomorphism $\alpha : U \to G$ is the composition of π_{ab} and a (unique) homomorphism $\alpha' : U_{\mathrm{ab}} \to G$. Hence, the condition in Definition 4.1 reduces to the following: the restriction of (α', π'_Δ) to the support of Δ is injective, where π'_Δ is the natural homomorphism $U_{\mathrm{ab}} \to U_{\mathrm{ab}}/t(U_{\mathrm{ab}})$. We will say that α' is *admissible* if this is satisfied.

Lemma 4.2. *Let Δ be an almost fine group grading (respectively, abelian group grading) on $\mathcal{A}$ and let G be a group (respectively, abelian group). Denote $T = \mathrm{Diag}(\Delta)^\circ$. Let $\Gamma = {}^\alpha\Delta$ be the G-grading induced by a homomorphism $\alpha : U(\Delta) \to G$ (respectively, $\alpha : U_{\mathrm{ab}}(\Delta) \to G$) . Then the following are equivalent:*

(i) *α is admissible;*
(ii) *T is a maximal torus in $\mathrm{Stab}(\Gamma)$ and the set of nonzero homogeneous components of Γ^*_T coincides with that of Δ (in particular, these gradings are equivalent).*

Proof. Recall from Subsection 3.2 that the torus $T = \mathrm{Diag}(\Delta)^\circ$ is isomorphic to the group of characters of $U_{\mathrm{ab}}/t(U_{\mathrm{ab}})$ where $U_{\mathrm{ab}} = U_{\mathrm{ab}}(\Delta)$. Hence, we obtain an evaluation homomorphism $\varepsilon : U = U(\Delta) \to \mathfrak{X}(T)$, which is the composition of π_Δ and the isomorphism $U_{\mathrm{ab}}/t(U_{\mathrm{ab}}) \to \mathfrak{X}(T)$. Explicitly, for any s in the support of Δ, $\varepsilon(s)$ is the character of T that maps each $\tau \in T$ to the scalar by which τ acts on the component $\mathcal{A}_s$ of Δ. It follows that the induced $G \times \mathfrak{X}(T)$-grading ${}^{(\alpha,\varepsilon)}\Delta$ coincides with the $G \times \mathfrak{X}(T)$-grading Γ' obtained from Γ by decomposing each of its components into eigenspaces with respect to the action of $T \subset \mathrm{Stab}(\Gamma)$.

Now, if (ii) holds, then $\Gamma' = \Gamma^*_T$ by definition and, hence, the coarsening ${}^{(\alpha,\varepsilon)}\Delta$ of Δ is not proper, so the restriction of (α, ε) to the support of Δ is injective. Since ε is the composition of π_Δ and an isomorphism, we get (i).

Conversely, assume (i). Then the restriction of (α, ε) to the support of Δ is injective, so $\Gamma' = {}^{(\alpha,\varepsilon)}\Delta$ has the same nonzero homogeneous components as Δ and, hence, $\mathrm{Stab}(\Gamma') = \mathrm{Stab}(\Delta)$. But $\mathrm{Stab}(\Gamma') = \mathrm{Cent}_{\mathrm{Stab}(\Gamma)}(T)$, so T is a maximal torus in $\mathrm{Cent}_{\mathrm{Stab}(\Gamma)}(T)$, since Δ is almost fine. It follows that T is a maximal torus in $\mathrm{Stab}(\Gamma)$. Since $\Gamma' = \Gamma^*_T$, we see that (ii) holds. □

Theorem 4.3. *Let $\{\Gamma_i\}_{i\in I}$ be a set of representatives of the equivalence classes of almost fine group (respectively, abelian group) gradings on $\mathcal{A}$. For any group (respectively, abelian group) G and a G-grading Γ on $\mathcal{A}$, there exists a unique $i \in I$ such that Γ is isomorphic to the induced grading ${}^\alpha\Gamma_i$ for some admissible homomorphism $\alpha : U(\Gamma_i) \to G$ (respectively, $\alpha : U_{\mathrm{ab}}(\Gamma_i) \to G$). Moreover, two such homomorphisms, α and α', induce isomorphic G-gradings if and only if there exists $w \in W(\Gamma_i)$ such that $\alpha = \alpha' \circ w$.*

Proof. Consider the canonical almost fine refinement $\Gamma^* = \Gamma^*_T$, for some maximal torus $T \subset \mathrm{Stab}(\Gamma)$. Since Γ^* is equivalent to some Γ_i, there exists an automorphism φ of $\mathcal{A}$ that moves the set of nonzero homogeneous components of Γ^* onto that of Γ_i. Hence, $\varphi(\Gamma)$ is a coarsening of Γ_i, so there exists a homomorphism $\alpha : U(\Gamma_i) \to G$ such that $\varphi(\Gamma) = {}^\alpha\Gamma_i$. Since $T = \mathrm{Diag}(\Gamma^*)^\circ$ by Lemma 3.11, we have $\varphi T \varphi^{-1} = \mathrm{Diag}(\Gamma_i)^\circ$ and can apply Lemma 4.2, with $\Delta = \Gamma_i$, to conclude that α is admissible.

Now, suppose that $\alpha : U(\Gamma_i) \to G$ and $\alpha' : U(\Gamma_j) \to G$ are admissible homomorphisms such that the induced G-gradings ${}^\alpha\Gamma_i$ and ${}^{\alpha'}\Gamma_j$ are isomorphic, i.e., there exists $\varphi \in \mathrm{Aut}(\mathcal{A})$ such that $\varphi({}^\alpha\Gamma_i) = {}^{\alpha'}\Gamma_j$. In particular, we have $\varphi\,\mathrm{Stab}({}^\alpha\Gamma_i)\varphi^{-1} = \mathrm{Stab}({}^{\alpha'}\Gamma_j)$. Let $T = \mathrm{Diag}(\Gamma_i)^\circ$ and $T' = \varphi^{-1}\,\mathrm{Diag}(\Gamma_j)^\circ\varphi$. Applying Lemma 4.2 to $\Delta = \Gamma_i$ and to $\Delta = \Gamma_j$, we see that T and T' are maximal tori of $\mathrm{Stab}({}^\alpha\Gamma_i)$, Γ_i is equivalent to $({}^\alpha\Gamma_i)^*_T$, and Γ_j is equivalent to $({}^\alpha\Gamma_i)^*_{T'}$. But $({}^\alpha\Gamma_i)^*_T$ and $({}^\alpha\Gamma_i)^*_{T'}$ are equivalent by Lemma 3.10, so Γ_i and Γ_j are equivalent, which forces $i = j$.

Finally, since T and $T' = \varphi^{-1}T\varphi$ are maximal tori of $\mathrm{Stab}({}^\alpha\Gamma_i)$, there exists $\psi \in \mathrm{Stab}({}^\alpha\Gamma_i)$ such that $T' = \psi T \psi^{-1}$. Replacing φ by the composition $\varphi\psi$, we get $T = \varphi^{-1}T\varphi$ and still $\varphi({}^\alpha\Gamma_i) = {}^{\alpha'}\Gamma_i$. But then φ moves the set of nonzero homogeneous components of $({}^\alpha\Gamma_i)^*_T$ onto that of $({}^{\alpha'}\Gamma_i)^*_T$. Since, by Lemma 4.2, these sets are both equal to the set of nonzero homogeneous components of Γ_i, we see that $\varphi \in \mathrm{Aut}(\Gamma_i)$ and, hence, φ determines an element $w \in W(\Gamma_i)$ by $\varphi(\mathcal{A}_s) = \mathcal{A}_{w(s)}$ for all s in the support S of Γ_i. Now, the homogeneous component of degree $g \in G$ in the grading ${}^{\alpha}\Gamma_i$ is, by definition, the direct sum $\bigoplus_{s\in\alpha^{-1}(g)} \mathcal{A}_s$, whereas in ${}^{\alpha'}\Gamma_i$ it is $\bigoplus_{s'\in\alpha'^{-1}(g)} \mathcal{A}_{s'}$. Since φ moves the former to the latter, we conclude that, for any $s \in S$, $\varphi(\mathcal{A}_s) \subset \bigoplus_{s'\in\alpha'^{-1}(\alpha(s))} \mathcal{A}_{s'}$ and, hence, $w(s) \in \alpha'^{-1}(\alpha(s))$. This implies $\alpha'(w(s)) = \alpha(s)$ for all $s \in S$, so $\alpha = \alpha' \circ w$. □

5. From fine to almost fine gradings

As we have seen in the previous section, the knowledge of almost fine group gradings on $\mathcal{A}$ up to equivalence, together with their universal and Weyl groups, yields a classification of all G-gradings on $\mathcal{A}$ up to isomorphism, for any group G. We will now discuss, in the abelian case, how to determine almost fine gradings if fine gradings are known, which can then be used to classify all G-gradings for abelian G.

Proposition 5.1. *Let Δ be a fine abelian group grading on $\mathcal{A}$, $U = U_{\mathrm{ab}}(\Delta)$, and let Γ be an abelian group grading that is a coarsening of Δ, so $U_{\mathrm{ab}}(\Gamma) = U/E$ for some subgroup $E \subset U$. Then Γ is almost fine if and only if $E \subset t(U)$ and* $\mathrm{tor.rank}(\Gamma) = \mathrm{tor.rank}(\Delta)$.

Proof. Since Δ is a fine abelian group grading, it is almost fine (as its canonical almost fine refinement cannot be proper), so we have $\mathrm{rank}(U) = \mathrm{tor.rank}(\Delta)$. Now, tensoring the short exact sequence of abelian groups $0 \to E \to U \to U/E \to 0$ by $\mathbb{Q}$ over $\mathbb{Z}$ yields $\mathrm{rank}(U/E) = \mathrm{rank}(U) - \mathrm{rank}(E)$. Since Γ is a coarsening of Δ, we also have $\mathrm{tor.rank}(\Delta) \le \mathrm{tor.rank}(\Gamma)$. Therefore, $\mathrm{rank}(U/E) = \mathrm{tor.rank}(\Gamma)$ if and only if $\mathrm{rank}(E) = 0$ and $\mathrm{tor.rank}(\Delta) = \mathrm{tor.rank}(\Gamma)$. □

We can now extend Proposition 2.1 and Corollary 2.2 to almost fine gradings:

Corollary 5.2. *Assume* $\mathbf{Aut}(\mathcal{A})$ *is smooth. If Γ is an almost fine abelian group grading on $\mathcal{A}$, then $U_{\mathrm{ab}}(\Gamma)$ has no p-torsion in the case* $\mathrm{char}\,\mathbb{F} = p$ *and, hence, Γ is the eigenspace decomposition with respect to* $\mathrm{Diag}(\Gamma)$ *in any characteristic.*

Corollary 5.3. *Any fine abelian group grading admits only finitely many almost fine coarsenings that are themselves abelian group gradings.*

Enumerating almost fine coarsenings of Δ is helped by the fact that the subgroups E of U that lie in the same $W(\Delta)$-orbit correspond to equivalent coarsenings.

Theorem 5.4. *Under the conditions of Proposition 5.1, assume that* $\mathbf{Aut}(\mathcal{A})$ *is reductive. Let Σ be the support of the U-grading on the Lie algebra* $\mathcal{D} = \mathrm{Der}(\mathcal{A})$ *induced by the U-grading Δ on $\mathcal{A}$. Then Γ is almost fine if and only if $E \subset t(U)$ and $E \cap \Sigma \subset \{e\}$.*

Proof. Under the additional assumption, we have $\mathrm{rank}(U) = \dim \mathcal{D}_e$ by Proposition 3.8. With respect to the U/E-grading on $\mathcal{D}$ induced by Γ, the identity component is $\mathcal{D}_E := \bigoplus_{g\in E} \mathcal{D}_g$. Hence, Γ is almost fine if and only if $\mathrm{rank}(U/E) = \dim \mathcal{D}_E$ (again by Proposition 3.8). Then we proceed as in the proof of Proposition 5.1, but the condition $\mathrm{tor.rank}(\Gamma) = \mathrm{tor.rank}(\Delta)$ is replaced with $\dim \mathcal{D}_E = \dim \mathcal{D}_e$, which is equivalent to $E \cap \Sigma \subset \{e\}$. □

Example 5.5. If $\mathrm{char}\,\mathbb{F} \neq 2$, the Lie algebra $\mathcal{L} = \mathfrak{sl}_4(\mathbb{F})$ is simple of type A_3 and has a fine $\mathbb{Z}_2^4$-grading Δ obtained by refining, by means of the outer automorphism $X \mapsto -X^T$, the $\mathbb{Z}_2^3$-grading induced from the Cartan grading by the "mod 2" map $\mathbb{Z}^3 \to \mathbb{Z}_2^3$ (see e.g. [EK13, Example 3.60]). Explicitly, Δ is the restriction to $\mathfrak{sl}_4(\mathbb{F})$ of the $\mathbb{Z}_2 \times (\mathbb{Z}_2^4)_0$-grading on $\mathfrak{gl}_4(\mathbb{F})$ defined by setting $\deg(E_{ij} - E_{ji}) = (\bar{0}, \varepsilon_i - \varepsilon_j)$

and $\deg(E_{ij}+E_{ji}) = (\bar{1}, \varepsilon_i - \varepsilon_j)$ where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ is the standard basis of $\mathbb{Z}_2^4$ and $(\mathbb{Z}_2^4)_0 \simeq \mathbb{Z}_2^3$ is the span of $\varepsilon_i - \varepsilon_j$. We have $\mathcal{L} \simeq \mathrm{Der}(\mathcal{L})$, $\mathcal{L}_e = 0$ and, moreover, $\mathcal{L}_g = 0$ for $g \in \mathbb{Z}_2 \times \{(\bar{1}, \bar{1}, \bar{1}, \bar{1})\}$. It follows that, for each of the two possible values of g, the $\mathbb{Z}_2^3$-grading on $\mathcal{L}$ induced by the natural homomorphism $\mathbb{Z}_2^4 \to \mathbb{Z}_2^4/\langle g \rangle$ is almost fine. In fact, these two almost fine gradings are equivalent, because the two values of g are in the same $W(\Delta)$-orbit (see [EK13, Example 3.63]).

6. Non-special gradings and root systems on semisimple Lie algebras

In this section $\mathcal{L}$ will be a semisimple finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. The aim is to show that a (possibly nonreduced) root system of rank r can be attached canonically to any abelian group grading Γ on $\mathcal{L}$ of toral rank $r \neq 0$, i.e., to any non-special Γ. We will take advantage of the results in [Eld15] that deal with the case when Γ is fine. For the definition of possibly nonreduced root systems, see e.g. [Bou02, Ch. VI,§1].

Let G be an abelian group and let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ be a G-grading on $\mathcal{L}$ with $\mathrm{tor.rank}(\Gamma) \geq 1$. Let T be a maximal torus in $\mathrm{Stab}(\Gamma)$. It induces a weight space decomposition:

$$\mathcal{L} = \bigoplus_{\lambda\in\mathfrak{X}(T)} \mathcal{L}(\lambda) \tag{6.1}$$

where $\mathcal{L}(\lambda) = \{x \in \mathcal{L} \mid \tau(x) = \lambda(\tau)x \ \forall \tau \in T\}$.

Let Γ_T^* be the associated almost fine grading, as in (3.2):

$$\Gamma_T^* : \mathcal{L} = \bigoplus_{(g,\lambda)\in G\times\mathfrak{X}(T)} \mathcal{L}_g \cap \mathcal{L}(\lambda). \tag{6.2}$$

Let $\mathcal{H}$ be the Lie algebra of T inside $\mathcal{L} \simeq \mathrm{Der}(\mathcal{L})$, so $\mathcal{H}$ is a Cartan subalgebra of the reductive Lie subalgebra $\mathcal{L}_e$. The adjoint action of $\mathcal{H}$ on any weight space $\mathcal{L}(\lambda)$ is given by the differential $\mathrm{d}\lambda \in \mathcal{H}^*$, which is therefore a weight of the adjoint action of $\mathcal{H}$ on $\mathcal{L}$. To simplify notation, we will use λ to denote its differential, too, and thus identify $\mathfrak{X}(T)$ with a subgroup of $\mathcal{H}^*$.

Denote by Φ the set of nonzero weights of $\mathcal{H}$ on $\mathcal{L}$:

$$\Phi = \{\lambda \in \mathcal{H}^* \setminus \{0\} \mid \mathcal{L}(\lambda) \neq 0\}.$$

Under the above identification, we have $\mathbb{Z}\Phi = \mathfrak{X}(T)$ and also $\mathcal{H} \subset \mathcal{L}_{(e,0)} = \mathcal{L}_e \cap \mathcal{L}(0)$, where 0 is (the differential of) the trivial character on T. Since Γ_T^* is almost fine, we have equality: $\mathcal{H} = \mathcal{L}_e \cap \mathcal{L}(0)$ by Proposition 3.8.

Theorem 6.1. *With the hypotheses above, Φ is a (possibly nonreduced) root system on $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}\Phi$. If $\mathcal{L}$ is simple, then Φ is an irreducible root system.*

Proof. Let Γ' be a refinement of Γ_T^* that is a fine abelian group grading, and let U be its universal abelian group. Lemma 3.11 and Remark 3.4 show that $T = \mathrm{Diag}(\Gamma_T^*)^\circ = \mathrm{Diag}(\Gamma')^\circ$ is a maximal torus in $\mathrm{Stab}(\Gamma')$, and hence $U/t(U) \simeq \mathfrak{X}(T)$. Now [Eld15, Theorem 4.4] (or [EK13, Theorem 6.61]) gives the result. □

Abelian group gradings on semisimple Lie algebras have been reduced to gradings on simple Lie algebras in [CE18]. For simple Lie algebras, Theorem 6.1 implies that any non-special abelian group grading is related to a grading by a root system. These gradings were first studied by S. Berman and R.V. Moody [BM92].

Definition 6.2. A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the reduced root system* Φ, or Φ-*graded*, if the following conditions are satisfied:

(i) $\mathcal{L}$ contains as a subalgebra a finite-dimensional simple Lie algebra whose root system relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is Φ: $\mathfrak{g} = \mathfrak{h} \oplus \big(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\big)$;

(ii) $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$;

(iii) $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

The subalgebra $\mathfrak{g}$ is said to be a *grading subalgebra* of $\mathcal{L}$.

The simply laced case (i.e., types A_r, D_r and E_r) was studeid in [BM92], and G. Benkart and E. Zelmanov considered the remaining cases in [BZ96].

As to nonreduced root systems, the definition works as follows (see [ABG02]):

Definition 6.3. Let Φ be the nonreduced root system BC_r ($r \geq 1$). A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by* Φ, or Φ*-graded*, if the following conditions are satisfied:

(i) $\mathcal{L}$ contains as a subalgebra a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{h}\oplus\big(\bigoplus_{\alpha\in\Phi'} \mathfrak{g}_\alpha\big)$ whose root system Φ' relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is the reduced subsystem of type B_r, C_r or D_r contained in Φ;

(ii) $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$;

(iii) $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

Again, the subalgebra $\mathfrak{g}$ is said to be a *grading subalgebra* of $\mathcal{L}$, and $\mathcal{L}$ is said to be BC_r*-graded with grading subalgebra of type* X_r, where X_r is the type of $\mathfrak{g}$.

Assume from now on that $\mathcal{L}$ is a finite-dimensional simple Lie algebra over our algebraically closed field $\mathbb{F}$ of characteristic 0 and let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ be a non-special grading by an abelian group G. As in the proof of Theorem 6.1, let Γ^*_T be the canonical almost fine refinement of Γ, and refine Γ^*_T to a fine abelian group grading $\Gamma' = \bigoplus_{u\in U} \mathcal{L}'_u$, where $U = U_{\mathrm{ab}}(\Gamma')$ (which coincides with $U(\Gamma')$ since $\mathcal{L}$ is simple [EK13, Corollary 1.21]). Then, for any $u \in U$, there is a unique $\alpha \in \Phi \cup \{0\}$ such that $\mathcal{L}'_u \subset \mathcal{L}(\alpha)$, and this induces a surjective group homomorphism $\pi : U \to \mathbb{Z}\Phi$ with kernel $t(U)$.

Fix a system Δ of simple roots of Φ. Then $\Phi = \Phi^+\cup\Phi^-$, with $\Phi^+ \subset \sum_{\alpha\in\Delta} \mathbb{Z}_{\geq0}\alpha$ and $\Phi^- = -\Phi^+$. As in [Eld15, §5], we choose, for any $\alpha \in \Delta$, an element $u_\alpha \in U$ such that $\pi(u_\alpha) = \alpha$. This gives us a section of the homomorphism π, so U becomes the direct product of the free abelian group U' generated by the elements u_α, $\alpha \in \Delta$, and its torsion subgroup $t(U)$. For any $\lambda \in \mathbb{Z}\Phi$, we will denote by u_λ the unique element of U' such that $\pi(u_\lambda) = \lambda$.

Now, [Eld15, Theorem 5.1] (or [EK13, Theorem 6.62]) shows that

$$\mathfrak{g} := \bigoplus_{u\in U'} \mathcal{L}'_u$$

is a simple Lie algebra with Cartan subalgebra $\mathfrak{h} = \mathcal{H}$ and a root system $\Phi' \subset \Phi$ such that Δ is a system of simple roots. Moreover, $\mathcal{L}$ is graded by the irreducible root system Φ with grading subalgebra $\mathfrak{g}$, and if Φ is nonreduced (type BC_r), then $\mathfrak{g}$ is simple of type B_r. By construction, not only $\mathfrak{g}$ but also the components of its triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ associated to Δ are graded subalgebras of $\mathcal{L}$ with respect to the fine grading Γ' and, hence, also for Γ^*_T and Γ.

In this situation, the adjoint action of $\mathfrak{g}$ on $\mathcal{L}$ decomposes $\mathcal{L}$ into a direct sum of irreducible submodules of only a few isomorphism classes. Collecting isomorphic submodules, we get the corresponding isotypic decomposition (see [ABG02]):

- If Φ is reduced, then the isotypic decomposition is

$$\mathcal{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathcal{W} \otimes \mathcal{C}) \oplus \mathcal{D},$$

  where $\mathcal{W} = 0$ is Φ is simply laced, and otherwise it is the irreducible module whose highest weight, relative to $\mathcal{H}$ and Δ, is the highest short root in Φ. The component $\mathcal{D}$ is the sum of trivial one-dimensional modules, so $\mathcal{D} = \mathrm{Cent}_{\mathcal{L}}(\mathfrak{g})$ is a subalgebra of $\mathcal{L}$. Note that $\mathcal{A}$ contains a distinguished element 1 that identifies the subalgebra $\mathfrak{g}$ with $\mathfrak{g}\otimes 1$. In this case $\mathfrak{a} := \mathcal{A}\oplus\mathcal{C}$ becomes the *coordinate algebra* with identity 1, whose product is determined by the bracket in $\mathcal{L}$. Depending on the type of Φ, different classes of algebras (associative, alternative, Jordan) may appear as coordinate algebras.

- If Φ is of type BC_r with grading subalgebra of type B_r and $r \geq 2$, then the isotypic decomposition is
$$\mathcal{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathfrak{s} \otimes \mathcal{B}) \oplus (\mathcal{W} \otimes \mathcal{C}) \oplus \mathcal{D},$$
where $\mathcal{W}$ is the natural module, of dimension $2r+1$, for the simple Lie algebra $\mathfrak{g} \simeq \mathfrak{so}_{2r+1}(\mathbb{F})$, so $\mathcal{W}$ is endowed with an invariant symmetric nondegenerate bilinear form $(\cdot \mid \cdot)$. Then
$$\mathfrak{s} = \{f \in \mathrm{End}_{\mathbb{F}}(\mathcal{W}) \mid (f(v) \mid w) = (v \mid f(w)) \;\; \forall u, v \in \mathcal{W},\ \mathrm{tr}(f) = 0\}.$$
The subalgebra $\mathcal{D}$ is again the centralizer of $\mathfrak{g}$, and the *coordinate algebra* is $\mathfrak{a} := \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$.
- If Φ is of type BC_1 with grading subalgebra of type B_1, then the adjoint module is isomorphic to the natural module and the isotypic decomposition reduces to
$$\mathcal{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathfrak{s} \otimes \mathcal{B}) \oplus \mathcal{D},$$
with *coordinate algebra* $\mathfrak{a} := \mathcal{A} \oplus \mathcal{B}$.

To simplify notation, we will write $\mathfrak{a} = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$ in all cases, with the understanding that $\mathcal{B}$ or $\mathcal{C}$ may be 0.

It is clear that $\mathcal{D} = \mathrm{Cent}_{\mathcal{L}}(\mathfrak{g})$ is a graded subalgebra with respect to Γ'. If λ is the highest root of $\mathfrak{g}$, then $\mathfrak{g}_\lambda \otimes \mathcal{A} = \{x \in \mathcal{L}(\lambda) \mid [\mathfrak{g}_+, x] = 0\}$ is a graded subspace of $\mathcal{L}$ with respect to Γ'. Since $\dim \mathfrak{g}_\lambda = 1$, this allows us to define a grading on $\mathcal{A}$ by the torsion subgroup $t(U)$ as follows: $\mathcal{A} = \bigoplus_{u \in t(U)} \mathcal{A}'_u$ where $\mathfrak{g}_\lambda \otimes \mathcal{A}'_u = (\mathfrak{g}_\lambda \otimes \mathcal{A}) \cap \mathcal{L}'_{u_\lambda u}$. Since $\mathfrak{g} \otimes \mathcal{A}$ is the $\mathfrak{g}$-submodule of $\mathcal{L}$ generated by $\mathfrak{g}_\lambda \otimes \mathcal{A}$, it follows that the isotypic component $\mathfrak{g} \otimes \mathcal{A}$ is graded and $\mathfrak{g}_\mu \otimes \mathcal{A}'_u = (\mathfrak{g} \otimes \mathcal{A}) \cap \mathcal{L}'_{u_\mu u}$ for all $\mu \in \mathbb{Z}\Phi$ and $u \in t(U)$. The same argument applies to the other possible isotypic components $\mathfrak{s} \otimes \mathcal{B}$ and $\mathcal{W} \otimes \mathcal{C}$, substituting for λ the highest weight of $\mathfrak{s}$ or $\mathcal{W}$. It follows that the coordinate algebra $\mathfrak{a}$ inherits a grading by $t(U)$.

Now let $\delta : U \to G$ be the group homomorphism obtained from the fact that Γ' is a refinement of Γ: $\mathcal{L}_g = \bigoplus_{u \in \delta^{-1}(g)} \mathcal{L}'_u$ for any $g \in G$. Then $\mathfrak{g}$ and the isotypic components are graded subspaces of $\mathcal{L}$ with respect to Γ, with the G-gradings induced by δ. We also define a G-grading on $\mathfrak{a}$ (and its pieces) via δ. For the (reductive) subalgebra $\mathcal{L}(0) = (\mathfrak{g}_0 \otimes \mathcal{A}) \oplus (\mathfrak{s}_0 \otimes \mathcal{B}) \oplus (\mathcal{W}_0 \otimes \mathcal{C}) \oplus \mathcal{D}$, the identity component with respect to Γ is $\mathcal{H} = \mathcal{L}(0)_e = (\mathfrak{g}_0 \otimes \mathcal{A}_e) \oplus (\mathfrak{s}_0 \otimes \mathcal{B}_e) \oplus (\mathcal{W}_0 \otimes \mathcal{C}_e) \oplus \mathcal{D}_e$. But $\mathfrak{g}_0 = \mathcal{H}$, so we conclude that $\mathcal{A}_e = \mathbb{F}1$ and $\mathcal{B}_e = \mathcal{C}_e = \mathcal{D}_e = 0$. On the other hand, since $\mathcal{L}(0) = \mathrm{Cent}_{\mathcal{L}}(\mathcal{H})$, we have $\mathcal{H} = \mathcal{L}(0)_e \subset Z(\mathcal{L}(0))$. Therefore, the restriction of Γ to the semisimple Lie algebra $[\mathcal{L}(0), \mathcal{L}(0)]$ is a special grading.

The fine grading Γ' is not uniquely determined by Γ. In fact, by Proposition 3.14, we can take as Γ' any fine refinement of Γ that has the same toral rank. We have obtained the following result:

Theorem 6.4. *Let $\mathcal{L}$ be a finite-dimensional simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a non-special grading on $\mathcal{L}$ by an abelian group G. There exists a fine refinement Γ' of Γ whose identity component is a Cartan subalgebra $\mathcal{H}$ of the reductive subalgebra $\mathcal{L}_e$. For any such Γ', let $U = U_{\mathrm{ab}}(\Gamma)$ and let $\pi : U \to \mathbb{Z}\Phi$ and $\delta : U \to G$ be homomorphisms defined by $\mathcal{L}'_u \subset \mathcal{L}(\pi(u))$ and $\mathcal{L}'_u \subset \mathcal{L}_{\delta(\alpha)}$, where Φ is the root system as in Theorem 6.1, associated to the decomposition $\mathcal{L} = \bigoplus_{\alpha \in \Phi \cup \{0\}} \mathcal{L}(\alpha)$ with respect to the adjoint action of $\mathcal{H}$. Then π is surjective with kernel $t(U)$, and any homomorphism $\lambda \mapsto u_\lambda$ splitting π defines a Φ-grading on $\mathcal{L}$ and a G-grading on $\mathfrak{a} = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$ such that the isotypic components are G-graded subspaces of $\mathcal{L}$ with $\deg(\mathfrak{g}_\alpha \otimes \mathcal{A}_g) = g_\alpha g$ for all $g \in G$, and similarly for $\mathfrak{s} \otimes \mathcal{B}$ and $\mathcal{W} \otimes \mathcal{C}$ if applicable, where $g_\alpha = \delta(u_\alpha)$.*

Moreover, the supports of the G-gradings on $\mathfrak{a}$ and $\mathcal{L}(0)$ are contained in $t(G)$, $\mathfrak{a}_e = \mathbb{F}1$, and the gradings on the subalgebras $\mathcal{D}$ and $[\mathcal{L}(0), \mathcal{L}(0)]$ of $\mathcal{L}(0)$ are special.

Example 6.5. The simple Lie algebra $\mathcal{L}$ of type E_8 is the Lie algebra obtained by means of the Tits construction using the Cayley algebra $\mathbb{O}$ and the Albert (i.e., exceptional simple Jordan) algebra $\mathbb{A}$: $\mathcal{L} = \mathrm{Der}(\mathbb{O}) \oplus (\mathbb{O}_0 \otimes \mathbb{A}_0) \oplus \mathrm{Der}(\mathbb{A})$ (see e.g. [EK13, §6.2]). The Cayley algebra is endowed with a $\mathbb{Z}_2^3$-grading (a division grading), and this induces naturally a $\mathbb{Z}_2^3$-grading Γ on $\mathcal{L}$. The group $\mathrm{Aut}(\mathbb{A})$ (simple of type F_4) embeds naturally in $\mathrm{Stab}(\Gamma) \subset \mathrm{Aut}(\mathcal{L})$, and any maximal torus T in $\mathrm{Aut}(\mathbb{A})$ is a maximal torus in $\mathrm{Stab}(\Gamma)$. The canonical almost fine refinement Γ_T^* is the $\mathbb{Z}^4 \times \mathbb{Z}_2^3$-grading obtained by combining the Cartan grading on $\mathbb{A}$ (induced by T) and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$. It happens in this case that Γ_T^* is fine.

Here the root system Φ is of type F_4 and the isotypic decomposition is given by the components in Tits construction: $\mathfrak{g} = \mathrm{Der}(\mathbb{A})$, $\mathcal{A} = \mathbb{F}1$, $\mathcal{W} = \mathbb{A}_0$, $\mathcal{B} = \mathbb{O}_0$, and $\mathcal{D} = \mathrm{Der}(\mathbb{O})$ ($\mathfrak{s} = 0$ in this case). The coordinate algebra is $\mathfrak{a} = \mathbb{F}1 \oplus \mathbb{O}_0$ is just the Cayley algebra $\mathbb{O}$. The reductive subalgebra $\mathcal{L}(0)$ is the direct sum of the Lie algebra of T (a Cartan subalgebra of $\mathrm{Der}(\mathbb{A})$) and the simple Lie algebra $\mathrm{Der}(\mathbb{O})$.

In conclusion, we note that the Φ-grading on $\mathcal{L}$ defined by a non-special fine grading Γ' allows us to restate the conditions in Theorem 5.4 and Definition 4.1 quite explicitly. Let $S_\mathfrak{a}$ and $S_\mathcal{D}$ be the supports of the $t(U)$-gradings on $\mathfrak{a}$ and $\mathcal{D}$, respectively, so the support of $\mathcal{L}(0)$ is $S = S_\mathfrak{a} \cup S_\mathcal{D}$. Then the almost fine coarsenings of Γ' are determined by the subgroups $E \subset t(U)$ that are generated by some elements of the form uv^{-1} with $u, v \in S$ (so U/E is the universal group of the coarsening [EK13, Corollary 1.26]) and satisfy $E \cap S = \{e\}$. A homomorphism $\gamma : U/E \to G$ is admissible if and only if its restriction to the support of each $\mathcal{L}(\alpha)$ is injective, which amounts to $\gamma|_S$ being injective.

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